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Bayesian Estimation and Prediction of the Rayleigh Distribution Based on Ordered Ranked Set Sampling under Type-II Doubly Censored Samples

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Abstract: A ranked set sampling (RSS) scheme is an efficient and cost-effective alternative to simple random sampling when the sampling units can be ranked visually or by any inexpensive method. In this study, we first use ordered ranked set sampling (ORSS) in obtaining Bayesian estimation for the parameter of the Rayleigh distribution under Type-II doubly censoring scheme. This is done with respect to both squared error loss and Al-Bayyati loss functions. We obtain the mean squared error (MSE) and bias of the derived estimates based on m-cycle ORSS and compare them with those based on the corresponding simple random sample (SRS) to appreciate the efficiency of the obtained estimators. Next, we present the two-sample Bayesian predictive density function (point and interval) for the s-th ordered lifetime from a future independent sample. Finally, simulation study and real data are analyzed for illustrative purpose.

Keywords: Ordered ranked set sampling, Bayes estimator, Bayesian prediction, Squared error loss function, Mean squared error, Bias, Rayleigh distribution, Type-II doubly censored data.

1 Introduction

Chen et al. [1] introduced a detailed review of RSS. Subsequently, Dell and Clutter [2] proved that even in the presence of ranking errors RSS, mean estimator based on RSS is not only unbiased but it is at least as efficient as the mean estimator with SRS. Recently, there has been considerable research on the inference procedures based on the RSS. Hassan [3] estimated the unknown parameters for exponentiated exponential distribution using maximum likelihood and Bayes estimators based on RSS. Helu et al. [4] used Bayes estimation for the parameters of Weibull distribution using different methods of sampling namely, SRS, RSS, and modified RSS. McIntyre [5] was the first to introduce RSS as a method to increase the precision of estimated yield without the bias of researcher-chosen 'representative' samples and reduce cost by using simple judgment or qualitative information. For a detailed review, see Mohie El-Din et al. [22], Muttlak [7] and Sadek et al. [8]. Takahasi and Wakimoto [9] formed the mathematical setup of the RSS scheme and showed that the mean estimator based on RSS is unbiased and it is more precise than the mean estimator based on SRS.

Suppose that $X_{\text{SRS}} = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ denote a SRS of size n from a continuous population with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. We use the following figure to describe a RSS of size n:

$$\begin{array}{cccccc}
\underline{X_{1(1:n)}} & X_{1(2:n)} & \cdots & X_{1(n:n)} & \rightarrow & X_{1(1:n)} \\
X_{2(1:n)} & \underline{X_{2(2:n)}} & \cdots & X_{2(n:n)} & \rightarrow & X_{2(2:n)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
X_{n(1:n)} & X_{n(2:n)} & \cdots & \underline{X_{n(n:n)}} & \rightarrow & X_{n(n:n)}
\end{array}$$

where $X_{j(i:n)}$ denotes the ith order statistic from the jth simple random sample of size n. Then the vector of observations $X_{\text{RSS}} = \left(X_{1(1:n)}, X_{2(2:n)}, \ldots, X_{n(n:n)}\right)$ completes a one-cycle RSS of size n. If this method is repeated m times, a RSS of

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size mn is obtained. Balakrishnan and Li [10] proposed the idea of order statistics from independent and non-identically distributed (INID) random variables to propose ORSS, and showed that the best linear unbiased estimators based on ORSS are more efficient than the best linear unbiased estimators based on RSS for the two-parameter exponential, normal and logistic distributions.

Now we consider the scheme of doubly Type-II censored sampling, by using the procedure of ORSS, let $Y_{\text{DORSS}} = (Y_{r:n}, Y_{r+1:n}, ..., Y_{s:n})$, $1 \leq r \leq s \leq n$ be the doubly Type-II censored sample from a sample of size n obtained from ORSS. Under this assumption, $Y_j \equiv Y_{j:n}$ has the same distribution as $X_{j:n}$, the pdf and cdf of Y_j, $j = r, \cdots, s$ can be written as

$$f_{j:n}(y) = \sum_{i=0}^{j-1} \widetilde{c}_{i,j}(n)(1-F(y))^{n+i-j} f(y), \tag{1}$$

and

$$F_{j:n}(y) = \sum_{i=j}^{n} \binom{n}{i} (F(y))^i (1-F(y))^{n-i}, \tag{2}$$

respectively, (see Arnold et al. [11] and David and Nagaraja [12]), where

$$\widetilde{c}_{i,j}(n) = (-1)^i j \binom{j-1}{i} \binom{n}{j}. \tag{3}$$

We can re-write the cdf (2), using the binomial expansion, in the following form

$$F_{j:n}(y) = \sum_{i=j}^{n} \sum_{\tau=0}^{i} c_{i,\tau}(n)(1-F(y))^{n+\tau-i}, \tag{4}$$

or, equivalently,

$$F_{j:n}(y) = 1 - \sum_{i=1}^{j} \omega_{i,j}(n) \left(1-F(y)\right)^{n+i-j}, \tag{5}$$

where

$$c_{i,\tau}(n) = \binom{n}{i} \binom{i}{\tau} (-1)^{\tau} \quad \text{and} \quad \omega_{i,j}(n) = \widetilde{c}_{i-1,j}(n)/(n+i-j). \tag{6}$$

The scheme of doubly Type-II censored sampling is an important method of obtaining data in life testing experiments. Some of the earlier works on doubly censored samples were conducted by Ariyawans and Templton [13], Domma et al. [14], Elfessi [15] and Pak et al. [16].

Based on the doubly Type-II censored sample, let $\mathbf{x} = (x_r, x_{r+1}, \cdots, x_s)$, $1 \leq r \leq s \leq n$ be the ordered observations remaining when the $r-1$ smallest failed observations and the $n-s$ largest observations have been censored. Then the joint density function is given by (see Fernéndez [17])

$$f_X(\mathbf{x}) = \frac{n!}{(r-1)!(n-s)!} \left(F(x_r)\right)^{r-1} \left(1-F(x_s)\right)^{n-s} \prod_{i=r}^{s} f(x_i). \tag{7}$$

Now, consider Rayleigh distribution was first introduced by Rayleigh [18]. The pdf and cdf of the one parameter Rayleigh distribution (denoted as Ray(α)) are

$$f(x;\alpha) = \frac{x}{\alpha^2} \exp\left(-\frac{x^2}{2\alpha^2}\right) \quad \text{and} \quad F(x;\alpha) = 1 - \exp\left(-\frac{x^2}{2\alpha^2}\right), \quad x \geq 0, \tag{8}$$

respectively, where $\alpha > 0$. Therefore, the reliability function $R(t)$, and the hazard function $h(t)$ at mission time t for the Ray(α) are given by

$$R(t;\alpha) = \exp\left(-\frac{t^2}{2\alpha^2}\right) \quad \text{and} \quad h(t;\alpha) = \frac{t}{\alpha^2}, \quad t \geq 0, \tag{9}$$

respectively. Rayleigh distribution has been widely used in lifetime data analysis, especially in reliability theory and survival analysis because of it's flexibility and simplicity. Some clinical studies dealing with cancer patients show that the survival pattern follows the Rayleigh distribution.

For Bayes estimation, the performance depends on the form of the prior distribution and the loss function must be assumed. It is remarkable that most of the Bayesian inference procedures have been developed under the usual squared error loss function which is symmetrical, and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. Bayes estimation has been discussed by many authors; see for example Bernardo and Smith [19], Fernéndez [17], Kim and Song [20], Kotb [21], Mohie El-Din et al. [6], Nigm et al. [23] and Yu [24]. The prediction problems are studied by several authors, for instance Dey and Dey [25], Kotb [26], Mohie El-Din et al. [27,22] and Nigm et al. [23].

The outline of the rest of the article is as follows. In Section 2, we present the Bayesian estimation and the two-sample Bayesian prediction bounds of the Rayleigh distribution based on m-cycle ORSS. The Bayes estimates are obtained using both the square error loss function(SEL) and Al-Bayyatis loss function (ALF). In Section 3, we derive the Bayesian estimatian and two-sample Bayesian prediction bounds based on SRS. Some numerical results for illustrating the results in Section 4. Finally, Section 5 concludes the paper.

2 Bayes estimation and prediction based on m-cycle ORSS

Suppose that $\mathbf{y} = (y_r, y_{r+1}, \cdots, y_s)$, where $y_r \le y_{r+1} \le \cdots \le y_s$ are $s-r+1$ (out of n) observed ORSS (one-cycle case). Then, using the results for order statistics from INID random variables (see Balakrishnan [28]), the joint density function based on Type-II doubly censored of $\mathbf{y}$ can be written as

$$L_1(\alpha|\mathbf{y}) \propto \sum_{\mathbf{P}} \left(\prod_{k=1}^{r-1} F_{i_k}(y_r) \prod_{k=r}^{s} f_{i_k}(y_k) \prod_{k=s+1}^{n} \left[1 - F_{i_k}(y_s)\right] \right), \tag{10}$$

where $\sum_{\mathbf{P}}$ denotes the summation over all $n!$ permutations $(i_1, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$. or, it can be written as

$$L_1(\theta|\mathbf{y}) = \frac{1}{(n-r)!} \mathrm{Per}\mathbf{B}, \tag{11}$$

where

$$\mathbf{B} = \begin{pmatrix} F_1(y_r) & F_2(y_r) & \cdots & F_n(y_r) \\ f_1(y_r) & f_2(y_r) & \cdots & f_n(y_r) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(y_s) & f_2(y_s) & \cdots & f_n(y_s) \\ 1-F_1(y_s) & 1-F_2(y_s) & \cdots & 1-F_n(y_s) \end{pmatrix} \begin{matrix} \}r-1 \quad \text{rows} \\ \\ \\ \\ \}n-s \quad \text{rows} \end{matrix} . \tag{12}$$

Substituting the expressions from (1), (4) and (5) into (10), the likelihood function (LF) based on Type-II doubly censored one cycle ORSS is given by

$$\begin{aligned} L_1(\alpha|\mathbf{y}) \propto \sum_{\mathbf{P}} \Bigg(& \left(\prod_{k=1}^{r-1} \sum_{\ell=i_k}^{n} \sum_{\tau=0}^{\ell} c_{\ell,\tau}(n) \left(1 - F(y_r)\right)^{n+\tau-\ell} \right) \left(\prod_{k=r}^{s} \sum_{\ell=0}^{i_k-1} \tilde{c}_{\ell,i_k}(n) \right. \\ & \times \left. \left(1 - F(y_k)\right)^{n+\ell-i_k} f(y_k) \right) \left(\prod_{k=s+1}^{n} \sum_{\ell=1}^{i_k} \omega_{\ell,i_k}(n) \left(1 - F(y_s)\right)^{n+\ell-i_k} \right) \Bigg). \end{aligned} \tag{13}$$

Suppose that $\underline{\mathbf{y}} = (y_{rj}, y_{(r+1)j}, \cdots, y_{sj})$, $j = 1, 2, \ldots, m$ be m-cycle ORSS from Ray(α). Then, the LF in this case is given by

$$\begin{aligned} L_2(\alpha|\underline{\mathbf{y}}) \propto \prod_{j=1}^{m} \sum_{\mathbf{P[j]}} \Bigg(& \left(\prod_{k=1}^{r-1} \sum_{\ell=i_k}^{n} \sum_{\tau=0}^{\ell} c_{\ell,\tau}(n) \left(1 - F(y_{rj})\right)^{n+\tau-\ell} \right) \left(\prod_{k=r}^{s} \sum_{\ell=0}^{i_k-1} \tilde{c}_{\ell,i_k}(n) \right. \\ & \times \left. \left(1 - F(y_{kj})\right)^{n+\ell-i_k} f(y_{kj}) \right) \left(\prod_{k=s+1}^{n} \sum_{\ell=1}^{i_k} \omega_{\ell,i_k}(n) \left(1 - F(y_{sj})\right)^{n+\ell-i_k} \right) \Bigg). \end{aligned} \tag{14}$$

We will need the following relations through the following subsections:

$$\left. \begin{aligned} \prod_{k=1}^{r-1} \sum_{\ell=i_k}^{n} \sum_{\tau_k=0}^{\ell} h_\ell(i_k) &= \sum_{t_1=i_1}^{n} \sum_{\tau_1=0}^{t_1} \sum_{t_2=i_2}^{n} \sum_{\tau_2=0}^{t_2} \cdots \sum_{t_{r-1}=i_{r-1}}^{n} \sum_{\tau_{r-1}=0}^{t_{r-1}} \prod_{k=1}^{r-1} h_{t_k}(i_k), \\ \prod_{k=r}^{s} \sum_{\ell=0}^{i_k-1} T_\ell(i_k) &= \sum_{v_r=0}^{i_r-1} \sum_{v_{r+1}=0}^{i_{r+1}-1} \cdots \sum_{v_s=0}^{i_s-1} \prod_{k=r}^{s} T_{v_k}(i_k), \\ \prod_{k=s+1}^{n} \sum_{\ell=1}^{i_k} W_\ell(i_k) &= \sum_{v_{s+1}=1}^{i_{s+1}} \sum_{v_{s+2}=1}^{i_{s+2}} \cdots \sum_{v_n=1}^{i_n} \prod_{k=s+1}^{n} W_{v_k}(i_k), \end{aligned} \right\} \tag{15}$$

where $i_1 < i_2 < \cdots < i_n$ and i_k, $k = 1, 2, \cdots, n$ are positive integers.

2.1 Bayes estimation

Let $\underline{\mathbf{y}}$ be the m-cycle ORSS from the Ray(α) in Eq. (8). Upon substituting (8) in (14), making use of (15), we obtain the LF as follows:

$$L_2(\alpha;\underline{\mathbf{y}}) \propto \prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n} T_{\underline{V}}(m)\left(\prod_{j=1}^{m}\prod_{k=r}^{s}\frac{y_{kj}}{\alpha^2}\right)\exp\left[\frac{-1}{2\alpha^2}\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2 + \sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+\nu_{kj}-i_{kj})y_{sj}^2\right)\right], \tag{16}$$

where $\underline{V}=(\underline{t},\underline{\tau},\underline{\nu},\underline{\nu})$,

$$\sum_{\underline{V}}^{r,s,n} = \sum_{t_{1j}=i_1}^{n}\sum_{\tau_{1j}=0}^{t_{1j}}\sum_{t_{2j}=i_2}^{n}\sum_{\tau_{2j}=0}^{t_{2j}}\cdots\sum_{t_{(r-1)j}=i_{r-1}}^{n}\sum_{\tau_{(r-1)j}=0}^{t_{(r-1)j}}\times\sum_{\nu_{rj}=0}^{i_r-1}\sum_{\nu_{(r+1)j}=0}^{i_{r+1}-1}\cdots\sum_{\nu_{sj}=0}^{i_s-1} \times\sum_{\nu_{(s+1)j}=1}^{i_{s+1}}\sum_{\nu_{(s+2)j}=1}^{i_{s+2}}\cdots\sum_{\nu_{nj}=1}^{i_n}, \tag{17}$$

and

$$T_{\underline{V}}(m)=\left(\prod_{j=1}^{m}\prod_{k=1}^{r-1}c_{t_{kj},\tau_{kj}}(n)\right)\left(\prod_{j=1}^{m}\prod_{k=r}^{s}\tilde{c}_{\nu_{kj},i_{kj}}(n)\right)\left(\prod_{j=1}^{m}\prod_{k=s+1}^{n}\omega_{\nu_{kj},i_{kj}}(n)\right). \tag{18}$$

For the Bayesian setting, the prior distribution of the unknown parameter of the model is the major component required. So we use the square root inverted gamma prior, suggested by Fernéndez[17], which is given by

$$\pi(\alpha;\delta)\propto\alpha^{-2b-1}\exp\left(-\frac{a}{2\alpha^2}\right),\quad \alpha>0, \tag{19}$$

where $\delta=(a,b)$ is a vector of prior hyper-parameters, the shape parameter $b>0$ and scale parameter $a>0$. This prior has especially important because of its analytical tractability and easy interoperability. Moreover, it is generalizes the Jeffreys prior (when $b=0$, $a=0$); and gamma prior ($1/\alpha^2=\theta$) with parameters b and $a/2$.

From Eqs.(16) and (19), the posterior density can be written as

$$\pi^*(\alpha|\underline{\mathbf{y}}) = A^{-1}\prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n} T_{\underline{V}}(m)\alpha^{-2q-1}\exp\left[\frac{-1}{2\alpha^2}\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2 + \sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+\nu_{kj}-i_{kj})y_{sj}^2+a\right)\right], \tag{20}$$

where $q=(s-r+1)m+b$ and the normalized constant A is given by

$$A = 2^{q-1}\Gamma(q)\prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n} T_{\underline{V}}(m)\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2 + \sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+\nu_{kj}-i_{kj})y_{sj}^2+a\right)^{-q}. \tag{21}$$

Since Bayesian estimators of α under a SEL function can be obtained from the mean of posterior distribution as

$$\begin{aligned}\tilde{\alpha}_{BS} &= E(\alpha|\underline{\mathbf{y}})\\ &= A^{-1}2^{q-3/2}\Gamma\left(q-\frac{1}{2}\right)\prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n} T_{\underline{V}}(m)\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2 + \sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+\nu_{kj}-i_{kj})y_{sj}^2+a\right)^{-q+1/2}.\end{aligned} \tag{22}$$

2.1.1 Bayes estimator under asymmetric loss function

In some cases, the use of symmetric loss function may be inappropriate as has been recognized by Basu and Ebrahimi [29]. So, we consider asymmetric loss functions that can lead to forecasting procedures far more sensitive to the real consequences of forecasting errors. A number of asymmetric loss functions have been shown to be functional, see Chandra [30], Varian [31], Zellner [32] etc. More specifically, we use a new loss function introduced by Al-Bayyati [33] of the form

$$L_A(\tilde{\alpha},\alpha)=\alpha^c(\tilde{\alpha}-\alpha)^2,\quad c\in R.$$

This loss function is frequently used because of its analytical tractability in Bayesian analysis. Under the ALF, we obtained the following risk function:

$$R(\tilde{\alpha})=\int_0^\infty \alpha^c(\tilde{\alpha}-\alpha)^2\pi_1^*(\alpha|\mathbf{y})d\alpha. \tag{23}$$

Hence, the Bayes estimator $\tilde{\alpha}_{BL}$ of α can be obtain by solving the following equation: $\partial R(\tilde{\alpha})/\partial\tilde{\alpha}=0$. An alternative formula of ALF for obtaining the Bayes estimator of α is

$$\tilde{\alpha}_{BL}=\frac{E\left(\alpha^{c+1}\right)}{E\left(\alpha^c\right)}=\frac{1}{\sqrt{2\pi}}B(q-(c+1)/2,1/2)\times\frac{\phi_{r,s,m}(c+1)}{\phi_{r,s,m}(c)}, \tag{24}$$

where

$$\begin{aligned}\phi_{r,s,m}(c)=&\prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n}T_{\underline{V}}(m)\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2+\sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2\right.\\&\left.+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+v_{kj}-i_{kj})y_{sj}^2+a\right)^{-(q-c/2)}.\end{aligned} \tag{25}$$

2.2 Two sample Bayesian prediction

In this subsection, we present the Bayesian predictive distribution for the future order statistics based on the observed m-cycle ORSS in the two sample case.

To predict the future random sample $\mathbf{z}=(z_1,z_2,\cdots,z_w)$ of size w based on the given observed sample $\underline{\mathbf{y}}$ from the same population. Then from (1) and (8), the density function of z_ι, $\iota=1,2,...,w$ is given by

$$f_{\iota:w}(z_\iota|\alpha)=\sum_{k_1=0}^{\iota-1}\widetilde{c}_{k_1,\iota}(w)\frac{z_\iota}{\alpha^2}\exp\left(-\frac{1}{2\alpha^2}(w+k_1-\iota+1)z_\iota^2\right). \tag{26}$$

By using Eqs.(20) and (26), the predictive density function of z_ι is given by

$$\begin{aligned}f(z_\iota|\underline{\mathbf{y}})&=\int_0^\infty f_{\iota:w}(z_\iota|\alpha)\pi^*(\alpha|\underline{\mathbf{y}})\mathrm{d}\alpha\\&=\frac{2^q}{A}\Gamma(q+1)\sum_{k_1=0}^{i-1}\widetilde{c}_{k_1,i}(w)z_i\prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n}T_{\underline{V}}(m)\\&\times\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2+\sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2\right.\\&\left.+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+v_{kj}-i_{kj})y_{sj}^2+a+(w+k_1-i+1)z_i^2\right)^{-(q+1)}.\end{aligned} \tag{27}$$

Hence, the predictive survival function is given by

$$\begin{aligned}
P[Z_\iota > \varepsilon|\underline{\mathbf{y}}] &= \int_{\varepsilon}^{\infty} f(z_\iota|\underline{\mathbf{y}})\,\mathrm{d}z_\iota \\
&= \frac{2^{q-1}}{A}\Gamma(q)\sum_{k_1=0}^{i-1}\widetilde{c}_{k_1,i}(w)\prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n}\frac{T_{\underline{V}}(m)}{w+k_1-\iota+1} \\
&\times\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2+\sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2\right. \\
&\left.+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+\nu_{kj}-i_{kj})y_{sj}^2+a+(w+k_1-i+1)\varepsilon^2\right)^{-q} \\
&= \begin{cases}(1+\tau)/2, \\ (1-\tau)/2.\end{cases}
\end{aligned} \tag{28}$$

So the lower and upper $100\tau\%$ prediction bounds for z_ι can be obtained by solving Eq.(28) with respect to ε. From Eq.(27), the predictive estimator of z_ι, $\iota = 1,2,...,w$ can be obtained as

$$\begin{aligned}
\tilde{z}_i &= E(z_i|\underline{\mathbf{y}}) = \int_{0}^{\infty} z_\iota f(z_\iota|\underline{\mathbf{y}})\mathrm{d}z_i \\
&= \frac{2^{q-2}}{A}\sqrt{\pi}\Gamma\left(q-\frac{1}{2}\right)\sum_{k_1=0}^{i-1}\widetilde{c}_{k_1,i}(w)\prod_{j=1}^{m}\sum_{\mathbf{P[j]}}\sum_{\underline{V}}^{r,s,n}\frac{T_{\underline{V}}(m)}{(w+k_1-\iota+1)^{3/2}} \\
&\times\left(\sum_{j=1}^{m}\sum_{k=1}^{r-1}(n+\tau_{kj}-t_{kj})y_{rj}^2+\sum_{j=1}^{m}\sum_{k=r}^{s}(n+\nu_{kj}-i_{kj}+1)y_{kj}^2\right. \\
&\left.+\sum_{j=1}^{m}\sum_{k=s+1}^{n}(n+\nu_{kj}-i_{kj})y_{sj}^2+a\right)^{-(q-1/2)}.
\end{aligned} \tag{29}$$

3 Bayes estimation and prediction based on SRS

3.1 Bayes estimation

Suppose that $\mathbf{x}_j = (x_{rj}, x_{(r+1)j}, \cdots, x_{sj})$, where $x_{rj} \le x_{(r+1)j} \le \cdots \le x_{sj}$, $j = 1,2,\cdots,m$ be independent m sets of Type-II doubly censored data of order statistics each of size $s-r+1$. From Eq.(7) we get the joint density of the SRS, due to the independence of $\mathbf{x}_j$, as follows

$$\begin{aligned}
L_3(\alpha|\underline{\mathbf{x}}) &= \prod_{j=1}^{m} f_X(\mathbf{x}_j) \\
&= \left(\frac{n!}{(r-1)!(n-s)!}\right)^m \prod_{j=1}^{m}\left((F(x_{rj}))^{r-1}(1-F(x_{sj}))^{n-s}\prod_{i=r}^{s} f(x_{ij})\right),
\end{aligned} \tag{30}$$

where $\underline{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m)$.

Substituting from (8) into (30), we obtain the LF as follows:

$$\begin{aligned}
L_3(\alpha|\underline{\mathbf{x}}) &= \left(\frac{n!}{(r-1)!(n-s)!}\right)^m \prod_{j=1}^{m}\sum_{\kappa_j=0}^{r-1} c_{r-1,\kappa_j}\left(\prod_{j=1}^{m}\prod_{i=r}^{s}\frac{x_{ij}}{\alpha^2}\right) \\
&\times \exp\left[\frac{-1}{2\alpha^2}\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2+(n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2\right)\right],
\end{aligned} \tag{31}$$

where $c_{r-1,\kappa_j} = \binom{r-1}{\kappa_j}(-1)^{\kappa_j}$.

From Eqs.(19) and (31), the posterior density function based on SRS can be written as

$$\pi^*(\alpha|\underline{\mathbf{x}}) = \Im_{r,s}^{-1}(b)\prod_{j=1}^{m}\sum_{\kappa_j=0}^{r-1}\frac{c_{r-1,\kappa_j}}{\alpha^{2q+1}}\exp\left[\frac{-1}{2\alpha^2}\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2 + (n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2+a\right)\right], \quad (32)$$

where

$$\Im_{r,s}(b) = 2^{q-1}\Gamma(q)\prod_{j=1}^{m}\sum_{\kappa_j=0}^{r-1}c_{r-1,\kappa_j}\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2+(n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2+a\right)^{-q}.$$

Hence, the Bayes estimate of α under the SEL function is given by

$$\tilde{\alpha}_{BS} = \Im_{r,s}^{-1}(b)2^{q-3/2}\Gamma\left(q-\frac{1}{2}\right)\prod_{j=1}^{m}\sum_{\kappa_j=0}^{r-1}c_{r-1,\kappa_j} \times\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2+(n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2+a\right)^{-(q-\frac{1}{2})}. \quad (33)$$

Under the ALF, the Bayesian estimates of α is given by

$$\tilde{\alpha}_{BL} = \frac{1}{\sqrt{2\pi}}B(q-(c+1)/2,1/2)\times\frac{\xi_{r,s,m}(c+1)}{\xi_{r,s,m}(c)}, \quad (34)$$

where

$$\xi_{r,s,m}(c) = \prod_{j=1}^{m}\sum_{\kappa_j=0}^{r-1}c_{r-1,\kappa_j}\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2+(n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2+a\right)^{-(q-\frac{c}{2})}. \quad (35)$$

3.2 Two sample Bayesian prediction

We can obtain the predictive survival function and the predictive density function of z_ι based on SRS by a similar method as in Subsection 2.2. By using Eqs. (26) and (32), the predictive density function of z_ι, $\iota = 1,2,...,w$ is given by

$$f(z_\iota|\underline{\mathbf{x}}) = \Im_{r,s}^{-1}(b)2^q\Gamma(q+1)\sum_{k_1=0}^{\iota-1}\widetilde{c}_{k_1,\iota}(w)z_\iota\prod_{j=1}^{m}\sum_{\kappa_j}^{r-1}c_{r-1,\kappa_j} \times\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2+(n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2+a+(w-\iota+k_1+1)z_\iota^2\right)^{-(q+1)}. \quad (36)$$

Hence, the predictive survival function is given by

$$P[Z_\iota>\varepsilon|\underline{\mathbf{x}}] = \Im_{r,s}^{-1}(b)2^{q-1}\Gamma(q)\sum_{k_1=0}^{\iota-1}\widetilde{c}_{k_1,\iota}(w)\prod_{j=1}^{m}\sum_{\kappa_j}^{r-1}\frac{c_{r-1,\kappa_j}}{w-\iota+k_1+1} \times\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2+(n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2+a+(w-\iota+k_1+1)\varepsilon^2\right)^{-q} = \begin{cases}(1+\tau)/2,\\(1-\tau)/2.\end{cases} \quad (37)$$

Consequently, the lower and upper $100\tau\%$ prediction bounds for z_ι can be obtained by solving Eq.(37) with respect to ε. From Eq.(36), the predictive estimator of z_ι can be obtained as

$$\tilde{z}_\iota = \Im_{r,s}^{-1}(b)2^{q-2}\Gamma\left(q-\frac{1}{2}\right)\sum_{k_1=0}^{\iota-1}\widetilde{c}_{k_1,\iota}(w)\prod_{j=1}^{m}\sum_{\kappa_j}^{r-1}\frac{c_{r-1,\kappa_j}}{(w-\iota+k_1+1)^{3/2}} \times\left(\sum_{j=1}^{m}\kappa_j x_{rj}^2+(n-s)\sum_{j=1}^{m}x_{sj}^2+\sum_{j=1}^{m}\sum_{i=r}^{s}x_{ij}^2+a\right)^{-(q-1/2)}. \quad (38)$$

Remark: Setting $m = 1$ in Sections 2 and 3, we have the Bayesian estimation and prediction of the Ray(α) based on one cycle ORSS (or SRS).

4 Illustrative example

In the following we give two examples to illustrate the theoretical results.

Example 1. In equation (19) we select the parameter values $a = 0.5$ and $b = 2$. By using the prior density function $\pi(\alpha;\delta)$ we generate $\alpha = 0.6269$. Based on the generated value of α, generate n random samples each of size n from the Ray(α), using the transformation $X_i = \sqrt{2\alpha^2 \ln\left(\frac{1}{1-U_i}\right)}$ where U_i from $U(0,1)$ for $i = 1, \cdots, n$. Using a one-cycle RSS procedure, we obtained Type-II doubly censored sample of size $n = 5, 7$ from the Ray(0.6269). The different Bayes estimates of α are computed through equations (22), (24), (33) and (34) based on ORSS and SRS. We repeat this process 1,000 times to obtain MSE and bias for each method was calculated, where $\text{MSE}(\alpha) = \sum_{i=1}^{1000}(\tilde{\alpha}_i - \alpha)^2/1000$, $\tilde{\alpha}_i$ is the estimator of α for the i-th simulated data and $\tilde{\alpha}_{\text{Bias}} = (\bar{\tilde{\alpha}} - \alpha)$, $\bar{\tilde{\alpha}}$ is the average of the 1000 estimates of α. The bias and MSE of all the estimates are displayed in Tables 1 and 2, respectively. Using our results in equations (28), (29), (37) and (38), the 95% Bayesian prediction bounds and the Bayes predictive estimate of Z_ι, $\iota = 1, \cdots, w$ are obtained and displayed in Tables 4 and 5 for SRS and ORSS when $m = 1, 2$.

Table 1: MSE of the Bayesian estimates for α based on SRS and ORSS when $m = 1, 2$.

				SRS			ORSS		
				$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{BS}$	$(\cdot)_{BL}$	
m	n	r	s		$c=1$	$c=-0.1$		$c=1$	$c=-0.1$
1	5	2	3	0.02168	0.02206	0.02167	0.01801	0.01767	0.01805
		2	4	0.01722	0.01723	0.01721	0.01060	0.01054	0.01067
		3	4	0.01756	0.01759	0.01756	0.01283	0.01276	0.01284
	7	2	5	0.01496	0.01506	0.01496	0.00744	0.07360	0.00745
			6	0.01242	0.01253	0.01242	0.00503	0.00501	0.00503
		3	5	0.01519	0.01520	0.01519	0.00859	0.00859	0.00860
			6	0.01313	0.01315	0.01313	0.00665	0.00661	0.00665
2	5	2	3	0.01239	0.01237	0.01240	0.00997	0.00982	0.00999
		2	4	0.01029	0.01029	0.0129	0.00601	0.00596	0.00602
		3	4	0.01088	0.01089	0.01088	0.00766	0.00761	0.00766
	7	2	5	0.00873	0.00872	0.00873	0.00429	0.00426	0.00429
			6	0.00716	0.00716	0.00716	0.00252	0.00251	0.00252
		3	5	0.00894	0.00896	0.00896	0.00483	0.00478	0.00483
			6	0.00740	0.00739	0.00741	0.00308	0.00307	0.00309

Table 2: Bias of the Bayesian estimates for α based on SRS and ORSS when $m = 1, 2$.

				SRS			ORSS		
				$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{BS}$	$(\cdot)_{BL}$	
m	n	r	s		$c=1$	$c=-0.1$		$c=1$	$c=-0.1$
1	5	2	3	-0.07138	-0.08120	-0.07128	-0.02132	-0.02745	-0.02068
		2	4	-0.04920	-0.05760	-0.04830	-0.02260	-0.02690	-0.02210
		3	4	-0.04790	-0.05659	-0.04697	-0.02873	-0.03347	-0.02824
	7	2	5	-0.04582	-0.05294	-0.04507	-0.00724	-0.01005	-0.00695
			6	-0.04087	-0.04707	-0.04023	-0.01087	-0.01316	-0.01064
		3	5	-0.04734	-0.04771	-0.00469	-0.01193	-0.01209	-0.01180
			6	-0.03707	-0.04341	-0.03641	-0.01251	-0.01492	-0.01227
2	5	2	3	-0.03188	-0.03831	-0.03121	-0.00571	-0.00915	-0.00535
		2	4	-0.02870	-0.03370	-0.02818	-0.00707	-0.00942	-0.00683
		3	4	-0.03059	-0.03571	-0.03006	-0.01427	-0.01688	-0.01401
	7	2	5	-0.02596	-0.03011	-0.02554	-0.00124	-0.00275	-0.00109
			6	-0.00203	-0.02389	-0.01996	-0.00197	-0.00317	-0.00185
		3	5	-0.02297	-0.02719	-0.02254	0.00037	-0.00126	0.00053
			6	-0.01966	-0.02326	-0.01929	-0.00401	-0.00528	-0.00389

Example 2. Hand et al. ([34], p. 107) presented data on the the weights of 13 mice weighed every 3 days after birth, see Rissanen [35]. There were 91 observations recorded.

0.10900	0.38800	0.62100	0.82300	1.07800	1.13200	1.19100	0.21800	0.39300
0.56800	0.72900	0.83900	0.85200	1.00400	0.21100	0.39400	0.54900	0.70000
0.78300	0.87000	0.92500	0.20900	0.41900	0.64500	0.85000	1.00100	1.02600
1.06900	0.19300	0.36200	0.52000	0.53000	0.64100	0.64000	0.75100	0.20100
0.36100	0.50200	0.53000	0.65700	0.76200	0.88800	0.20200	0.37000	0.49800
0.65000	0.79500	0.85800	0.91000	0.19000	0.35000	0.51000	0.66600	0.81900
0.87900	0.92900	0.21900	0.39900	0.57800	0.69900	0.70900	0.82200	0.95300
0.25500	0.40000	0.54500	0.69000	0.79600	0.82500	0.83600	0.22400	0.38100
0.57700	0.75600	0.86900	0.93900	0.99900	0.18700	0.32900	0.44100	0.52500
0.58900	0.62100	0.79600	0.27800	0.47100	0.60600	0.77000	0.88800	1.00100
1.10500								

To check whether the Ray(α) is suitable for this data, Cramér von Mises test is used to test the null hypothesis $H_0 : F(x) = \text{Ray}(\alpha)$, vs $H_1 : F(x) \neq \text{Ray}(\alpha)$. H_0 is rejected at a significance level of $\gamma = 0.05$ if probability value $p-\text{value} < \gamma$. The Cramer von Mises test statistic is 0.10089 with an associated $p-\text{value} = 0.10822 > 0.05$, so Ray$(\alpha)$ is fitted to the above real data set.

The procedure of a one- and two- cycle RSS can be described in Table 6. Hence, SRS and RSS for the cases when $m = 1$ and 2 are displayed in Table 7. Now we consider the case when the data are Type-II doubly censored. It is assumed that the experimenter failed to observe the two smallest failure times and the experiment was terminated at the time of the s-th failure. Hence $n = 5,6$, $r = 2,3$ and $s = 3,4,5$. So, the Bayes estimates based on SEL and ALF of α are computed and the results are presented in Table 8. The 95% two-sample Bayesian prediction intervals and Bayesian predictive estimators for $z_\imath$, $\imath = 1,\cdots,w$ $(w = s-r+1)$ based on ORSS and SRS for one and two-cycle are obtained and displayed in Tables 9 and 10.

Table 3: The data of RSS and SRS for one- and two- cycle.

m	n	Samples						
				RSS				
1	5	0.46693	0.34230	0.54901	0.34969	0.92832		
	7	0.21614	0.23416	0.33515	0.32707	0.66766	0.55453	0.86097
2	5	0.14326	0.29492	0.53543	0.5335	0.70084		
	7	0.39783	0.23927	0.43625	0.46539	0.59002	0.77086	0.80819
				SRS				
1	5	0.46693	0.50810	0.52832	0.57139	0.61249		
	7	0.21614	0.25264	0.32014	0.46005	0.47078	0.57222	0.85671
2	5	0.14326	0.22356	0.55768	0.73413	0.87065		
	7	0.39783	0.48695	0.50682	0.58426	0.62940	0.72101	1.28801

Table 4: The Bayesian prediction bounds for z_ι and $\iota = 1, \cdots, w$ based on SRS and ORSS for one cycle.

				SRS				ORSS			
n	r	s	ι	$\tilde{z}_\iota$	Lower	Upper	Width	$\tilde{z}_\iota$	Lower	Upper	Width
5	2	3	1	0.36827	0.05976	0.92369	0.86393	0.38063	0.06513	0.88597	0.82084
			2	0.67965	0.21110	1.47842	1.26732	0.69595	0.23485	1.39231	1.15746
	2	4	1	0.21675	0.03593	0.52758	0.49165	0.22814	0.03939	0.52361	0.48422
			2	0.36289	0.11877	0.75873	0.63996	0.38196	0.13238	0.73972	0.60734
			3	0.54663	0.21683	1.09223	0.87541	0.57536	0.24412	1.05857	0.81445
	3	4	1	0.27267	0.04489	0.66875	0.62386	0.27985	0.04766	0.65378	0.60612
			2	0.49855	0.15928	1.06404	0.90476	0.51169	0.17133	1.02719	0.85587
7	2	5	1	0.25111	0.04224	0.59868	0.55644	0.24202	0.04256	0.54086	0.49830
			2	0.40650	0.13630	0.82469	0.68839	0.39178	0.14004	0.72931	0.58927
			3	0.56442	0.23439	1.07607	0.84168	0.54398	0.24403	0.94097	0.69694
			4	0.78686	0.35432	1.47718	1.12286	0.75837	0.37197	1.29021	0.91824
	2	6	1	0.18824	0.03198	0.44252	0.41054	0.19037	0.03357	0.42370	0.39013
			2	0.29934	0.10191	0.59527	0.49336	0.30273	0.10881	0.55932	0.45051
			3	0.40273	0.17111	0.74769	0.57659	0.40729	0.18484	0.69475	0.50991
			4	0.51993	0.24476	0.93428	0.68953	0.52582	0.26661	0.86299	0.59638
			5	0.69437	0.33895	1.24723	0.90828	0.70224	0.37104	1.15459	0.78355
	3	5	1	0.29196	0.04910	0.69625	0.64715	0.28366	0.04984	0.63474	0.58490
			2	0.48880	0.16323	0.99543	0.83220	0.47492	0.16889	0.88944	0.72055
			3	0.73629	0.29899	1.43018	1.13119	0.71538	0.31318	1.26959	0.95641
	3	6	1	0.21201	0.03600	0.49878	0.46258	0.21500	0.03787	0.47917	0.44130
			2	0.34320	0.11661	0.68385	0.56724	0.34804	0.12472	0.64543	0.52071
			3	0.47652	0.20103	0.89034	0.68932	0.48325	0.21744	0.83223	0.61479
			4	0.66432	0.30433	1.22182	0.91750	0.67371	0.33155	1.14105	0.80950

Table 5: The Bayesian prediction bounds for z_ι and $\iota = 1, \cdots, w$ based on SRS and ORSS for two cycles.

				SRS				ORSS			
n	r	s	ι	$\tilde{z}_\iota$	Lower	Upper	Width	$\tilde{z}_\iota$	Lower	Upper	Width
5	2	3	1	0.36435	0.06188	0.85661	0.79472	0.37102	0.06499	0.83415	0.76916
			2	0.66618	0.22239	1.34928	1.12689	0.67838	0.23676	1.29838	1.06162
	2	4	1	0.29879	0.05138	0.68963	0.63825	0.24765	0.04363	0.55176	0.50813
			2	0.50024	0.17248	0.97621	0.80373	0.41462	0.14807	0.77203	0.62396
			3	0.75351	0.31779	1.39789	1.08010	0.62455	0.27483	1.10146	0.82663
	3	4	1	0.36945	0.06348	0.85383	0.79035	0.31497	0.05521	0.70681	0.65160
			2	0.67551	0.22925	1.33876	1.10951	0.57589	0.23098	1.09931	0.86833
7	2	5	1	0.24947	0.04326	0.56909	0.52583	0.24615	0.04372	0.54200	0.49828
			2	0.40385	0.14141	0.77283	0.63141	0.39849	0.14450	0.72692	0.58242
			3	0.56074	0.24528	1.00094	0.75565	0.55329	0.25288	0.93508	0.68220
			4	0.78173	0.37138	1.36694	0.99556	0.77135	0.38641	1.28202	0.89561
	2	6	1	0.20999	0.03662	0.47511	0.43849	0.21162	0.03765	0.46476	0.42711
			2	0.33392	0.11807	0.63093	0.51286	0.33651	0.12260	0.61053	0.48793
			3	0.44927	0.19981	0.78655	0.58674	0.45274	0.20894	0.75602	0.54708
			4	0.58000	0.28741	0.97893	0.69152	0.58449	0.30220	0.93756	0.63536
			5	0.77461	0.39930	1.30855	0.90925	0.78059	0.42125	1.25548	0.83424
	3	5	1	0.29174	0.05059	0.66558	0.61499	0.29167	0.05177	0.64299	0.59122
			2	0.48843	0.17037	0.93864	0.76827	0.48833	0.17630	0.89642	0.72011
			3	0.73573	0.31462	1.34253	1.02791	0.73557	0.32806	1.27754	0.94948
	3	6	1	0.23838	0.04156	0.53950	0.49794	0.24151	0.04295	0.53089	0.48794
			2	0.38589	0.13620	0.73066	0.59446	0.39098	0.14205	0.71155	0.56950
			3	0.53580	0.23663	0.94494	0.70831	0.54286	0.24861	0.91495	0.66634
			4	0.74697	0.36000	1.29582	0.93582	0.75681	0.37999	1.25442	0.87444

Table 6: A ranked set sample design with sample size $n = 5,6$ when $m = 1,2$.

	$m=1$						$m=2$						
n	*RSS*						*RSS*						
5	0.10900	0.38800	0.62100	0.82300	1.07800		0.19300	0.36200	1.00100	1.02600	1.06900		
	0.21800	0.39300	0.56800	1.13200	1.19100		0.52000	0.53000	0.64000	0.64100	0.75100		
	0.21100	0.72900	0.83900	0.85200	1.00400		0.20100	0.36100	0.50200	0.53000	0.65700		
	0.39400	0.54900	0.70000	0.78300	0.87000		0.20200	0.37000	0.49800	0.76200	0.88800		
	0.20900	0.41900	0.64500	0.85000	0.92500		0.19000	0.65000	0.79500	0.85800	0.91000		
6	0.10900	0.38800	0.62100	0.82300	1.07800	1.13200		0.36100	0.50200	0.53000	0.65700	0.76200	0.88800
	0.21800	0.39300	0.56800	0.72900	0.83900	1.19100		0.20200	0.37000	0.49800	0.65000	0.79500	0.85800
	0.21100	0.39400	0.54900	0.70000	0.85200	1.00400		0.19000	0.35000	0.51000	0.66600	0.81900	0.91000
	0.20900	0.41900	0.64500	0.78300	0.87000	0.92500		0.21900	0.39900	0.57800	0.69900	0.87900	0.92900
	0.19300	0.36200	0.85000	1.00100	1.02600	1.06900		0.25500	0.40000	0.54500	0.70900	0.82200	0.95300
	0.20100	0.52000	0.53000	0.64000	0.64100	0.75100		0.22400	0.38100	0.69000	0.79600	0.82500	0.83600

Table 7: The data of RSS and SRS for one- and two- cycle.

m	n	Samples					
		RSS					
1	5	0.10900	0.3930	0.8390	0.78300	0.92500	
	6	0.10900	0.3930	0.5490	0.78300	1.02600	0.75100
2	5	0.19300	0.5300	0.5020	0.76200	0.91000	
	6	0.36100	0.3700	0.5100	0.69900	0.82200	0.83600
		SRS					
1	5	0.10900	0.3880	0.62100	0.82300	1.07800	
	6	0.10900	0.3880	0.62100	0.82300	1.07800	1.13200
2	5	0.19300	0.3620	1.00100	1.02600	1.06900	
	6	0.36100	0.5020	0.53000	0.65700	0.76200	0.88800

Table 8: Bayesian estimates for α based on SRS and ORSS when $m = 1,2$.

				SRS			ORSS		
				$(\cdot)_{BS}$	$(\cdot)_{BL}$		$(\cdot)_{BS}$	$(\cdot)_{BL}$	
m	n	r	s		$c=1$	$c=-0.1$		$c=1$	$c=-0.1$
1	5	2	3	0.65925	0.64626	0.65939	0.47014	0.46630	0.47054
		2	4	0.60137	0.59259	0.60230	0.48719	0.48447	0.48747
		3	4	0.64553	0.63504	0.64664	0.45156	0.44913	0.45181
	6	2	4	0.54652	0.53923	0.54729	0.53995	0.53703	0.54024
			5	0.57316	0.56646	0.57386	0.55934	0.55692	0.55959
		3	4	0.55657	0.54894	0.55738	0.54537	0.54218	0.54570
			5	0.58154	0.57459	0.58227	0.56457	0.56197	0.56483
2	5	2	3	0.59172	0.58558	0.59236	0.55039	0.54782	0.55065
		2	4	0.55809	0.55382	0.55853	0.51124	0.50970	0.51140
		3	4	0.59001	0.58506	0.59052	0.51046	0.50867	0.51064
	6	2	4	0.52407	0.52029	0.52446	0.52062	0.51920	0.52077
			5	0.53122	0.52804	0.53155	0.52727	0.52615	0.52738
		3	4	0.52414	0.52032	0.52454	0.52609	0.52453	0.52625
			5	0.53139	0.52818	0.53172	0.53189	0.53068	0.53201

5 Conclusion

We use ORSS based on Type-II doubly censored to improve Bayesian estimation for the parameter of the Ray(α) and Bayesian prediction intervals. The Bayes estimators are obtained using both squared error loss and Al-Bayyati loss functions. The MSEs and bias of the derived Bayesian estimates are computed. To illustrate the use of Bayesian estimation and prediction, simulation study and real data are analyzed. From the results in Section 4, we observed that the Bayesian estimates based on ORSS have a much smaller (MSE) than the corresponding Bayesian estimates based on SRS, as well as the width of the Bayesian prediction intervals based on ORSS better than of SRS. This clearly demonstrates that the ORSS is more efficient than the SRS for all cases considered in this study. It clear that the lengths of the prediction intervals are increasing with increasing ι and decrease with increasing n and $s-r+1$ for SRS and ORSS in two cases $m = 1$ and $m = 2$. Also, the Bayesian estimates and the Bayesian prediction intervals based on both SRS and ORSS with two-cycle ($m = 2$) are better than the Bayesian estimates and the Bayesian prediction intervals with one-cycle ($m = 1$). In general, the better results are obtained by use of a large number of cycles. Moreover, the Bayes estimators under the ALF are better than their corresponding the estimators under the SEL function. This is one of the useful properties of working with the ALF.

Table 9: The Bayesian prediction bounds for z_ι and $\iota = 1, \cdots, w$ based on SRS and ORSS for one cycle.

				SRS				ORSS			
n	r	s	ι	$\tilde{z}_\iota$	Lower	Upper	Width	$\tilde{z}_\iota$	Lower	Upper	Width
5	2	3	1	0.62581	0.10158	1.56922	1.46763	0.41619	0.07078	0.97715	0.90637
			2	0.85691	0.23769	1.86467	1.62700	0.76097	0.25453	1.53884	1.28431
	2	4	1	0.45399	0.07531	1.10423	1.02893	0.35338	0.06124	0.80668	0.74543
			2	0.76009	0.24901	1.58781	1.33879	0.59164	0.20620	1.13775	0.93156
			3	1.14493	0.45469	2.28564	1.83096	0.89120	0.38070	1.62736	1.24666
	3	4	1	0.60321	0.09969	1.47317	1.37348	0.39780	0.06882	0.91007	0.84125
			2	0.80771	0.35432	2.34220	1.98787	0.72736	0.24919	1.42284	1.17365
6	2	4	1	0.40670	0.06746	0.98926	0.92180	0.39458	0.06871	0.89481	0.82610
			2	0.68091	0.22306	1.42250	1.19944	0.66062	0.23190	1.25963	1.02773
			3	1.02566	0.40729	2.04769	1.64040	0.99510	0.42885	1.80066	1.37181
	2	5	1	0.36976	0.06220	0.88157	0.81937	0.35371	0.06202	0.79368	0.73165
			2	0.59856	0.20069	1.21437	1.01368	0.57258	0.20379	1.07166	0.86786
			3	0.83109	0.34512	1.58454	1.23942	0.79502	0.35474	1.38366	1.02892
			4	1.15864	0.52170	2.17518	1.65348	1.10835	0.54034	1.89724	1.35690
	3	4	1	0.50888	0.08432	1.23924	1.15492	0.48876	0.08491	1.11210	1.02720
			2	0.93045	0.30006	1.96928	1.66923	0.89367	0.30811	1.73658	1.42847
	3	5	1	0.43404	0.07297	1.03553	0.96256	0.41259	0.07226	0.92734	0.85508
			2	0.72669	0.24253	1.48063	1.23810	0.69077	0.24448	1.30126	1.05678
			3	1.09461	0.44420	2.12735	1.68315	1.04051	0.45284	1.85818	1.40534

Table 10: The Bayesian prediction bounds for z_ι and $\iota = 1, \cdots, w$ based on SRS and ORSS for two cycle.

				SRS				ORSS			
n	r	s	ι	$\tilde{z}_\iota$	Lower	Upper	Width	$\tilde{z}_\iota$	Lower	Upper	Width
5	2	3	1	0.53915	0.09158	1.26745	1.17587	0.49232	0.08614	1.10861	1.02247
			2	0.98580	0.32914	1.99639	1.66725	0.90016	0.31362	1.72625	1.41263
	2	4	1	0.41056	0.07064	0.94739	0.87676	0.37119	0.06547	0.82594	0.76047
			2	0.68738	0.23709	1.34096	1.10387	0.62146	0.22226	1.15516	0.93290
			3	1.03541	0.43686	1.92024	1.48338	0.93612	0.41266	1.64787	1.23521
	3	4	1	0.53433	0.09177	1.23578	1.14401	0.45410	0.07985	1.01472	0.93487
			2	0.97698	0.33130	1.93793	1.60663	0.83029	0.29137	1.57645	1.28508
6	2	4	1	0.38335	0.06595	0.88468	0.81873	0.37812	0.06683	0.83881	0.77197
			2	0.64181	0.22134	1.25222	1.03087	0.63307	0.22714	1.17199	0.94485
			3	0.96677	0.40784	1.79318	1.38534	0.95360	0.42205	1.67140	1.24935
	2	5	1	0.33617	0.05830	0.76676	0.70846	0.33145	0.05880	0.73128	0.67248
			2	0.54419	0.19059	1.04123	0.85064	0.53656	0.19429	0.98147	0.78719
			3	0.75560	0.33058	1.34854	1.01796	0.74500	0.33959	1.26301	0.92342
			4	1.05339	0.50237	1.84950	1.34713	1.03862	0.51871	1.73164	1.21293
	3	4	1	0.46960	0.08075	1.08445	1.00370	0.46828	0.08266	1.04076	0.95810
			2	0.85863	0.29168	1.70006	1.40838	0.85622	0.30220	1.61459	1.31239
	3	5	1	0.38833	0.06732	0.88610	0.81878	0.38624	0.06847	0.85300	0.78453
			2	0.65015	0.22673	1.24968	1.02295	0.64665	0.23301	1.18991	0.95690
			3	0.97933	0.41869	1.78743	1.36875	0.97406	0.43337	1.69609	1.26272

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References

[1] Z. Chen, Z. Bai, B.K Sinha, Ranked set sampling: Theory and Application, *Springer, New York* 176 *2004.*

[2] *T.R Dell, J.L Clutter, Ranked set sampling with order statistics background, Biometrics **28**, 545–555 (1972).*

[3] *A.S Hassan, Maximum likelihood and Bayes estimators of the unknown parameters for exponentiated exponential distribution using ranked set sampling, International Journal of Engineering Research and Applications **3**, 720-725 (2013).*

[4] *A. Helu, M. Abu-salih, O. Alkam, Bayes estimation of Weibull distribution parameters using ranked set sampling, Communication in Statistic Theory and Methods **39**, 2533-2551 (2010).*

[5] *G.A McIntyre, A method for unbiased selective sampling using ranked sets, Australian Journal of Agricultural Research **3**, 385-390 (1952).*

[6] *M.M Mohie El-Din, E.F Abd-Elfattah, M.S Kotb, H.A Newer, Bayesian inference and prediction of the Pareto distribution based on ordered ranked set sampling, Communication in Statistic Theory and Methods 1-17 (2017).*

[7] *H.A Muttlak, Median ranked set sampling, Journal of Applied Statistical Sciences **6**(4), 577-586 (1997).*

[8] *A. Sadek, K.S Sultan, N. Balakrishnan, Bayesian estimation based on ranked set sampling using asymmetric loss function, Bulletin of the Malaysian Mathematical Sciences Society **38**, 707-718 (2015).*

[9] *K. Takahasi, K. Wakimoto, On unbiased estimates of the population mean based on the sample stratified by means of ordering, Annals of Institute of Statistical Mathematics **20**, 1-31 (1968).*

[10] *N. Balakrishnan, T. Li, Ordered ranked set samples and applications to inference, Journal of Statistical Planning and Inference **138**, 3512–3524 (2008).*

[11] *B.C Arnold, N. Balakrishnan, H.N Nagaraja, A first course in order statistics, John Wiley and Sons, New York (1992).*

[12] *H.A David, H.N Nagaraja, Order statistics third edition Wiley, New York 2003.*

[13] *K.A Ariyawans, J.G.C Templton, Structural inference on the parameter of the Rayleigh distribution from doubly censored samples, Statistische Hefte **25**, 181–199 (1984).*

[14] *F. Domma, S. Giordano, M. Zenga, The Fisher information matrix on a Type-II doubly censored sample from a Dagum distribution, Applied Mathematical Sciences **7**(75), 3715–3729 (2013).*

[15] *A. Elfessi, Estimation of a linear function of the parameters of an exponential distribution from doubly censored samples, Statistics & Probability Letters **36**, 251–259 (1997).*

[16] *A. Pak, G.A Parham, M. Saraj, On estimation of Rayleigh scale parameter under doubly Type-II censoring from imprecise data, Journal of Data Science **11**, 305-322 (2013).*

[17] *A.J Fernéndez, Bayesian inference from Type-II doubly censored Rayleigh data, Statistics & Probability Letters **48**, 393–399 (2000).*

[18] *J. Rayleigh, On the resultant of a large number of vibrations of the some pitch and of arbitrary phase, Philosophical Magazine **10**, 73-78 (1980).*

[19] *J.M Bernardo, A.F.M Smith, Bayesian Theory, Wiley, New York 1994.*

[20] *C. Kim, S. Song, Bayesian estimation of the parameters of the generalized exponential distribution from doubly censored samples, Statistical Papers **51**, 583-597 (2010).*

[21] *M.S Kotb, Bayesian inference and prediction for modified Weibull distribution under generalized order statistics, Journal of Statistics & Management Systems **17**(5 & 6), 547-578 (2014).*

[22] *M.M Mohie El-Din, M.S Kotb, H.A Newer, Bayesian estimation and prediction for Pareto distribution based on ranked set sampling, Journal of Statistics Applications & Probability **4(2)**, 1-11 (2015).*

[23] *A.M Nigm, E.K AL-Hussaini, Z.F Jaheen, Bayesian one-sample prediction of future observations under Pareto distribution, Statistics **37**(6), 527-536 (2003).*

[24] *B. Yu, A Bayesian MCMC approach to survival analysis with doubly-censored data, Computational Statistics and Data Analysis **54** , 1921-1929 (2010).*

[25] *S. Dey, T. Dey, Bayesian estimation and prediction intervals for a Rayleigh distribution under a conjugate prior, Journal of Statistical Computation and Simulation **82**(11), 1651–1660 (2012).*

[26] *M.S Kotb, Bayesian prediction bounds for the exponential-type distribution based on ordered ranked set sampling, Economic Quality Control **31**(1), 45-54 (2016).*

[27] *M.M Mohie El-Din, Y. Abdel-Aty, A.R Shafay, Two sample Bayesian prediction intervals for order statistics based on the inverse exponential-type distributions using right censored sample, Journal of the Egyptian Mathematics Society **19**, 102-105 (2011).*

[28] *N. Balakrishnan, Permanents, order statistics, outliers, and robustness, Revista Matemática Complutense **20**, 7–107 (2008).*

[29] *A.P Basu, N. Ebrahimi, Bayesian approach to life testing and reliability estimation using asymmetric loss function, Journal of Statistical Planning and Inference **29**, 21–31 (1991).*

[30] *M.J Chandra, Statistical quality control, CRC Press, Boca Raton (2001).*

[31] *H.R Varian, A Bayesian approach to real estate assessment, North Holland, Amsterdam 195-208 (1975).*

[32] *A. Zellner Bayesian estimation and prediction using asymmetric loss function, Journal of the American Statistical Association **81**, 446-451 (1986).*

[33] *H.N Al-Bayyati, Comparing methods of estimating Weibull failure models using simulation. Ph.D. Thesis, College of Administration and Economics, Baghdad University, Iraq (2002).*

[34] *D.J Hand, F. Daly, A.D Lunn, K.J McConway, E. Ostrowski, A Handbook of small data sets, Chapman and Hall (1994).*

[35] *J. Rissanen, Stochastic complexity in statistical enquiry, Singapore, World Scientific 1989.*