Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110114

Regular Double *MS*-Algebras

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Received: 1 Aug. 2015, Revised: 15 Oct. 2015, Accepted: 24 Oct. 2015
Published online: 1 Jan. 2017

Abstract: The propose of this paper is to extend the construction due to T. Katriňák of regular double Stone algebras [1] to a certain subclass of the class of regular double *MS*-algebras. According to this construction we investigate many properties of these algebras deal with subalgebras, homomorphisms, congruences and permutable congruences.

Keywords: De Morgan algebras; *MS*-algebras; Double *MS*-algebras; Homomorphisms; Congruences; Permutable congruences.

1 Introduction

An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of Ockham algebras contains the well-known classes as de Morgan algebras and Stone algebras [2]. T. S. Blyth and J. C. Varlet [3] defined a subclass of Ockham algebras so called *MS*-algebras denoted by **MS** which generalizes both de Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by J. Berman [4]. The class **MS** of all *MS*-algebras is equational. T. S. Blyth and J. C. Varlet [5] characterized the subvarieties of **MS**. Also, T. S. Blyth and J. C. Varlet [6] introduced the class of double *MS*-algebras and they showed that every de Morgan algebra M can be represented non-trivially as the skeleton of the double *MS*-algebra $M^{[2]} = \{(a,b) \in M \times M : a \leq b\}$. The class of double *MS*-algebras satisfying the complement property have been introduced by Luo Congwen [7].

In 2012, A. Badawy, D. Guffova and M. Haviar [8] introduced and characterized the class of principal *MS*-algebras and the class of decomposable *MS*-algebras by means of triples. In 2015, A. Badawy [9] studied the notion of d_L-filters of principal *MS*-algebras. A. Badawy [10] presented the notion of de Morgan filters of decomposable *MS*-algebras. Also he established the relationship between congruences and de Morgan filters of a decomposable *MS*-algebra in [11]. In 2014 [12] A. Badawy and M. Sambasiva Rao considered the notion of closure ideals of *MS*-algebras. Recently, A Badawy [13] gave the first quadruple construction of modular generalized *MS*-algebras. Also, A. Badawy [14] presented a certain triple construction of principal generalized *MS*-algebras.

Regular double Stone algebras have been characterized by T. Katriňák [1] in terms of pairs (B,F), where B is a Boolean algebra and F is a filter of B. Also, he derived that every regular double Stone algebra L is uniquely determined by the pair $(B(L), D(L)^{++})$, where $B(L)$ and $D(L)$ are the center and the dense set of L, respectively.

In this paper we introduce the class of double *MS*-algebras satisfying the generalized complement property (briefly DMS^{gc}-algebras). Many related properties and examples are given. The main result of this article is to extend the construction of regular double Stone algebras due to Katriňák [1] to the class of DMS^{gc}-algebras; instead of Boolean algebras and the filters $D(L)$ used in the representation of [1], de Morgan algebras and the filters $[L^{\vee})$, respectively, are used in our representation (Theorem 3.7). We give an example (Example 3.9) to illustrate the construction of DMS^{gc}-algebras. Also, we prove that every DMS^{gc}-algebra L is uniquely determined by the pair $(L^{\circ\circ}, [L^{\vee})^{++})$.

Many applications of the construction Theorem (Theorem 3.7) are presented in section 4. We introduce and characterize subalgebras of DMS^{gc}-algebras by means of pairs (M,F). We investigate a special family of subalgebras of a DMS^{gc}-algebra $M^{[2]}$, where M is a de

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Morgan algebra. Homomorphisms of DMS^{gc}-algebras are characterized in terms of pairs (M,F). Finally, we discuss the concepts of congruences and permutability of congruences of DMS^{gc}-algebras using the construction Theorem. It is observed that the congruence lattices of a DMS^{gc}-algebra $L=(M,F)$ and the de Morgan algebra M are isomorphic. Also, we prove that a DMS^{gc}-algebra $L=(M,F)$ has permutable congruences if and only if the de Morgan algebra M has permutable congruences.

2 Preliminaries

A Stone algebra is a universal algebra $(L,\vee,\wedge,^{*},0,1)$ of type $(2,2,1,0,0)$, where $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice and the unary operation * has the properties that $x\wedge a=0 \Leftrightarrow x\le a^{*}$ and $x^{**}\vee x^{*}=1$.

A dual Stone algebra is a universal algebra $(L,\vee,\wedge,^{+},0,1)$ of type $(2,2,1,0,0)$, where $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice and the unary operation $^{+}$ has the properties that $x\vee a=1 \Leftrightarrow x\ge a^{+}$ and $x^{++}\vee x^{+}=1$.

A double Stone algebra is an algebra $(L,^{*},^{+})$ such that $(L,^{*})$ is a Stone algebra, $(L,^{+})$ is a dual Stone algebra and for every $x\in L, x^{*+}=x^{**}, x^{+*}=x^{++}$.

A double Stone algebra $(L,^{*},^{+})$ is called regular if

$$x^{*}=y^{*} \text{ and } x^{+}=y^{+} \text{ imply } x=y.$$

A de Morgan algebra is an algebra $(L,\vee,\wedge,\bar{\ },0,1)$ of type (2,2,1,0,0) where $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice and $\bar{\ }$ the unary operation of involution satisfies:

$$\bar{\bar{x}}=x, \overline{(x\vee y)}=\bar{x}\wedge\bar{y}, \overline{(x\wedge y)}=\bar{x}\vee\bar{y}.$$

An *MS*-algebra is an algebra $(L,\vee,\wedge,^{\circ},0,1)$ of type (2,2,1,0,0) where $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice and a unary operation $^{\circ}$ satisfies:

$$x\le x^{\circ\circ}, (x\wedge y)^{\circ}=x^{\circ}\vee y^{\circ}, 1^{\circ}=0.$$

A dual *MS*-algebra is an algebra $(L,\vee,\wedge,^{+},0,1)$ of type (2,2,1,0,0) where $(L,\vee,\wedge,0,1)$ is a bounded distributive lattice and a unary operation $^{+}$ satisfies:

$$x\ge x^{++}, (x\wedge y)^{+}=x^{+}\vee y^{+}, 0^{+}=1.$$

The class **M** of de Morgan algebra is a subvariety of **MS** and is defined by the identity $x^{\circ\circ}=x$. The member of the subvariety **K** of **M** defined by the inequality $x\wedge x^{\circ}\le y\vee y^{\circ}$ are called Kleene algebras. The subvariety $\mathbf{K_2}$ of **MS** defined by the additional two identities:

$$x\wedge x^{\circ}=x^{\circ\circ}\wedge x^{\circ}, (x\wedge x^{\circ})\vee y\vee y^{\circ}=y\vee y^{\circ}.$$

The subvariety $\mathbf{K_2}\vee\mathbf{K_3}$ of **MS** defined by the following two identities:

$$(x\wedge x^{\circ})\vee y^{\circ}\vee y^{\circ\circ}=y^{\circ}\vee y^{\circ\circ},$$
$$(x\vee x^{+})\wedge y^{+}\wedge y^{++}=y^{+}\wedge y^{++}.$$

The class **S** of Stone algebras is a subvariety of **MS** and is characterized by the identity $x\wedge x^{\circ}=0$. The subvariety **B** of **MS** characterized by the identity $x\vee x^{\circ}=1$ is the class of Boolean algebras.

A double *MS*-algebra is an algebra $(L,^{\circ},^{+})$ such that $(L,^{\circ})$ is an *MS*-algebra, $(L,^{+})$ is a dual *MS*-algebra and for every $x\in L, x^{\circ+}=x^{\circ\circ}, x^{+\circ}=x^{++}$.

The class **DS** of all double Stone algebras is a subclass of the class **DMS** of all double *MS*-algebras.

Theorem 2.1.
Let L be a double *MS*-algebra. Then

(1) the skeleton $L^{\circ\circ}=\{x\in L: x^{\circ\circ}=x\}=\{x\in L: x^{++}=x\}=L^{++}$ is a de Morgan subalgebra of L,
(2) $L^{\vee}=\{x\vee x^{\circ}: x\in L\}$ is an order filter (increasing subset) of L,
(3) $L^{\wedge}=\{x\wedge x^{\circ}: x\in L\}$ is an order ideal (decreasing subset) of L,
(4) the dense set $D(L)=\{x\in L: x^{\circ}=0\}$ is a filter of L,
(5) the dual dense set $\overline{D(L)}=\{x\in L: x^{+}=1\}$ is an ideal of L.

The elements of $L^{\circ\circ}$ are called the closed elements of L and the elements of $D(L)$ are called the dense elements of L.

Now we recall the following result from [7].

Theorem 2.2. [*Theorem* 2.1, 7]
A double *MS*-algebra L satisfies the complement property if and only if

(1) Given $a,b\in L$ such that $a^{\circ\circ}=b^{\circ\circ}, a^{++}=b^{++}$, then $a=b$,
(2) Given $a,b\in L$ such that $a=a^{\circ\circ}, b=b^{\circ\circ}, a\le b$ there exists an element $x\in L$ such that $x^{++}=a, x^{\circ\circ}=b$.

A (0,1)-homomorphism from a bounded lattice into another one is a lattice homomorphism taking 0 into 0 and 1 into 1. A mapping $f: M\to C$ of a de Morgan algebra M into a de Morgan algebra C is called a de Morgan algebra homomorphism if f is a lattice homomorphism satisfying $\overline{f(x)}=f(\bar{x})$ for every $x\in M$. A mapping $f: L\to L_1$ of a double *MS*-algebra L into a double *MS*-algebra L_1 is called a double *MS*-algebra homomorphism if f is a lattice homomorphism satisfying $(f(x))^{\circ}=f(x^{\circ})$ and $(f(x))^{+}=f(x^{+})$ for every $x\in L$.

Let L be a double *MS*-algebra. A lattice congruence θ on L is a congruence if $x\equiv y(\theta)$, then $x^{\circ}\equiv y^{\circ}$ and $x^{+}\equiv y^{+}$. We denote by $Con(L)$ the congruence lattice of L.

Let A be an algebra. We say that $\theta,\psi\in Con(A)$ permute if $x\equiv y(\theta)$ and $y\equiv z(\psi)$ imply $x\equiv r(\psi)$ and $r\equiv z(\theta)$, for some $y,r\in A$. The algebra A is congruence permutable if every pair of congruences in $Con(A)$ permutes.

For the basic properties of distributive lattices we refer to [15] and for MS-algebras and double MS-algebras, we refer to [2,3,5,6] and [8].

3 The Construction

In this section the concept of regularity on the class of double MS-algebras is considered. Many related properties and examples are given. A construction of a double MS-algebra L satisfying the generalized complement property from a suitable de Morgan algebra M and a filter F of M containing $M^{\vee}$ is investigated. Every double MS-algebra L satisfying the generalized complement property can be uniquely determined by the pair $(L^{\circ\circ},[L^{\vee})^{++})$.

Let $(L,^{\circ},^{+})$ be a double MS-algebra. Then for $H\subseteq L$, consider H^{+} and H^{++} as follows:

$$H^{+}=\{x^{+}:x\in H\} \text{ and } H^{++}=\{x^{++}:x\in H\}$$

Lemma 3.1.
Let F be a filter of a double MS-algebra L. Then F^{++} is a filter of $L^{\circ\circ}$.

Proof
Clearly, $1\in F^{++}$. Let $x,y\in F^{++}$. Then $x=a^{++},y=b^{++}$ for some $a,b\in F$. Hence $x\wedge y=a^{++}\wedge b^{++}=(a\wedge b)^{++}\in F^{++}$, as $a\wedge b\in F$. Again, let $x\in F^{++}$ and $z\in L^{\circ\circ}$ be such that $z\geq x$. Then $x=a^{++}$ for some $a\in F$. Thus $z=z\vee x=z^{++}\vee a^{++}=(z\vee a)^{++}\in F^{++}$ as $z\vee a\in F$. Then F^{++} is a filter of $L^{\circ\circ}$.

Corollary 3.2.

(1) If L is a double MS-algebra from $\mathbf{K}_2$, then $L^{\vee++}=\{d^{++}:d\in L^{\vee}\}$ is a filter of $L^{\circ\circ}$,
(2) If L is a double Stone algebra, then $D(L)^{++}=\{d^{++}:d\in D(L)\}$ is a filter of $L^{\circ\circ}$.

Proof
(1). Since $L\in\mathbf{K_2}$, then $L^{\vee}$ is a filter of L. Thus $L^{\vee++}$ is a filter of $L^{\circ\circ}$ by lemma 3.1 and

$$\begin{aligned}L^{\vee++}&=\{(x\vee x^{\circ})^{++}:x\in L\}\\&=\{d^{++}:d=x\vee x^{\circ}\in L^{\vee}\}.\end{aligned}$$

(2). Since L is a Stone algebra, then $L^{\vee}=D(L)$ is a filter of L and $L^{\circ\circ}$ is a Boolean algebra which is usually denoted by $B(L)$. Thus $D(L)^{++}$ is a filter of $B(L)$ by lemma 3.1 and

$$\begin{aligned}D(L)^{++}&=\{(x\vee x^{\circ})^{++}:x\in L\}\\&=\{d^{++}:d=x\vee x^{\circ}\in D(L)\}.\end{aligned}$$

The concept of regular double MS-algebras is given as follows:

Definition 3.3. A double MS-algebra is called regular if

$$x^{\circ}=y^{\circ} \text{ and } x^{+}=y^{+} \text{ imply } x=y.$$

Let us denote by **RDMS** the class of all regular double MS-algebras and **RDS** the class of all regular double Stone algebras. Tt is easy to show that the class **RDS** is a subclass of the class **RDMS**.

Now, we present double MS-algebras satisfying the generalized complement property generalizing double MS-algebras satisfying the complement property due to L. Congwen [7].

Definition 3.4. A double MS-algebra L satisfying the generalized complement property (or DMS^{gc}-algebra) is a double MS-algebra satisfying the following two conditions:

(1) L is a regular double MS-algebra,
(2) Given $a,b\in L^{\circ\circ}$ and a filter F of $L^{\circ\circ}$ containing $L^{\circ\circ\vee}$ such that $a\leq b$ and $a\vee b^{\circ}\in F$, then there exists an element $x\in L$ such that $x^{++}=a$ and $x^{\circ\circ}=b$.

We shall denote by $\mathbf{DMS^{gc}}$ the class of all DMS^{gc}-algebras and by $\mathbf{DMS^{c}}$ the class of all double MS-algebras satisfying the complement property (briefly DMS^{c}-algebras).

Example 3.5.

(1) Every regular double Stone algebra $L=(L,\vee,\wedge,^{*},^{+},0,1)$ is a DMS^{gc}-algebra. Since for any filter F of L and for any $a,b\in B(L)$ such that $a\leq b,a\vee b^{*}\in F$, there exists an element $x\in L$ such that $x^{++}=a$ and $x^{**}=b$ (see [Lemma 2, 7]).
(2) Every DMS^{c}-algebra L is a DMS^{gc}-algebra by considering $F=L^{\circ\circ}$.

Now we illustrate two examples to show that the class of $\mathbf{DMS^{c}}$ is a proper subclass of the class of $\mathbf{DMS^{gc}}$ and the later is a proper subclass of the class of **RDMS**.

Example 3.6.

(1) Consider $L=\{0<c<a<d<1\}$ and $a=a^{\circ}=c^{\circ}=a^{+}=d^{+},d^{\circ}=1^{\circ}=0,0^{+}=c^{+}=1$. Clearly $(L,^{\circ},^{+})$ is double MS-algebra and $F=\{a,1\}$ is a filter of $L^{\circ\circ}$ containing $L^{\circ\circ\vee}$. It is observed that $L\in\mathbf{DMS^{gc}}$. Now $0<1$ but there is no $x\in L$ such that $x^{++}=0,x^{\circ\circ}=1$. Therefore L does not satisfy the complement property. Then $L\notin\mathbf{DMS^{c}}$.
(2) Let $L=\{0<a<d<1\}$ be a four element chain and $a^{\circ}=a=a^{+}=d^{+},d^{\circ}=0$. Obviously $(L,^{\circ},^{+})$ is a regular double MS-algebra. L does not satisfy the condition (2) of Definition 3.4 because of $0<a$ and $0\vee a^{\circ}=a\in L^{\circ\circ\vee}=L^{\vee++}$ but there is no an element $x\in L$ such that $x^{++}=0$ and $x^{\circ\circ}=a$. Then $L\notin\mathbf{DMS^{gc}}$.

Now, we introduce a construction of a DMS^{gc}-algebra L from a suitable de Morgan algebra M and a filter F of

M containing $M^\vee$.

Theorem 3.7. (**Construction Theorem**)
Let $(M,\wedge,\vee,\bar{},0,1)$ be a de Morgan algebra and F be a filter of M containing $M^\vee$. Then

$$L=(M,F)=\{(a,b):a\le b, a\vee\bar{b}\in F\}$$

is a DMS^{gc}-algebra if we define

$$\begin{aligned}(a,b)\wedge(c,d)&=(a\wedge c,b\wedge d),\\ (a,b)\vee(c,d)&=(a\vee c,b\vee d),\\ (a,b)^\circ&=(\bar{b},\bar{b}),\\ (a,b)^+&=(\bar{a},\bar{a}),\\ 1_L&=(1,1),\\ 0_L&=(0,0).\end{aligned}$$

Furthermore, $L^{\circ\circ}\cong M$ as de Morgan algebras, $D(L)\cong F\cong D(L)^{++}$ as lattices and $L^{\circ\circ\vee}\subseteq D(L)^{++}$.

Proof
Let $(a,b),(c,d)\in(M,F)$. Then $a\le b, c\le d$ and $a\vee\bar{b}, c\vee\bar{d}\in F$. Hence

$$(a,b)\wedge(c,d)=(a\wedge c,b\wedge d)\in L \text{ and } (a,b)\vee(c,d)=(a\vee c,b\vee d)\in L$$

because of

$$\begin{aligned}&(a\wedge c)\vee\overline{(b\wedge d)}=(a\wedge c)\vee(\bar{b}\vee\bar{d})\\ &=(a\vee\bar{b}\vee\bar{d})\wedge(c\vee\bar{b}\vee\bar{d})\in F \text{ by distributivity of } M,\\ &(a\vee c)\vee\overline{(b\vee d)}=(a\vee c)\vee(\bar{b}\wedge\bar{d})\\ &=(a\vee c\vee\bar{b})\wedge(a\vee c\vee\bar{d})\in F.\end{aligned}$$

Clearly $(0,0),(1,1)\in L$. Then L is a (0,1) sublattice of $M\times M$. Therefore L is a bounded distributive lattice. Now we have

$$\begin{aligned}(a,b)^{\circ\circ}&=(b,b)\ge(a,b) \text{ as } b\ge a,\\ ((a,b)\wedge(c,d))^\circ&=(a\wedge c,b\wedge d)^\circ\\ &=(\bar{b}\vee\bar{d},\bar{b}\vee\bar{d})\\ &=(\bar{b},\bar{b})\vee(\bar{d},\bar{d})\\ &=(a,b)^\circ\vee(c,d)^\circ,\\ (1,1)^\circ&=(0,0)\end{aligned}$$

Then $(L,^\circ)$ is an *MS*-algebra. Also, we have

$$\begin{aligned}(a,b)^{++}&=(a,a)\le(a,b) \text{ as } a\le b,\\ ((a,b)\wedge(c,d))^+&=(a\wedge c,b\wedge d)^+\\ &=(\bar{a}\vee\bar{c},\bar{a}\vee\bar{c})\\ &=(\bar{a},\bar{a})\vee(\bar{c},\bar{c})\\ &=(a,b)^+\vee(c,d)^+,\\ (0,0)^+&=(1,1).\end{aligned}$$

Thus $(L,^+)$ is a dual MS-algebra. We observe that $(a,b)^{\circ+}=(b,b)=(a,b)^{\circ\circ}$ and $(a,b)^{+\circ}=(a,a)=(a,b)^{++}$. Therefore $(L,^\circ,^+)$ is a double MS-algebra. For regularity of L, let $(a,b)^\circ=(c,d)^\circ$ and $(a,b)^+=(c,d)^+$. Then $(\bar{b},\bar{b})=(\bar{d},\bar{d})$ and $(\bar{a},\bar{a})=(\bar{c},\bar{c})$ implies $b=d$ and $a=c$, respectively. Thus $(a,b)=(c,d)$.
Moreover

$$\begin{aligned}L^{\circ\circ}&=\{(a,b)\in L:(a,b)^{\circ\circ}=(a,b)\}\\ &=\{(a,b)\in L:a=b\}\\ &=\{(a,a):a\in M\},\\ D(L)&=\{(a,b)\in L:(a,b)^\circ=(0,0)\}\\ &=\{(a,1):a\in F\},\\ \overline{D(L)}&=\{(a,b)\in L:(a,b)^+=(1,1)\}\\ &=\{(0,b)\in L:\bar{b}\in F\},\\ D(L)^{++}&=\{(a,1)^{++}:(a,1)\in D(L)\}\\ &=\{(a,a):a\in F\}, F \text{ is a filter of } M,\\ L^{\circ\circ\vee}&=\{(a,a):a\in M^\vee\subseteq F\}\subseteq D(L)^{++}\end{aligned}$$

It is obviously that the mappings $f:M\to L^{\circ\circ}$, $g:F\longrightarrow D(L)$ and $h:F\to D(L)^{++}$ such that $f(a)=(a,a)$, $g(x)=(x,1)$ and $h(x)=(x,x)$ are isomorphisms. Now we have to prove that L satisfies condition (2) of Definition 3.4. Let $(a,a)\le(b,b)$ be such that $\underline{(a,a)}\vee(b,b)^\circ\in\underline{D}(L)^{++}$. Then $(a\vee\bar{b},a\vee\bar{b})\in D(L)^{++}$ implies $a\vee\bar{b}\in F$. So $(a,b)\in L$ such that $(a,b)^{++}=(a,a)$ and $(a,b)^{\circ\circ}=(b,b)$. Then L is a *DMS*-algebra satisfying the generalized complement property.

We shall say that the regular double MS^{gc}-algebra L from Theorem 3.7 is associated with the pair (M,F).

Two special cases are considered in the following corollary.

Corollary 3.8.

(1) If M is a Kleene algebra, then L described by Theorem 3.7 is a DMS^{gc}-algebra from $\mathbf{K_2}\vee\mathbf{K_3}$,
(2) If M is a Boolean, then L described by Theorem 3.7 is a regular double Stone algebra.

Proof
(1). Let $x=(a,b), y=(c,d)\in L$. We have to show that if $M\in\mathbf{K}$, then $(x\wedge x^\circ)\vee y^\circ\vee y^{\circ\circ}=y^\circ\vee y^{\circ\circ}$ and $(x\vee x^+)\wedge y^+\wedge y^{++}=y^+\wedge y^{++}$. Now

$$\begin{aligned}&[(a,b)\wedge(a,b)^\circ]\vee(c,d)^\circ\vee(c,d)^{\circ\circ}\\ &=[(a,b)\wedge(\bar{b},\bar{b})]\vee(\bar{d},\bar{d})\vee(d,d)\\ &=(a\wedge\bar{b},b\wedge\bar{b})\vee(\bar{d}\vee d,\bar{d}\vee d)\\ &=((a\wedge\bar{b})\vee(d\vee\bar{d}),(b\wedge\bar{b})\vee(d\vee\bar{d}))\\ &=(d\vee\bar{d},d\vee\bar{d}) \text{ as } a\wedge\bar{b}\le b\wedge\bar{b}\le d\vee\bar{d},\end{aligned}$$

and

$$\begin{aligned}(c,d)^\circ\vee(c,d)^{\circ\circ}&=(\bar{d},\bar{d})\vee(d,d)\\ &=(\bar{d}\vee d,\bar{d}\vee d).\end{aligned}$$

Also,

$$
\begin{aligned}
& [(a,b)\vee(a,b)^{+}]\wedge(c,d)^{+}\wedge(c,d)^{++} \\
&= [(a,b)\vee(\bar{a},\bar{a})]\wedge(\bar{c},\bar{c})\vee(c,c) \\
&= (a\vee\bar{a},b\vee\bar{a})\wedge(\bar{c}\wedge c,\bar{c}\wedge c) \\
&= ((a\vee\bar{a})\wedge(\bar{c}\wedge c),(b\vee\bar{a})\wedge(\bar{c}\wedge c)) \\
&= (\bar{c}\wedge c,\bar{c}\wedge c) \text{ as } \bar{c}\wedge c\leq\bar{a}\vee a\leq\bar{a}\vee b, \\
(c,d)^{+}\wedge(c,d)^{++} &= (\bar{c},\bar{c})\vee(c,c) \\
&= (\bar{c}\wedge c,\bar{c}\wedge c).
\end{aligned}
$$

Then L is a DMS^{gc}-algebra from the subclass $\mathbf{K_2}\vee\mathbf{K_3}$.
(2). Since M is a Boolean algebra, then $a\wedge\bar{a}=0$ and $a\vee\bar{a}=1$ for every $a\in M$. For every $(a,b)\in L$, we have $(a,b)\wedge(a,b)^{\circ}=(a\wedge\bar{b},b\wedge\bar{b})=(0,0)$ as $a\wedge\bar{b}\leq b\wedge\bar{b}=0$ and $(a,b)\vee(a,b)^{+}=(a,b)\vee(\bar{a},\bar{a})=(a\vee\bar{a},b\vee\bar{a})=(1,1)$ as $b\vee\bar{a}\geq a\vee\bar{a}=1$. Then $L=(M,F)$ is a regular double Stone algebra.

We illustrate the construction of DMS^{gc}-algebras on the following example.

Example 3.9.
Consider $M=\{0<a=a^{\circ}<1\}$ be the three element kleene algebra and $F=\{a,1\}=M^{\vee}$ be a filter of M. Using the construction Theorem, we can construct a DMS^{gc}-algebra $L=(M,F)$ as follows:

$$L=(M,F)=\{(0,0)<(0,a)<(a,a)<(a,1)<(1,1)\}$$

$$(0,a)^{\circ}=(a,a)^{\circ}=(a,a)=(a,a)^{+}=(a,1)^{+},(0,a)^{+}=(1,1),(a,1)^{\circ}=(0,0)$$

Notice that

$$L^{\circ\circ}=\{(0,0),(a,a),(1,1)\}\cong M,D(L)=\{(a,1),(1,1)\}\cong F$$

and

$$D(L)^{++}=\{a,a),(1,1)\}=L^{\circ\circ\vee}\cong F.$$

The following Theorem shows that each element x of a DMS^{gc}-algebra L is uniquely described by the greatest closed element below x and the smallest closed element above x.

Theorem 3.10.
Let L be a DMS^{gc}-algebra, $M=L^{\circ\circ}$ and $F=[L^{\vee})^{++}$. Then the mapping $\psi:L\to(M,F)$ defined by $\psi(x)=(x^{++},x^{\circ\circ})$ is an isomorphism.

Proof
For every $x\in L$, we have $x^{++}\leq x^{\circ\circ}$ and $x^{++}\vee x^{\circ\circ\circ}=x^{++}\vee x^{\circ}=(x\vee x^{\circ})^{++}\in[L^{\vee})^{++}$ as $x\vee x^{\circ}\in L^{\vee}$. Then $(x^{++},x^{\circ\circ})\in(M,F)$ and ψ is a well defined map. Now, we prove that ψ is a (0,1) lattice homomorphism. It is clear that $\psi(0)=(0,0)$ and $\psi(1)=(1,1)$. For every $x,y\in L$, we get

$$
\begin{aligned}
\psi(x\wedge y) &= ((x\wedge y)^{++},(x\wedge y)^{\circ\circ}) \\
&= (x^{++}\wedge y^{++},x^{\circ\circ}\wedge y^{\circ\circ}) \\
&= (x^{++},x^{\circ\circ})\wedge(y^{++},y^{\circ\circ}) \\
&= \psi(x)\wedge\psi(y), \\
\psi(x\vee y) &= ((x\vee y)^{++},(x\vee y)^{\circ\circ}) \\
&= (x^{++}\vee y^{++},x^{\circ\circ}\vee y^{\circ\circ}) \\
&= (x^{++},x^{\circ\circ})\vee(y^{++},y^{\circ\circ}) \\
&= \psi(x)\vee\psi(y)
\end{aligned}
$$

Obviously $\psi(x^{\circ})=(\psi(x))^{\circ}$ and $\psi(x^{+})=(\psi(x))^{+}$. Thus ψ is a double MS-algebra homomorphism. To show that ψ is an injective mapping, let $\psi(x)=\psi(y)$. Then $(x^{++},x^{\circ\circ})=(y^{++},y^{\circ\circ})$ implies $x^{\circ}=y^{\circ}$ and $x^{+}=y^{+}$. By regularity of L we get $x=y$. It remains to prove that ψ is surjective. Let $(a,b)\in(M,F)$. According to condition (2) of Definition 3.4, there exists $x\in L$ such that $x^{++}=a\leq b=x^{\circ\circ}$ and $x^{++}\vee x^{\circ\circ\circ}=x^{++}\vee x^{\circ}=a\vee b^{\circ}\in F$. Thus $(x^{++},x^{\circ\circ})\in(M,F)$ and $\psi(x)=(x^{++},x^{\circ\circ})=(a,b)$. Therefore ψ is a double MS-algebra isomorphism.

4 Applications

Many applications of the construction Theorem (Theorem 3.7) are given in the following two subsections.

4.1 Subalgebras and homomorphisms

Using the construction of a DMS^{gc}-algebra from the pair (M,F), where M is a de Morgan algebra and F is a filter of M containing $M^{\vee}$, we characterize subalgebras of a DMS^{gc}-algebra L associated with (M,F). A description of special subalgebras of a DMS^{gc}-algebra $M^{[2]}$ is given. Also we characterize homomorphisms of DMS^{gc}-algebras in terms of pairs (M,F).

Theorem 4.1.
If $L=(M,F),H=(C,G)$ be DMS^{gc}-algebras. Then L is a subalgebra of H if and only if M is a subalgebra of C and F is a sublattice of G with 1.

Proof
Suppose L is a subalgebra of H. Then by Theorem 3.7, $L^{\circ\circ}=\{(a,a):a\in M\}$, $H^{\circ\circ}=\{(a,a):a\in C\}$, $D(L)=\{(x,1):x\in F\}$ and $D(H)=\{(y,1):y\in G\}$. Clearly $L^{\circ\circ}$ is a subalgebra of $H^{\circ\circ}$ and $D(L)$ is a sublattice of $D(H)$ containing $(1,1)$. Let $a\in M$. Thus $(a,a)\in L^{\circ\circ}\subseteq H^{\circ\circ}$. Then $(a,a)\in H^{\circ\circ}$ implies $a\in C$. So $M\subseteq C$. Since $(0,0),(1,1)\in L^{\circ\circ}$. Then $0,1\in M$. Let $x,y\in M$. Then we get

$$
\begin{aligned}
x,y\in M &\Rightarrow (x,x),(y,y)\in L^{\circ\circ} \\
&\Rightarrow (x\wedge y,x\wedge y),(x\vee y,x\vee y)\in L^{\circ\circ} \\
&\Rightarrow x\wedge y,x\vee y\in M.
\end{aligned}
$$

Therefore M is a bounded sublattice of de Morgan algebra C. For every $x \in M$, $(x,x) \in L^{\circ\circ}$. Then $(\overline{x},\overline{x}) = (x,x)^{\circ} \in L^{\circ\circ}$ implies $\overline{x} \in M$. Therefore M is a subalgebra of C. Let $x \in F$. Then $(x,1) \in D(L) \subseteq D(H)$ implies $x \in G$. Thus $F \subseteq G$. Clearly $1 \in F$. Let $x,y \in F$, so $(x,1),(y,1) \in D(L)$. Then $(x \wedge y,1),(x \vee y,1) \in D(L)$ imply $x \wedge y, x \vee y \in F$. Therefore F is a sublattice of G with 1.

Conversely, suppose M is a subalgebra of C and F is a sublattice of G with 1. Again by Theorem 3.7, for every $(a,b) \in L$, we have $a \leq b$ and $a \vee \overline{b} \in F \subseteq G$. This gives $(a,b) \in H$. Therefore $L \subseteq H$. Since L and H are DMS^{gc}-algebras, then L is a subalgebra of H.

Let M be a de Morgan algebra, $F(M)$ be the lattice of all filters of M and $F_{M^{\vee}} = \{F : F \in F(M), M^{\vee} \subseteq F\}$ be the family of filters of M containing $M^{\vee}$. We will write R_F instead of a DMS^{gc}-algebra (M,F). Let $R_{F_{M^{\vee}}} = \{R_F : F \in F_{M^{\vee}}\}$ be the family of all DMS^{gc}-algebras constructing from (M,F) for all $F \in F_{M^{\vee}}$. Many properties of $R_{F_{M^{\vee}}}$ are investigated in the following two Theorems.

Theorem 4.2.
Let $M = (M, \wedge, \overline{\ }, 0, 1)$ be a de Morgan algebra. Then for any $F,G \in F_{M^{\vee}}$ we have

(1) $R_F \subseteq R_G$ if and only if $(R_F)^{\circ\circ} = (R_g)^{\circ\circ}$ and $D(R_F) \subseteq D(R_G)$,
(2) $F \subseteq G$ if and only if $R_F \subseteq R_G$,
(3) R_F is a subalgebra of $M^{[2]}$.

Proof

(1) Let $R_F \subseteq R_G$. Clearly $(R_F)^{\circ\circ} \subseteq (R_G)^{\circ\circ}$. Since $(R_F)^{\circ\circ} \cong M \cong (R_G)^{\circ\circ}$, then $(R_F)^{\circ\circ} = (R_G)^{\circ\circ}$. Now, let $(x,1) \in D(R_F)$. Then $(x,1) \in R_G$. Thus $(x,1) \in D(R_G)$ as $(x,1)^{\circ} = (0,0)$. Conversely, Let $(a,b) \in R_F$. Then $a \leq b$ and $a \vee \overline{b} \in F$. Hence $(a \vee \overline{b},1) \in D(R_F) \subseteq D(R_G)$ and $a \vee \overline{b} \in G$. Therefore $(a,b) \in R_F$.

–(2) Let $F \subseteq G$ and $(a,b) \in R_F$. Thus $a \vee \overline{b} \in F$. Then $a \vee \overline{b} \in G$ implies $(a,b) \in R_G$. Then $R_F \subseteq R_G$. Conversely, let $R_F \subseteq R_G$ and $x \in F$. Then $x = (x,1) \in R_F$ and $(x,1) \in D(R_F) \subseteq D(R_G)$. Therefore $x \in G$.

–(3) One can easily verify that R_F is a subalgebra of $M^{[2]}$ for every $F \in F_{M^{\vee}}$.

Theorem 4.3.
Let M be a de Morgan algebra. Then for any $F,G \in F_{M^{\vee}}$ we have

(1) $F_{M^{\vee}}$ is a bounded distributive lattice on its own,
(2) the family $R_{F_{M^{\vee}}}$ is a bounded distributive lattice on its own,
(3) $F_{M^{\vee}} \cong R_{F_{M^{\vee}}}$.

Proof
(1) Let $F,G \in F_{M^{\vee}}$. Clearly $F \cap G \in F_{M^{\vee}}$ and $F \vee G = \{x = f \wedge g, f \in F, g \in G\} \in F_{M^{\vee}}$. Then $F_{M^{\vee}}$ is a sublattice of $F(M)$. Obviously $M, [M^{\vee})$ are the greatest and the smallest elements of $F_{M^{\vee}}$ respectively. Therefore $(F_{M^{\vee}}, \cap, \vee, M, [M^{\vee}))$ is a bounded distributive lattice.

(2) Clearly $R_{F_{M^{\vee}}}$ is a partially ordered set with respect to the set inclusion. Now for any two DMS^{gc}-algebras R_F and R_G in $R_{F_{M^{\vee}}}$, define the operations $\cap$ and $\sqcup$ on $R_{F_{M^{\vee}}}$ as follows:

$$R_F \cap R_G = R_{F\cap G} \text{ and } R_F \sqcup R_G = R_{F\vee G}$$

Clearly $R_{F\cap G}$ is the infimum of both R_F, R_G in $R_{F_{M^{\vee}}}$. Obviously $R_{F\vee G}$ is an upper bound of R_F and R_G. Suppose $R_F \subseteq R_H, R_G \subseteq R_H$ for some $H \in F_{M^{\vee}}$. Then H is an upper bound of both F and G in $F_{M^{\vee}}$. Hence $F \vee G \subseteq H$. Then $R_{F\vee G} \subseteq R_H$. Therefore $R_{F\vee G}$ is the supermum of both R_F and R_G in $R_{F_{M^{\vee}}}$. Consequently $(R_{F_{M^{\vee}}}, \cap, \sqcup)$ is a lattice. We observe that $M^{[2]} = R_M$ is the greatest member in $R_{F_{M^{\vee}}}$ and $R_{[M^{\vee})}$ is the smallest member in $R_{F_{M^{\vee}}}$. This deduce that $R_{F_{M^{\vee}}}$ is a bounded lattice. It can be easily obtained that $(R_{F_{M^{\vee}}}, \cap, \sqcup, R_{[M^{\vee})}, M^{[2]})$ is a distributive lattice.

(3) Define the map $\pi : F_{M^{\vee}} \to R_{F_{M^{\vee}}}$ by $\pi(F) = R_F$. It is clear that $\pi([M^{\vee})) = R_{[M^{\vee})}$ and $\pi((M) = M^{[2]}$. Let $F,G \in F_{M^{\vee}}$. Then we get

$$\begin{aligned}\pi(F\cap G) &= R_{F\cap G}\\ &= R_F \cap R_G\\ &= \pi(F) \cap \pi(G),\\ \pi(F\vee G) &= R_{F\vee G}\\ &= R_F \sqcup R_G\\ &= \pi(F) \sqcup \pi(G).\end{aligned}$$

Then π is a (0,1) lattice homomorphism. To show that π is an injective map, let $\pi(F) = \pi(G)$. Then $R_F = R_G$ implies $F = G$. It is clear that π is a surjective map. Therefore π is a lattice isomorphism.

Now, we characterize homomorphisms of DMS^{gc}-algebras in terms of pairs (M,F).

Theorem 4.4.
Let $L = (M,F)$ and $L_1 = (M_1,F_1)$ be DMS^{gc}-algebras and let $h : L \to L_1$ be a double *MS*-algebra homomorphism. Then $S(h) : L^{\circ\circ} \to L_1^{\circ\circ}$ defined by $S(h)(a) = h(a)$ for each $a \in L^{\circ\circ}$ is a de Morgan algebra homomorphism and $h(F) \subseteq F_1$. Conversely, if $h : M \to M_1$ is a de Morgan homomorphism and $h(F) \subseteq F_1$, then h can be uniquely extended to a double *MS*-algebra homomorphism from $L = (M,F)$ into $L_1 = (M_1,F_1)$.

Proof
For every $a \in L^{\circ\circ}$, $S(h)(a) \in L_1^{\circ\circ}$ as $(h(a))^{\circ\circ} = h(a^{\circ\circ}) = h(a)$. It is easy to check that $S(h)$ is a de Morgan algebra homomorphism. Let $y \in h(F)$. Then $y = h(x)$ for some $x \in F$. So $(x,1) \in D(L)$ and

$(y,1) = (h(x),1) \in h(D(L_1))$ as $(y,1)^\circ = (0,0)$. Thus $y \in F_1$ and $h(F) \subseteq F_1$. Conversely, define $R(h) : L \to L_1$ by $R(h)(a,b) = (h(a),h(b)),(a,b) \in L$. Then $a \leq b$ and $a \vee b \in F$ imply $h(a) \leq h(b)$ and $h(a) \vee h(b) = h(a \vee b) \in h(F) \subseteq F_1$. Hence $R(h)(a,b) \in L_1$ and $R(h)$ is well defined mapping. Now, for every $(a,b),(c,d) \in L$ we get

$$\begin{aligned}
R(h)((a,b)\wedge(c,d)) &= R(h)(a\wedge c, b\wedge d)\\
&= (h(a\wedge c), h(b\wedge d)\\
&= (h(a)\wedge h(c), h(c)\wedge h(d))\\
&= (h(a),h(b))\wedge(h(b),h(c))\\
&= R(h)(a,b)\wedge R(h)(c,d)),\\
R(h)((a,b)\vee(c,d)) &= R(h)(a\vee c, b\vee d)\\
&= (h(a\vee c), h(b\vee d)\\
&= (h(a)\vee h(c), h(c)\vee h(d))\\
&= (h(a),h(b))\vee(h(b),h(c))\\
&= R(h)(a,b)\vee R(h)(c,d)),
\end{aligned}$$

and

$$\begin{aligned}
(R(h)(a,b))^\circ &= (h(a),h(b))^\circ\\
&= (h(\bar{b}),h(\bar{b}))\\
&= R(h)(\bar{b},\bar{b})\\
&= R(h)(a,b)^\circ,\\
(R(h)(a,b))^+ &= (h(a),h(b))^+\\
&= (h(\bar{a}),h(\bar{a}))\\
&= R(h)(\bar{a},\bar{a})\\
&= R(h)(a,b)^+,
\end{aligned}$$

$R(h)(1,1) = (1,1)$ and $R(h)(0,0) = (0,0)$.

Consequently $R(h)$ is a double MS-algebra homomorphism.

4.2 Congruence relations

A DMS^{gc}-algebra $L = (M,F)$ regards as an extension of the de Morgan algebra M. The construction of regular double MS^{gc}-algebras from de Morgan algebras leads us to show that the congruence lattices of $L = (M,F)$ and M are isomorphic. Also, we prove that a regular double MS^{gc}-algebra $L = (M,F)$ has permutable congruences if and only if M has permutable congruences.

Theorem 5.1.

Let $(M,\vee,\wedge,\bar{\ },0,1)$ be a de Morgan algebra. Let L be a DMS^{gc}-algebra associated with the pair (M,F) for some filter F of M containing $M^\vee$. Then there exists a one-to-one correspondence between $Con(L)$ and $Con(M)$.

Proof

We have $L^{\circ\circ} = \{(a,a) : a \in M\} \cong M$ (see Theorem 3.7). Firstly, let $\theta \in Con(L)$. Define a relation ψ on M as follows:

$$a \equiv b(\psi) \Leftrightarrow (a,a) \equiv (b,b)(\theta)$$

It is clear that ψ is a lattice congruence on M. Let $a \equiv b(\psi)$. Then $(a,a) \equiv (b,b)(\theta)$ implies $(\bar{a},\bar{a}) = (a,a)^\circ \equiv (b,b)^\circ(\theta) = (\bar{b},\bar{b})$. Thus $\bar{a} \equiv \bar{b}(\psi)$ and $\psi \in Con(M)$. Conversely, let $\psi \in Con(M)$. Define a relation θ on L as follows:

$$(a,b) \equiv (c,d)(\theta) \Leftrightarrow a \equiv c(\psi) \text{ and } b \equiv d(\psi)$$

Clearly θ is a lattice congruence on L. It remains to show that θ preserves the operations $^\circ,^+$ on L. Let $(a,b) \equiv (c,d)(\theta)$. Then $a \equiv c(\psi), b \equiv d(\psi)$ imply $\bar{a} \equiv \bar{c}(\psi), \bar{b} \equiv \bar{d}(\psi)$. This gives $(a,b)^\circ = (\bar{b},\bar{b}) \equiv (\bar{d},\bar{d})(\theta) = (c,d)^\circ$ and $(a,b)^+ = (\bar{a},\bar{a}) \equiv (\bar{c},\bar{c})(\theta) = (c,d)^+$. Then $\theta \in Con(L)$.

In closing this paper, we introduce an important result concerning the permutability of congruences of DMS^{gc}-algebras.

Theorem 5.2.

Let L be a DMS^{gc}-algebra associated with (M,F) for a filter F of M containing $M^\vee$. Then L is a congruence permutable if and only if M is a congruence permutable.

Proof

Assume that L is a congruence permutable. Let $x,y,z \in L$. Then $x = (a,b), y = (c,d)$ and $z = (g,h)$ for some $a,b,c,d,g,h \in M$. Suppose that $\theta,\psi \in Con(L)$ are respectively corresponding to $\acute{\theta},\acute{\psi} \in Con(M)$. Let $x \equiv y(\theta)$ and $y \equiv z(\psi)$. Then by Theorem 5.1, we have

$$\begin{aligned}
&(a,b) \equiv (c,d)(\theta) \text{ and } (c,d) \equiv (g,h)(\psi)\\
&\quad\Rightarrow a \equiv c(\acute{\theta}), b \equiv d(\acute{\theta}) \text{ and } c \equiv g(\acute{\psi}), d \equiv h(\acute{\psi})\\
&\quad\Rightarrow a \equiv c(\acute{\theta}), c \equiv g(\acute{\psi}) \text{ and } b \equiv d(\acute{\theta}), d \equiv h(\acute{\psi})
\end{aligned}$$

Since M is a congruence permutable, then there exist $r,n \in M$ such that

$$\begin{aligned}
&a \equiv r(\acute{\psi}), r \equiv g(\acute{\theta}) \text{ and } b \equiv n(\acute{\psi}), n \equiv h(\acute{\theta})\\
&\Rightarrow (a,b) \equiv (r,n)(\psi) \text{ and } (r,n) \equiv (g,h)(\theta)\\
&\quad \text{for some } (r,n) \in L
\end{aligned}$$

Therefore θ,ψ are permute. Conversely, let L be a congruence permutable and let $\bar{\theta},\bar{\psi} \in Con(M)$. Then $a \equiv b(\bar{\theta})$ and $b \equiv c(\bar{\psi})$ implies $(a,a) \equiv (b,b)(\theta)$ and $(b,b) \equiv (c,c)(\psi)$, respectively. Thus there exists $(r,n) \in L$ such that

$$\begin{aligned}
&(a,a) \equiv (r,n)(\psi) \text{ and } (r,n) \equiv (c,c)(\theta)\\
&\qquad\Rightarrow a \equiv r(\bar{\psi}), r \equiv c(\bar{\theta}) \text{ for some } r \in M
\end{aligned}$$

Therefore $\bar{\theta},\bar{\psi}$ are permute. This deduce that M is a congruence permutable.

5 Conclusion

In this paper we introduced a class of so called double MS-algebras satisfying the generalized complement property (briefly DMS^{gc}-algebras) that includes the class of double *MS*-algebras satisfying the complement property. We illustrated two examples to show that the class of *DMS*-algebras satisfying the complement property is a proper subclass of the class of DMS^{gc}-algebras and the later is a proper subclass of the class of regular double *MS*-algebras. We presented an important construction (see Theorem 3.7) of DMS^{gc}-algebras from the pairs (M,F), where M is a de Morgan algebra and F is a filter of M containing $M^{\vee}$, generalizing the construction of regular double Stone algebras [1] presented by T. Katriňák. Further, we derived that every DMS^{gc}-algebra L is uniquely determined by the pair $(L^{\circ\circ},[L^{\vee})^{++})$.

Many applications of our construction are given in section 4. A characterization of homomorphisms and subalgebras of DMS^{gc}-algebras using the construction Theorem are obtained. Also, using the construction Theorem we investigated interesting descriptions of the notions of congruences and permutability of congruences of DMS^{gc}-algebras. For every DMS^{gc}-algebra $L=(M,F)$, we derived that $Con(L)$ and $Con(M)$ are isomorphic. Also, we proved that a DMS^{gc}-algebra $L=(M,F)$ has permutable congruences if and only if the de Morgan algebra M has permutable congruences. As a future work on this topic, we hope to study the perfect (also called canonical) extensions of DMS^{gc}-algebras in sense of [16] due to S. D. Comer by using our representation.

Acknowledgement

The author would like to thank the editors and referees for their valuable comments and suggestions to improve this presentation.

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