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Harmonic Variational Inequalities

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Abstract: In this paper, we consider a new class of variational inequalities, which is called the harmonic variational inequality. It is shown that that the minimum of a differentiable harmonic convex function on the harmonic convex set can be characterized by the harmonic variational inequality. We use the auxiliary principle technique to discuss the existence of a solution of the harmonic variational inequality. Results proved in this paper may stimulate further research in this field.

Keywords: Harmonic convex functions, variational inequalities, Auxiliary principle technique, Existence.

1 Introduction

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of harmonic functions. Anderson et al [1] have investigated several aspects of the harmonic convex functions. Iscan [6] and Noor et al [15,16,17] have derived several Hermite-Hadmard, Simpson, Trapezoid, Choloswki type integral inequalities for the harmonic convex functions and their variant form. It is well known that the convex functions are closely related with the variational inequalities theory, which was introduced and studied by Stampacchia [19] in 1964. For the recent applications, formulation, numerical methods and other aspects of variational inequalities, see [2,4,5,7,9,10,11,12,13,14] and the references therein. Variational inequalities represent the optimality conditions for the differentiable convex functions on the convex sets. To the best of our knowledge no such type of the characterization exists for harmonic convex functions and its variant forms. Inspired by the recent activities in this area, we show that the minimum of a differentiable harmonic convex functions can be characterized by a class of variational inequalities, which is called the harmonic variational inequality. This has motivated us to introduce and investigate the harmonic variational inequalities. Harmonic variational inequalities are quite different other type of variational inequalities and their variant form forms. We remark that the projection and resolvent operator techniques can not be used to study the existence of a solution of the harmonic variational inequalities. In this paper, we use the auxiliary principle technique,which does not use the projection or resolvent. This technique is mainly due to to Lions and Stampacchia [19]. Glowinski et al [4] used this technique to study the existence of a solution of the mixed variational inequalities. Noor [11,13] used this technique to develop some iterative methods for solving variational inequalities. We show that this technique can be used to study the existence of a solution of the harmonic variational inequalities. Interested readers are encouraged to find the novel applications of harmonic variational inequalities in various fields of pure and applied sciences. The development of an implementable algorithm for finding the approximate solution of the harmonic variational inequalities is an interesting problem. For the recent developments, see [18].

2 Preliminaries

Let K_h be a nonempty closed and harmonic convex set in the real Hilbert space H. We denote by $\langle .,. \rangle$ and $\|.\|$ be the inner product and norm, respectively.

For a given nonlinear operator T, consider the problem of finding $u \in K_h$, such that

$$\langle Tu, \frac{uv}{u-v} \rangle \geq 0, \quad \forall v \in K_h. \tag{1}$$

The inequality (1) is called the harmonic variational inequality. We now show that the minimum of a

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differentiable harmonic convex function on the harmonic convex set can be characterized by the harmonic variational inequality (1).

For this purpose, we recall the following well known concepts.

Definitions 2.1 [1,6]. A set $K_h \subset H\setminus\{0\}$ is said to be a harmonic convex set, if

$$\frac{uv}{v+t(u-v)} \in K_h, \quad \forall u,v\in K_h, \quad t\in[0,1].$$

Definition 2.2 [1,6]. A function $f: K_h \subset H\setminus\{0\} \to \mathbb{R}$ is said to be harmonic convex function, if

$$f\left(\frac{uv}{v+t(u-v)}\right) \le (1-t)f(u)+tf(v),$$
$$\forall u,v\in K_h, \quad t\in[0,1].$$

The function f is said to be harmonic concave, if and only if $-f$ is harmonic convex.

Definition 2.3. The differentiable function f on K_h is said to be an harmonic invex function, if

$$f(v)-f(u) \ge \langle f'(u), \frac{uv}{u-v}\rangle, \quad \forall u,v\in K_h,$$

where $f'(u)$ is the differential of f at u. The concept of the harmonic invex functions is a new one.

Definition 2.4. A function $f: K_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be quasi harmonic, if

$$f\left(\frac{uv}{v+t(u-v)}\right) \le max\{f(u),f(v)\}, \forall u,v\in K_h, t\in[0,1].$$

Definition 2.5 [1,6]. A function $f: K_h \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}_+$ is said to be logarithmic harmonic convex, if

$$f\left(\frac{uv}{v+t(u-v)}\right) \le (f(u))^{1-t}(f(v))^t, u,v\in K_h, t\in[0,1].$$

where $f(.) > 0$.

From the above definitions, we have

$$\begin{aligned} f\left(\frac{uv}{v+t(u-v)}\right) &\le (f(u))^{1-t}(f(v))^t \\ &\le (1-t)f(u)+tf(v) \\ &\le \max\{f(u),f(v)\}. \end{aligned}$$

This shows that harmonic log-convex functions are harmonic convex functions and harmonic convex functions are quasi-harmonic convex functions. However, the converse is not true.

From definition 2.5, we have

$$\log f\left(\frac{uv}{v+t(u-v)}\right)$$
$$\le (1-t)\log(f(u))+t\log(f(v)), \forall u,v\in K_h, t\in[0,1].$$

We now show that the minimum of a differentiable harmonic convex function on the harmonic convex set K_h can be characterized by the harmonic variational inequality (1) and this is main motivation of our next result.

Theorem 2.1. Let f be a differentiable harmonic convex function on the harmonic convex set K_h. Then $u\in K_h$ is a minimum of f, if and only if, $u\in K_h$ is the solution of the inequality

$$\langle f'(u), \frac{uv}{u-v}\rangle \ge 0, \quad \forall v\in K_h, \tag{2}$$

which is called the harmonic variational inequality.

Proof. Let $u\in K_h$ be a minimum of a harmonic convex function f. Then

$$f(u) \le f(v)(, \quad \forall v\in K_h. \tag{3}$$

Since K_h is a harmonic convex set, so $\forall u,v\in K_h$, and $t\in[0,1]$, $\quad v_t = \frac{uv}{v+t(u-v)} \in H_k$.

Replacing v by v_t in (3), we have

$$\frac{f(\frac{uv}{v+t(u-v)} - f(u)}{t} \ge 0.$$

Since f is a differentiable function, taking the limit in the above inequality, as $t \longrightarrow 0$, we have

$$\langle f'(u), \frac{uv}{u-v}\rangle \ge 0, \quad \forall v\in K_h,$$

the required (3).

Conversely, let $u\in K_h$ satisfy (3). Then, we have to show that $u\in K_h$ is the minimum of the function f on the harmonic convex set. Since f is harmonic convex function, we have

$$f(\frac{uv}{v+t(u-v)}) \le (1-t)f(u)+tf(v), \quad \forall u,v\in K_h, t\in[0,1],$$

from which it follows that

$$f(v)-f(u) \ge \frac{f(\frac{uv}{v+t(u-v)})-f(u)}{t}.$$

Since f is a differentiable function, so taking the limit in the above inequality as $t \longrightarrow 0$, we have

$$f(v)-f(u) \ge \langle f'(u), \frac{uv}{u-v}\rangle \ge 0. \quad \text{using (3)}$$

Thus, it follows that

$$f(u) \le f(v), \quad \forall v\in K_,$$

which implies that $u\in K_h$ is the minimum of f.. This completes the proof. □

Theorem 2.1 implies that harmonic convex programming problem can be studied via the harmonic variational inequality (1).

We now consider some other properties of the differentiable harmonic convex functions. In this respect, we have following.

Theorem 2.2. Let f be a differentiable harmonic convex functions on the harmonic convex set K_h. Then

$$(i). \qquad f(v)-f(u) \geq \langle f'(u), \frac{uv}{u-v}\rangle, \quad \forall u,v \in K_h.$$
$$(ii). \qquad \langle f'(u)-f'(v), \frac{uv}{u-v}\rangle \leq 0, \quad \forall u,v \in K_h,$$

where $f'(u)$ is the differential of f at u in the direction $\frac{uv}{u-v}$.

Proof. (i). Let f be a harmonic convex function. Then

$$f\left(\frac{uv}{v+t(u-v)}\right) \leq (1-t)f(u)+tf(v),$$

from which, we have

$$f(v)-f(u) \geq \frac{f(\frac{uv}{v+t(u-v)})-f(u)}{t}.$$

Since f is a differentiable function, so taking the limit in the above inequality as $t \longrightarrow 0$, we have

$$f(v)-f(u) \geq \langle f'(u), \frac{uv}{u-v)}\rangle, \quad \forall u,v \in K_h, \tag{4}$$

the required (i).

(ii). Changing the role of v and u, in (4), we obtain

$$f(u)-f(v) \geq \langle f'(v), \frac{uv}{v-u)}\rangle, \quad \forall u,v \in K_h, \tag{5}$$

Adding (4) and (5), we we obtain

$$\langle f'(u)-f'(v), \frac{uv}{u-v}\rangle \leq 0, \quad \forall u,v \in K_h,$$

the required (ii). □

Definition 2.6. For all $u,v \in H$, an operator T is said to be:

(i). Strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu-Tv, u-v\rangle \geq \|u-v\|^2.$$

(ii). Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Tu-Tv\| \leq \beta\|u-v\|.$$

3 Main Results

In this Section, we consider the existence of a solution of the harmonic variational inequality (1) using the auxiliary principle technique, which is mainly due to Glowinski et al [4] as developed by Noor [11,13].

Theorem 3.1. Let K_h be nonempty, closed and harmonic convex set in H. Let the operator T be

$$0 < \rho \leq \frac{2\alpha}{\beta^2}, \tag{6}$$

then there exists a solution of the harmonic variational inequality (5).

Proof. We use the auxiliary principle technique to prove the existence of a solution of (1). For a given $u \in K_h$ satisfying (1), consider the problem of finding $w \in K_h$ such that

$$\langle \rho Tu + w - u, \frac{vw}{w-v}\rangle \geq 0, \quad \forall v \in K_h, \tag{7}$$

which is called the auxiliary harmonic variational inequality. The relation (7) defines a mapping $u \longrightarrow w$. It is sufficient to show that the mapping $u \longrightarrow w$ defined by (7) has a fixed point belong to K_h satisfying (1). In other words, it is enough to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta\|u_1 - u_2\|$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2.

Let $w_1 \neq w_2 \in K_h$, be two solutions of (7) corresponding to $u_1 \neq u_2 \in K_h$. Then

$$\langle \rho Tu_1 + w_1 - u_1, \frac{vw_1}{w_1-v}\rangle \geq 0, \quad \forall v \in K_h, \tag{8}$$
$$\langle \rho Tu_2 + w_2 - u_2, \frac{vw_2}{w_2-v}\rangle \geq 0, \quad \forall v \in K_h, \tag{9}$$

Taking $v = w_2$ in (8), $v = w_1$ in (9) and adding the resultant, we have

$$\langle w_1 - w_2, \frac{w_1w_2}{w_2-w_1}\rangle \leq \langle u_1 - u_2 - \rho(Tu_1 - Tu_2), \frac{w_1w_2}{w_2-w_1}\rangle,$$

from which it follows

$$\|w_1 - w_2\| \leq \|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|. \tag{10}$$

Using the strongly monotonicity of the operator T with constant $\alpha > 0$ and Lipschitz continuity with constant $\beta > 0$. respectively, we have

$$\begin{aligned} &\|u_1 - u_2 - \rho(Tu_1 - Tu_2)\|^2 \\ &= \langle u_1 - u_2 - \rho(Tu_2 - Tu_2), u_1 - u_2 - \rho(Tu_2 - Tu_2)\rangle \\ &= \|u_1 - u_2\|^2 - 2\rho\langle Tu_1 - Tu_2, u_1 - u_2\rangle + \rho^2\|Tu_1 - Tu_2\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \end{aligned} \tag{11}$$

From (10) and (11), we have

$$\begin{aligned} \|w_1 - w_2\| &\leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned}$$

where

$$\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}.$$

From (6), it follows that $\theta < 1$. Thus the mapping w is a contraction mapping and consequently has a fixed point $w(u) = u \in K_h$ satisfying the harmonic variational inequality (1), the required result □.

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