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On the oscillation of third order neutral delay differential equations

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Abstract: The aim of this paper is to study the oscillatory behavior of general third order neutral delay differential equations by using a generalized Riccati transformation. New sufficient conditions for oscillations of solutions are established.The obtained results extend and improve some known results in the literature. Illustrative examples are given to support our main results.

Keywords: Third order, neutral differential equation, oscillation criteria, Riccati transformation

1 Introduction

There has been considerable interest in studying the oscillation of solutions of neutral delay differential equations in the last two decades. Although the oscillation of third-order equations has received less attentions relatively comparing with those of second-order, however there is an increasing interest in studying the oscillation of neutral delay third-order equations (see [2], [4], [8], [10]) The aim of this paper is to study the oscillation of solutions of the third order neutral differential equation

$$(r(t)z''(t))' + f(t,z(t),z'(t)) = 0 \qquad (1.1)$$

where $z(t) = x(t) + p(t)x(\tau(t))$ under the assumptions
(H_1) $r(t), p(t) \in C([t_0,\infty),(0,\infty)), r'(t) \geq 0, r''(t) > 0, \int^{\infty} r^{-1}(s)ds = \infty,$ *and* $0 \leq p(t) \leq p_0 < \infty.$
(H_2) $\tau(t) \in C^1([t_0,\infty),R),$ for $t \geq t_0, \tau(t) \leq t, \lim_{t\to\infty} \tau(t) = \infty.$
(H_3) $f \in C(R\times R^2,R), \frac{f(t,u,v)}{v} \geq K > 0.$
Many efforts were done to deduce sufficient conditions for the oscillation of differential equations of the type (1.1) (see [5], [10], [12],[13]). To the best of our knowledge most of those papers considered a common condition on the nonlinear function f ,namely

$$\frac{f(x)}{x} \geq K > 0 \qquad (c)$$

One of our main goal of this paper is to establish new oscillation criteria for Eq.(1.1) without the traditional condition (c) . The paper is organized as follows .In sec. 2 we give our main results, we establish sufficient conditions guarantee the oscillation of Eq.(1.1). In sec.3 we give some examples for which our criteria applied while some of the others in the literature fail.
We say that a function $\phi(t,s,l)$ belongs to the function class Ω, denoted by $\phi \in \Omega$ if $\phi \in C(E,R)$, where $E = \{(t,s,l) : t_0 \leq l \leq s \leq t < \infty\}$, which satisfies $\phi(t,t,l) = 0$,$\phi(t,l,l) > 0$, and $\phi(t,s,l) > 0$ for $l < s < t$, and has the partial derivative $\frac{\partial\phi}{\partial s}$, defined by

$$\frac{\partial\phi}{\partial s} = \varphi(t,s,l)\phi(t,s,l), \varphi \in \Omega \qquad (1.2)$$

Further we define the operator $A[.;l,t]$ by

$$A[g;l,t] := \int_l^t \phi^2(t,s,l)g(s)ds \qquad (1.3)$$

$for\ t \geq s \geq l \geq t_0, where\ g(s) \in C[t_0,\infty).$

It is easy to see that $A[.;l,t]$ is a linear operator and satisfies

$A[g';l,t] = -2A[g\varphi;l,t]\ for \quad g(s) \in C^1[t_0,\infty).$ (see [11]). In what follows we use the notation $D = \{(t,s) : t_0 \leq s < t < \infty\}$ and

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$D_0 = \{(t,s) : t_0 \le s \le t < \infty\}$ .We say that a continuous function $H : D \to [0,\infty)$ belongs to the function class X denoted by $H \in X$ if
(i)$H(t,t) = 0, H(t,s) > 0$ for $(t,s) \in D_0$.
(ii)$H(t,s)$ has a continuous partial derivative with respect to s defined by

$$\frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)} \quad for some\, h \in C(D_0,R).(1.4)$$

The key idea in our proofs makes use of the idea used in [9,11].

2 Main Results

Before starting our main results, we begin with Following the lemma Which depends on [2] and will play an important role in the proof of main results.

Lemma 2.1 *If $x(t)$ is a nonoscillatory solution of Eq. (1.1), then either $z(t) > 0, z'(t) > 0, r(t)z''(t) > 0$ eventually or $z(t) < 0, z'(t) < 0, r(t)z''(t) < 0$ eventually where $z(t)$ be as defined in Eq.(1.1).*

Proof. Without loss of generality we may assume that $x(t) > 0, x(\tau(t)) > 0$ for $t \ge t_1 \ge t_0$. Then $z(t) > 0$. Firstly we claim that $r(t)z''(t)$ is monotone and of one signe. If $r(t_1)z''(t_1) = 0$ for some $t_1 \ge t_0$ .Then $(r(t)z''(t))' |_{t=t_1} = -f(t_1, z(t_1), z'(t_1)) < 0$ from which we can prove that $r(t)z''(t)$ cannot have another zero after it vanishes once (see [7]) . Hence $r(t)z''(t)$ is monotone and of one sign, so either $r(t)z''(t) > 0$, or $r(t)z''(t) < 0$ for $t \ge t_1$. If $r(t)z''(t) < 0$, then there exists $t_2 \ge t_1$ such that $r(t)z''(t) \le r(t_2)z''(t_2) < 0$
$= -c,\ c > 0 \quad \forall\, t \ge t_2$
Thus $z''(t) = \frac{-c}{r(t)}, t \ge t_2$, by integration, we get
$z'(t) = z'(t_2) - c\int_{t_2}^{t} \frac{ds}{r(s)}, t \ge t_2$, therefore we see that $z'(t) \to -\infty$ as $t \to \infty$, for which it follows that $z(t)$ is eventually negative which contradicts the fact that $z(t) > 0$. Hence we conclude that $r(t)z''(t) > 0 \quad \forall\, t \ge t_2$. Now we claim that $z'(t) > 0$. Since $r(t)z''(t) > 0$ and $r(t) > 0$, then $z''(t) > 0$. Thus $z'(t)$ is monotone and of one signe (i.e. $z'(t) > 0$ or $z'(t) < 0$). Assume that $z'(t) < 0$. Since $r(t)z''(t) > 0$, then $r(t)z''(t) > c > 0$. Thus
$z''(t) \ge \frac{c}{r(t)}, t \ge t_2$.
By integration, we get
$z'(t) \ge z'(t_2) + c\int_{t_2}^{t} \frac{ds}{r(s)}, t \ge t_2$. Therefore we see that $z'(t) \to \infty$ as $t \to \infty$. This contradicts with $z'(t) < 0$. Hence $z'(t) > 0$. The case when $x(t) < 0$ is similar.

Theorem 2.1 *Suppose that for each $l \ge t_0$ there exists a function $\phi(t,s,l) \in \Omega$ such that*

$$\limsup_{t\to\infty} A[K - r\varphi^2; l,t] > 0 \qquad (2.1)$$

where the operator $A[.;l,t]$ is defined by (1.3) and the function $\varphi = \varphi(t,s,l)$ is defined by (1.2). Then Eq. (1.1) is oscillatory.

Proof. Suppose that there exists a nonoscillatory solution $x(t)$ such that $x(t) > 0 \forall t \ge t_1 \ge t_0$. Then $z(t) > 0$. Now, from Eq. (1.1) and (H_3) , we get

$$(r(t)z''(t))' \le -Kz'(t) \qquad (2.2)$$

Define $w(t) = \frac{r(t)z''(t)}{z'(t)} > 0$
Differentiating $w(t)$, we obtain

$$w'(t) = \frac{(r(t)z''(t))'}{z'(t)} - r(t)\frac{z''(t)}{z'(t)}^2, \forall t \ge t_1$$

Using (2.2), we have

$$w'(t) \le -K - \frac{w^2(t)}{r(t)}, \forall t \ge t_1 \qquad (2.3)$$

Applying the operator $A[.;l,t]$ to (2.3), we get

$$A[w'(s); l,t] \le -A[K + \frac{w^2(s)}{r(s)}; l,t], \forall t \ge t_1$$

Thus, by the properties of the operator $A[.;l,t]$, we obtain

$$A[K; l,t] \le -A[\frac{w^2(s)}{r(s)} - 2w(s)\varphi; l,t], \forall t \ge t_1$$

$$= -A[(\sqrt{\frac{1}{r(s)}}w(s) - \sqrt{r(s)}\varphi)^2; l,t] + A[r\varphi^2; l,t]$$

$$\le A[r\varphi^2; l,t].$$

Therefore

$$A[K - r\varphi^2; l,t] \le 0$$

$$\limsup_{t\to\infty} A[K - r\varphi^2; l,t] \le 0$$

This contradicts with (2.1). Hence Eq. (1.1) is oscillatory.

Theorem 2.2 *Assume that there exists a function $g \in C^1([t_0,\infty), R)$ such that for some $H, h \in X$*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} (H(t,s)\psi(s) - \frac{1}{4}r(s)v(s)h^2(t,s))ds = \infty$$

where $\psi(t) = v(t)[K + r(t)g^2(t) - (r(t)g(t))'], v(t) = \exp(-2\int_{t_0}^{t} g(s)ds)$. Then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), then there exists $t_1 \ge t_0$ such that $x(t) > 0, x(\tau(t)) > 0,\ \forall t \ge t_1$. As in the proof of Theorem 2.2 we arrive (2.2). Now, define
$u(t) = v(t)r(t)[\frac{z''(t)}{z'(t)} + g(t)] > 0,\ t \ge t_1$
Differentiating $u(t)$, we obtain
$u'(t) = v(t)\frac{(r(t)z''(t))'}{z'(t)} - 2v(t)g(t)[\frac{u(t)}{v(t)} - r(t)g(t)] -$

$r(t)v(t)[\frac{u(t)}{r(t)v(t)} - g(t)]^2 + v(t)[r(t)g(t)]' - 2v(t)r(t)g^2(t)$
Using (2.2), we get

$$u'(t) \leq -\psi(t) - \frac{u^2(t)}{r(t)v(t)} \qquad (2.5)$$

Multiplying (2.5) by $H(t,s)$ and integrating with respect to s from T to t , we get

$$\int_T^t H(t,s)\psi(s)ds \leq H(t,T)u(T) - \int_T^t (\sqrt{\frac{H(t,s)}{v(s)r(s)}}u(s)$$

$$+\frac{1}{2}\sqrt{v(s)r(s)}h(t,s))^2 ds + \int_T^t \frac{1}{4}v(s)r(s)h^2(t,s)ds$$

$$\int_T^t [H(t,s)\psi(s) - \frac{1}{4}v(s)r(s)h(t,s)^2]ds \leq H(t,T)u(T)$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t (H(t,s)\psi(s)$$

$$-\frac{1}{4}r(s)v(s)h^2(t,s))ds \leq |u(t_3)| < \infty, t_3 \geq T \geq t_0$$

This contradicts (2.4). Hence Eq. (1.1) is oscillatory.
In Theorem 2.2, if we take $H(t,s) = (t-s)^n - 1$ for $(t,s) \in D, n > 2$, then $h(t,s) = -(n-1)(t-s)^(n-3)/2$. Hence we get the following result.

Corollary 2.3 *Suppose that there exists a function* $g \in C^1([t_0,\infty),R)$ *such that for some integer* $n > 2$ *if*

$$\limsup_{t\to\infty} \frac{1}{t^n - 1}$$

$$\int_{t_0}^t (t-s)^(n-3)[(t-s)^2\psi(s) - \frac{(n-1)^2}{4}r(s)v(s)]ds = \infty$$

where $\psi(s), v(s)$ *are as in Theorem 2.3, then every solution of Eq. (1.1) is oscillatory.*

3 Examples

Example 3.1 Consider the D.E.

$$(\frac{1}{t^2}z''(t))' + z'\sin^2(t) + (z')^3 = 0, t \geq 1 \qquad (3.1)$$

Here $r(t) = \frac{1}{t^2}, f(t,z,z') = z'\sin^2(t) + (z')^3$
Choosing $\phi(t,s,l) = t-s$
It is clear that $\frac{f(t,z,z')}{z'} = \sin^2(t) + (z')^2 \geq K{>}0$ and $\varphi(t,s,l) = \frac{-1}{t-s}$.
Now

$$A[K - r\varphi^2; l,t] = \int_l^t (t-s)^2[1 - \frac{1}{s^2(t-s)^2}]ds$$

$$= \frac{1}{t} + \frac{(t-l)^3}{3} - \frac{1}{l}$$

$$\limsup_{t\to\infty} A[K - r\varphi^2; l,t] = \infty > 0$$

Hence by Theorem 2.2 Eq. (3.1) is oscillatory.
One may note that Theorem 1 in [2] fails to apply to Eq. (3.1) with $q(t) = 1, \gamma = 1$.

Example 3.2 Consider the D.E.

$$(\frac{1}{t^4}z''(t))' + z'\sin^2(t) + (z')^3 = 0, t \geq 1 \qquad (3.2)$$

Here $r(t) = \frac{1}{t^4}, f(t,z,z') = z'\sin^2(t) + (z')^3$
Choosing $g(t) = \frac{-1}{t}, n = 3$ in Corollary 2.4, we get
$v(t) = t^2, \psi(t) = t^2 - \frac{4}{t^4}$
It is clear that $\frac{f(t,z,z')}{z'} = \sin^2(t) + (z')^2 \geq K{>}0$
Now

$$\frac{1}{t^n - 1}\int_{t_0}^t (t-s)^(n-3)[(t-s)^2\psi(s) - \frac{(n-1)^2}{4}r(s)v(s)]ds$$

$$= \frac{1}{t^2}[\frac{t^5}{30} - \frac{5t^2}{3} + \frac{9t}{2} + \frac{9}{t} - \frac{26}{5}]$$

$$\limsup_{t\to\infty} \frac{1}{t^n - 1}$$

$$\int_{t_0}^t (t-s)^(n-3)[(t-s)^2\psi(s) - \frac{(n-1)^2}{4}r(s)v(s)]ds = \infty$$

Hence by Corollary 2.4 Eq. (3.2) is oscillatory.
Note that, if we put $q(t) = 1, \alpha = \beta = K = L = 1, \rho(t) = t^2$, then we see that Theorem 1 in [13] fails to apply to Eq.(3.2).

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