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Multifarious Implicit Summation Formulae of Hermite-Based Poly-Daehee Ploynomials

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Abstract: In this paper, we introduce the generating function of Hermite-based poly-Daehee numbers and polynomials. By making use of this generating function, we investigate some new and interesting identities for the Hermite-based poly-Daehee numbers and polynomials including recurrence relations, addition property and correlations with poly-Bernoulli polynomials of second kind. We then derive diverse implicit summation formula for Hermite-based poly-Daehee numbers and polynomials by applying the series manipulation methods.

Keywords: Hermite polynomials, Bernoulli polynomials, Daehee polynomials, Hermite-based poly-Daehee polynomials, Generating function, Cauchy product, Summation formulas.

1 Introduction

In recent years, the Daehee polynomials and Bernoulli polynomials (known closely related each other) in conjunction with their diverse generalizations have been studied by many authors (*cf.* [1,2,4-22]). For example, Dattoli *et al.* [4] introduced new forms of Bernoulli numbers and polynomials, which are exploited to derive further classes of partial sums involving generalized several index many variable polynomials. Haroon *et al.* [5] performed to classify fully degenerate Hermite-Bernoulli polynomials with formulation in terms of p-adic fermionic integrals on $\mathbb{Z}_p$ and also illustrated novel properties with Daehee polynomials in a consolidated and generalized form. Khan *et al.* [7] introduced a new class of Hermite multiple-poly-Bernoulli numbers and polynomials of the second kind and investigate some properties for these polynomials, and then derived several implicit summation formulae and general symmetry identities by using different analytical means. Kim *et al.* [8] studied λ-Daehee polynomials and investigated their properties arising from the p-adic integral equations. Kim *et al.* [9] considered the Witt-type formula for Daehee numbers and polynomials and derived assorted relationships for these polynomials and numbers including close relations with higher-order Bernoulli numbers and those of the second kind. Kim *et al.* [10] acquired multifarious formulas for expressing any polynomial as linear combinations of two kinds of higher order Daehee polynomial basis and then used these formulas in order to certain polynomials to obtain novel and quirky identities involving higher-order Daehee polynomials of the first and the second kinds. Kim *et al.* [11], by considering Barnes-type Daehee polynomials of the first kind as well as poly-Cauchy polynomials of the first kind, introduced mixed-type polynomials of these polynomials and examined their some properties arising from umbral calculus. Kim *et al.* [12] considered the Daehee numbers and polynomials of order k and gave various relationship

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between Daehee polynomials of order k and some special polynomials. Kim *et al.* [13] studied q-extension of the Daehee polynomials and numbers. Kim *et al.* [14] considered the poly-Bernoulli numbers and polynomials of the second kind and presented new and explicit formulas for calculating the poly-Bernoulli numbers of the second kind and the Stirling numbers of the second kind. Kwon *et al.* [15] considered Appell-type Daehee polynomials and derived many identities and formulas. Lim *et al.* [16] defined the poly-Daehee numbers and attained explicit identities for those numbers and polynomials related to poly-Bernoulli numbers, polynomials and those of the second kind. Moon *et al.* [17] considered the generalized q-Daehee numbers and polynomials of higher order and stated diverse interesting formulas and a representation for them as the sums of products of the generalized q-Daehee polynomials and numbers. Park [18] provided a p-adic integral representation of the twisted Daehee polynomials with a q-parameter and developed some interesting properties. Park *et al.* [19] presented Witt-type formula for the twisted Daehee polynomials and investigated their various properties. Pathan *et al.* [20] introduced a new class of generalized Hermite-Bernoulli polynomials and derived many implicit summation formulae and symmetric identities. Seo *et al.* [21] defined generalized Daehee numbers of higher order and represented them as the sums of products of generalized Daehee numbers. There are various applications of the aforementioned polynomials and numbers in many branches of not only in mathematics and mathematical physics, but also in computer and engineering science with real world problems including the combinatorial sums, combinatorial numbers such as the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers of first and second kinds, the Changhee numbers and polynomials, etc. (see [1-22]).

We now begin with recalling some known numbers and polynomials as follows. Assuming that $\mathbb{N}$ denotes the set of natural numbers with the associated set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let $H_n(x,y)$ be the 2-variable Kampé de Fériet generalization of the Hermite polynomials given by means of the following generating function (*cf.* [3], [4]):

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = e^{xt+yt^2} \tag{1}$$

satisfying the following property

$$H_n(2x,-1) = H_n(x),$$

where $H_n(x)$ are called the ordinary Hermite polynomials (*cf.* [1]). For $k \in \mathbb{N}$ with $k > 1$, the k-th polylogarithm function is defined by

$$Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad (z \in \mathbb{C} \text{ with } |z| < 1). \tag{2}$$

Notice that if $k = 1$, then $Li_1(z) = -\log(1-z)$, *cf.* [6], [7], [11], [14], [16].

The Daehee polynomials $D_n(x)$ are defined by means of the following generating function (*cf.* [8-14]):

$$\sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}(1+t)^x. \tag{3}$$

In case when $x = 0$, $D_n := D_n(0)$ stands for the Daehee numbers. The first few Daehee numbers D_n are as follows.

$$D_0 = 1, D_1 = -\frac{1}{2}, D_2 = \frac{1}{3}, D_3 = -\frac{1}{4}, D_4 = \frac{1}{5}, \cdots.$$

The Bernoulli polynomials $B_n(x)$ are defined via the following exponential generating function (*cf.* [5],[7]):

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt} \qquad (|t| < 2\pi) \tag{4}$$

where $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers.

The Bernoulli polynomials of the second kind $b_n(x)$ are defined by the following generating function to be (see [5],[7]):

$$\sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}(1+t)^x. \tag{5}$$

The poly-Bernoulli numbers $B_n^{(k)}$ and polynomials $B_n^{(k)}(x)$ are respectively defined by (*cf.* [6], [7], [14], [16]):

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1} = \sum_{n=0}^{\infty} B_n^{(k)}\frac{t^n}{n!} \text{ and } \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!}. \tag{6}$$

from which if we letting $k = 1$ in Eq. (6), it then yields

$$B_n^{(1)} := B_n \text{ and } B_n^{(1)}(x) := B_n(x).$$

Recently, Khan *et al.* [7] introduced the 3-variable Hermite multi poly-Bernoulli polynomials of the second kind via the following generating function:

$$\sum_{n=0}^{\infty} {}_Hb_n^{(k_1,\cdots,k_r)}(x,y,z)\frac{t^n}{n!} = \frac{r!\mathrm{Li}_{k_1,\cdots,k_r}(1-e^{-t})}{(\log(1+t))^r}(1+t)^x e^{yt+zt^2} \tag{7}$$

where

$$\mathrm{Li}_{k_1,\cdots,k_r}(z) = \sum_{0<m_1<m_2<\cdots<m_r}^{\infty} z^{m_r}\prod_{i=1}^{r} m_i^{-k_i}$$

is the multiple polylogarithm.

In this paper, we consider the Hermite-based poly-Daehee numbers and polynomials. We then derive explicit identities for those numbers and polynomials which are related to poly-Bernoulli numbers and polynomials. We also investigate some implicit summation formula for the foregoing numbers and polynomials by using the series manipulation methods.

2 On the properties of Hermite-based poly-Daehee polynomials

In this part, we start by defining Hermite-based poly-Daehee polynomials ${}_HD_n^{(k)}(x,y,z)$ as follows.

Definition 1. *Let $n \in \mathbb{N}_0$. Then,*

$$\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!} = \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{yt+zt^2}, \quad (8)$$

where if we take $x=y=z=0$, then $D_n^{(k)} := {}_HD_n^{(k)}(0,0,0)$ stands for the poly-Daehee numbers.

Remark. Upon setting $k=1$ in Eq. (8), one can easily derive

$${}_HD_n^{(1)}(x,y,z) := {}_HD_n(x,y,z). \quad (9)$$

Remark. On setting $y=z=0$ in Eq. (8), it reduces to the poly-Daehee polynomials given by Lim and Kwon in [16, p. 220].

Remark. Taking $z=0$ in Eq. (8), we have ${}_HD_n^{(k)}(x,y,0) := {}_HD_n^{(k)}(x,y)$ that will be used in Theorem 8.

Theorem 1. *The following result holds true for $n \in \mathbb{N}_0$:*

$${}_HD_n^{(k)}(x,y,z) = \sum_{m=0}^{n}\binom{n}{m} D_{n-m}^{(k)}(x)H_m(y,z).$$

Proof. Using (1) and (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!} &= \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{yt+zt^2}\\ &= \left(\sum_{n=0}^{\infty} D_n^{(k)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(y,z)\frac{t^m}{m!}\right).\end{aligned}$$

Replacing n by $n-m$ in above equation and comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we arrive at the desired result.

Theorem 2. *Let $n \in \mathbb{N}_0$. Hermite-based poly-Daehee polynomials have the following relation:*

$${}_HD_n^{(k)}(x,y,z) = \frac{{}_HD_{n+1}^{(k)}(x+1,y,z) - {}_HD_{n+1}^{(k)}(x,y,z)}{n+1}.$$

Proof. It follows from Eq. (8) that

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x+1,y,z)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!}\\ &= \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^{x+1} e^{yt+zt^2} - \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{yt+zt^2}\\ &= \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{yt+zt^2} t\\ &= \sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^{n+1}}{n!}.\end{aligned}$$

Now comparing the coefficients of t^n on the both sides, we complete the proof.

Theorem 3. *Hermite based poly-Daehee polynomials satisfy the following addition identity for $n \in \mathbb{N}_0$;*

$${}_HD_n^{(k)}(x+w,y,z) = \sum_{m=0}^{n}\binom{n}{m} {}_HD_{n-m}^{(k)}(x,y,z)(w)_m,$$

where $(w)_m$ is well known as falling factorial defined as $w(w-1)\cdots(w-m+1)$.

Proof. By Definition (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x+w,y,z)\frac{t^n}{n!} &= \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^{x+w} e^{yt+zt^2}\\ &= \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{yt+zt^2}(1+t)^w\\ &= \left(\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} (w)_m\frac{t^m}{m!}\right).\end{aligned}$$

Replacing n by $n-m$ in above equation and comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get the required result.

Theorem 4. *The following correlations holds true for $n \in \mathbb{N}_0$;*

$$\sum_{m=0}^{n}\binom{n}{m} B_m\, {}_HD_{n-m}(x,y,z) = \sum_{m=0}^{n}\binom{n}{m} B_m^{(k)}\, {}_HD_{n-m}^{(k)}(x,y,z).$$

Proof. Combining Eq. (8) with Eq. (6), it becomes

$$\begin{aligned}\frac{\log(1+t)}{e^t-1}(1+t)^x e^{yt+zt^2} &= \frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{yt+zt^2} \quad (10)\\ &= \left(\frac{\mathrm{Li}_k(1-e^{-t})}{e^t-1}\right)\left(\frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{yt+zt^2}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} B_m^{(k)}\, {}_HD_{n-m}^{(k)}(x,y,z)\right)\frac{t^n}{n!}. \quad (11)\end{aligned}$$

By the left-hand side of Eq. (10), using Eq. (4) and Eq. (7), we have

$$\begin{aligned}\frac{\log(1+t)}{e^t-1}(1+t)^x e^{yt+zt^2} &= \frac{t}{e^t-1}\frac{\log(1+t)}{t}(1+t)^x e^{yt+zt^2}\\ &= \left(\sum_{m=0}^{\infty} B_m\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} {}_HD_n(x,y,z)\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} B_m\, {}_HD_{n-m}(x,y,z)\right)\frac{t^n}{n!}. \quad (12)\end{aligned}$$

Therefore, by Eq. (11) and Eq. (12), we arrive at the desired result.

Upon setting $r=1$ and $y=z=0$ in Eq. (7), we then obtain poly-Bernoulli polynomials of the second kind given below:

$$\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^x.$$

We here give a correlation including classical Hermite polynomials, Hermite-based poly-Daehee polynomials and poly-Bernoulli polynomials of the second kind.

Theorem 5. *The following relation is valid for $n \in \mathbb{N}_0$;*

$$H_n(y,z) = \sum_{m=0}^{n} \binom{n}{m} \, {}_HD_{n-m}^{(k)}(x,y,z) b_m^{(k)}(-x).$$

Proof. From Eq. (8) and Eq. (1), we have the following applications:

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(y,z)\frac{t^n}{n!} &= e^{yt+zt^2} \\ &= \frac{\mathrm{Li}_k(1-e^{-t})}{\log(1+t)}(1+t)^{-x}\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!} \\ &= \left(\sum_{m=0}^{\infty} b_m^{(k)}(-x)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} {}_HD_{n-m}^{(k)}(x,y,z)b_m^{(k)}(-x)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ of both sides above, we get the required result.

3 Implicit summation formulae for Hermite-based poly-Daehee polynomials

In this section, we investigate various implicit summation formulae of Hermite-based poly-Daehee polynomials.

Theorem 6. *The following implicit summation formula for Hermite-based poly-Daehee polynomials ${}_HD_n^{(k)}(x,y,z)$ holds true;*

$${}_HD_{q+l}^{(k)}(x,w,z) = \sum_{n,p=0}^{q,l}\binom{q}{n}\binom{l}{p}(w-y)^{n+p} \, {}_HD_{q+l-p-n}^{(k)}(x,y,z).$$

Proof. We first need the following series manipulation formula:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \qquad (13)$$

which can be found in [22, p.52 (2)]. We now consider the following generating function which is obtained by changing t to $t+u$ and from (13) in (8):

$$\frac{\log(1+t+u)}{\mathrm{Li}_k(1-e^{-t+u})}(1+(t+u))^x e^{z(t+u)^2} = e^{-y(t+u)}\sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!}.$$

After replacing y by w in Eq. (14), we equate obtained result with Eq. (14). It then becomes

$$e^{(w-y)(t+u)}\sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \qquad (14)$$

On expanding exponential function in Eq. (14) gives

$$\sum_{N=0}^{\infty}\frac{[(w-y)(t+u)]^N}{N!}\sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \qquad (15)$$

From (13) and (15), we see

$$\sum_{n,p=0}^{\infty}\frac{(w-y)^{n+p}t^n u^p}{n!p!}\sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \qquad (16)$$

Now replacing q by $q-n$, l by $l-p$ in the left hand side of Eq. (16), we get

$$\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l}\frac{(w-y)^{n+p}}{n!p!} {}_HD_{q+l-n-p}^{(k)}(x,y,z)\frac{t^q}{(q-n)!}\frac{u^l}{(l-p)!} = \sum_{q,l=0}^{\infty} {}_HD_{q+l}^{(k)}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}.$$

Finally, on equating the coefficients of the like powers of t^q and u^l in the above equation, we get the claimed result.

By substituting $l=0$ in Theorem 6, we immediately obtain the following corollary.

Corollary 1. *The following formula is valid;*

$${}_HD_q^{(k)}(x,w,z) = \sum_{n=0}^{q}\binom{q}{n}(w-y)^n \, {}_HD_{q-n}^{(k)}(x,y,z).$$

Corollary 2. *On replacing w by $w+y$ and setting $x=0$ in Theorem 6, we get the following result involving Hermite-based poly-Daehee polynomials of one variable;*

$${}_HD_{q+l}^{(k)}(w+y,z) = \sum_{n,p=0}^{q,l}\binom{q}{n}\binom{l}{p}w^{n+p} {}_HD_{q+l-p-n}^{(k)}(z).$$

Theorem 7. *Hermite-based poly-Daehee polynomials satisfy the following implicit summation formula;*

$${}_HD_n^{(k)}(x,y+u,z+w) = \sum_{s=0}^{n}\binom{n}{s} {}_HD_{n-s}^{(k)}(x,y,z)H_s(u,w).$$

Proof. Replacing y by $y+u$ and z by $z+w$ in Eq. (8) and using Eq. (3), we then have

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y+u,z+w)\frac{t^n}{n!} &= \frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{(y+u)t+(z+w)t^2} \\ &= \left(\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} H_n(u,w)\frac{t^n}{n!}\right).\end{aligned}$$

Now changing n by $n-s$ in left-hand side and comparing the coefficients of t^n, we acquire the required identity.

Theorem 8. *The following correlations holds true;*

$${}_HD_n^{(k)}(x,y,z) = \sum_{s=0}^{n}\binom{n}{s}D_{n-s}^{(k)}(x,y-w)H_s(w,z).$$

Proof. By Eq. (8), we have

$$\frac{\log(1+t)}{\mathrm{Li}_k(1-e^{-t})}(1+t)^x e^{(y-w)t}e^{wt+zt^2} = \left(\sum_{n=0}^{\infty} D_n^{(k)}(x,y-w)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} H_n(w,z)\frac{t^n}{n!}\right). \qquad (17)$$

By applying Cauchy product to right-hand side of (17), we get

$$\sum_{n=0}^{\infty} {}_HD_n^{(k)}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{s=0}^{n}\binom{n}{s}D_{n-s}^{(k)}(x,y-w)H_s(w,z)\right)\frac{t^n}{n!}.$$

Equating the coefficients of t^n on the both sides above, we complete the proof of theorem.

Theorem 9. *Hermite-based poly-Daehee polynomials fulfill the following implicit summation formula;*

$$ {}_{H}D_n^{(k)}(x,y+1,z) = \sum_{s=0}^{n} \binom{n}{s} {}_{H}D_{n-s}^{(k)}(x,y,z). $$

Proof. By changing the variable y to $y+1$ in (8), and by simple calculations, our assertion follows immediately. Therefore, we omit the proof.

4 Conclusion and Observation

In this paper, we have considered the generating function of Daehee polynomials as

$$ \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} = \frac{\log(1+t)}{t} (1+t)^x \tag{18} $$

which was introduced by Kim *et al.* [8-14]. Firstly, we have multiplied the right-hand side of (18) with e^{yt+zt^2}, then it became

$$ \sum_{n=0}^{\infty} {}_{H}D_n(x,y,z) \frac{t^n}{n!} = \frac{\log(1+t)}{t} (1+t)^x e^{yt+zt^2} \tag{19} $$

which was called Hermite-based Daehee polynomials. Secondly, since

$$ Li_1(1-e^{-t}) = t $$

we have considered (19) as

$$ \sum_{n=0}^{\infty} {}_{H}D_n^{(k)}(x,y,z) \frac{t^n}{n!} = \frac{\log(1+t)}{Li_k(1-e^{-t})} (1+t)^x e^{yt+zt^2} \tag{20} $$

satisfying ${}_{H}D_n^{(1)}(x,y,z) :={}_{H} D_n(x,y,z)$. Thus, by (20), we have introduced the generating function of Hermite-based poly-Daehee polynomials and derived their new properties. Also, by applying the series manipulation methods to the generating function of Hermite-based poly-Daehee polynomials, we have obtained some interesting implicit summation formulae.

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