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Stress-based Finite Element Analysis of Sliding Beams

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Abstract: This paper developed a stress-based finite element method for solving a sliding beam problem. Conventionally, the displacement-based finite element method is usually in computational solid mechanics, but the method cannot satisfy the compatibility of inter-element stress fields and the stress boundary conditions. These problems yield the inaccuracy of approximated solutions. However, the stress-based finite element method can solve these problems. Besides, the numerical examples demonstrate that the accuracy of this method is higher than that of the displacement-based method based on the same number of degrees of freedom. The sliding beam problems have diverse engineering applications. The most challenging task for this problem is necessarily to construct a varying-length beam element. This paper presents a varying-length beam element for the stress-based finite element method. The dynamic simulations reveal that the results are in good agreement with those in literature.

Keywords: Stress-based finite element method, sliding beam, error analysis

1 Introduction

The displacement-based finite element method is mostly applied to computational solid mechanics. This method usually uses polynomials to approximate displacement fields, so the inter-element stresses are discontinuous while low-order elements are employed. Besides, the stress boundary conditions cannot be satisfied. These problems cause the inaccuracy of approximated solutions. In order to increase the accuracy, the h-version finite element method increases the number of elements, but this method also increases computational cost. In contrast to the displacement-based finite element method, a so-called stress-based finite element method approximates stress fields as assumed functions. This method can keep the inter-element stresses continuous and the stress boundary conditions satisfied, but it is difficult to construct simple stress functions for two- and three-dimensional solid mechanics problems.

The stress-based finite element method was first introduced by Veubeke and Zienkiewicz [1,2]. To keep equilibrium equations satisfied, Taylor and Zienkiewicz [3] used the penalty functions in the stress-based finite element method. To have simple stress functions, Gallagher et al. [4] utilized the Airy stress function to construct assumed stress functions. To satisfy boundary conditions, Vallabhan et al. [5] treated them as constraints and incorporated them with the Lagrange multipliers. In addition, some researchers developed the alternative stress-based finite element methods by incorporating stochastic perturbation approach and the boundary element method [6]. Also, some researches developed the so-called the hybrid method, which uses both assumed displacement and stress functions to obtain the benefits of the displacement- and stress-based finite element methods [7,8]. Huang et al. used the stress-based finite element method to investigate the seismic response analysis of the deep saturated soil deposits [9]. Kuss and Lebon used the stress-based finite element method to solve contact problems [10]. Kuo et al. developed the curvature-based finite element method for linkage problems including four bar mechanisms and slider-crank mechanisms [11,12].

Sliding beams can be applied to diverse mechanical problems, such as spacecraft antennas, telescope robotic manipulators, and high-speed magnetic drives. Tabarrok et al. [13,14] derived the equations of motion, which consist of continuity equations, momentum equations, and mass-tension relations. Then, they solved the equations of motion by the method of characteristics. Stylianou et al. [15] used the finite element method to develop time-varying elements and to study the dynamics and stability analysis of the flexible and extendible sliding beams under general configurations. Behdinan et al. [16] considered the geometrically nonlinearity in flexible sliding beams, which can be deployed or retrieved

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through a rigid channel. The formulations were based on an extended Hamiltons principle. Kuo et al. tried to develop a relative-error-based technique to obtain an optimal finite element mesh, and the technique was applied to the sliding beam problem [17]. Ibrahim et al. [18] investigated the efficiency of a sliding beam as a nonlinear vibration isolator. Two approaches can be considered in the finite element method to deal with sliding beams. One approach is to employ beam elements of fixed lengths. Thus, the number of elements increases or decreases as the motion of the beam varies. This approach is impossible to be practically implemented. The other approach is to use a fixed number of elements, but it is necessary to establish a beam element of variable space-domain beam. This paper will use the second approach to develop a variable spatial-domain beam element for the stress-based finite element method.

This paper presents a stress-based finite element method for sliding beams. The formulations of the stress-based finite element method is derived first, and then the varying-length beam element for this method is established. Numerical examples are presented to show the validity of the proposed approach. The organization of this paper is summarized as follows. Section 2 presents the formulation of the sliding beam problem. Section 3. introduces the formulation of the stress-based finite element method. Section 4 compares the displacement- and stress-based finite element methods. Section 5 demonstrates numerical examples, and the conclusions are summarized in Section 6.

2 Formulation of sliding beams

The finite element modeling of sliding beams is based on the Euler-Bernoulli beam theory. Since the beam length is changing with time, it is necessary to construct a variable-length beam element by utilizing the Liebnitzs rule. One considers a uniform but inextensible sliding beam with length L_0 (see Figure 1). The beam with a specified axially moving velocity $V(t)$ is clamped the rigid wall and vibrating in the inertial coordinate system $X-Y$. At time t, the length of the beam outside the rigid wall is $L(t)$, and the inside beam is assumed as rigid but still has an axial motion. Thus, the transverse displacement is represented by $v(X,t)$. Therefore, the Lagrangian of the sliding beam may be given as [14]

$$L_g = \int_0^{L(t)} [\frac{1}{2}\rho A(\frac{\partial v}{\partial t} + V\frac{\partial v}{\partial X})^2 - \frac{1}{2}EI(\frac{\partial^2 v}{\partial X^2})^2 dX] + \frac{1}{2}\rho A L_0 V^2 \quad (1)$$

where the three terms of the Lagrangian are the kinetic energy, the potential energy, and the longitudinal kinetic energy (a prescribed quantity), respectively.

To develop a variable-length beam element, the beam element defined on the $x-y$ coordinate system (see Figure 2). One defines Li as the displacement from the rigid wall to the left end of the beam element. Then, the axial velocity of any point on the beam element with respect to the beam coordinate system is written as

$$\dot{x} = \dot{X} - \dot{L}_i = (1 - \frac{L_i}{L})\dot{L} \quad (2)$$

Based on Equations (1) and (2), the Lagrangian of a beam element is

$$L_e = \int_0^{l(t)} \{\frac{1}{2}\rho A[v_t + (1 - \frac{L_i}{L})\dot{L}v_x]^2 - \frac{1}{2}EIv_{xx}^2\}dx + \frac{1}{2}\rho A l \dot{L}^2 \quad (3)$$

where l is the length of a variable-length beam element.

An approximated transverse displacement for the finite element method can be expressed as

$$v = N(x, l(t))\phi_e \quad (4)$$

where $N(x)$ is a row vector in terms of shape functions; ϕ_e is a column vector in terms of nodal values. It is noted that the beam length is a function of time, so the time derivative of the transverse displacement is given as

$$\dot{v} = \dot{N}(x,l)\phi_e + N(x,l)\dot{\phi}_e = N_l(x,l)\dot{l}\phi_e + N(x,l)\dot{\phi}_e \quad (5)$$

where $N_l(x,l)$ is a derivative with respect to the element length.

Applying the Lagrange's equation leads to the finite element equations as

$$M_e\ddot{\phi}_e + C_e\dot{\phi}_e + K_e\phi_e = F_e \quad (6)$$

where M_e, C_e and K_e are mass, equivalent damping and equivalent stiffness matrices of elements, F_e is a load vector of elements, and ϕ_e is a vector of element variables. A physical proportional damping matrix is taken into account, so the global system equations are obtained as

$$M\ddot{\phi} + C\dot{\phi} + K\phi = F \quad (7)$$

where M, C, K are global mass, equivalent damping and stiffness matrices, F is a global load vector, which is also a zero vector, and ϕ is a vector of global variables.

3 Stress-based finite element method

The bending stress of Euler-Bernoulli beams is proportional to the second derivative of the transverse displacement with respect to the axial position. Thus, the second derivative of the transverse displacement is approximated as polynomials and is expressed as the

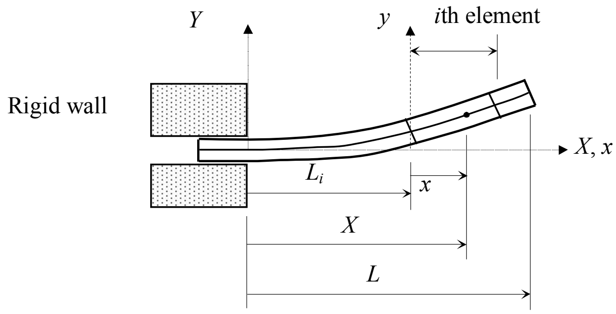

Fig. 1: A beam element coordinate system

product of shape functions $P(x,l)$ and nodal variables. Next, double integrating the approximated second derivative of displacement leads to the approximated transverse displacement. Thus, the approximated second derivative of displacement, the approximated first derivative of displacement, and the transverse displacement for the ith element can be written as

$$w^{(i)} = \left[v_{xx}\ v_x\ v\right]^T = A(x,l)S^{(i)}\phi + B(x)c^{(i)} \tag{8}$$

where ϕ is a column vector of the global variables, and each variable represents the curvature at each node; $S^{(i)}$ is a matrix, which transforms a set of nodal variables to a set of global variables; $c^{(i)}$ is a 2×1 matrix of two integration constants; $A(x,l)$ and $B(x)$ are respectively expressed as

$$A(x,l) = \left[P(x,l)\ \int P(x,l)dx\ \iint P(x,l)dxdx\right]^T \tag{9}$$

$$B(x) = \begin{bmatrix} 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}^T \tag{10}$$

Using the compatibility condition, Equation (8) can be rewritten as

$$w^{(i)} = N^{(i)}(x,l)\phi \tag{11}$$

where $N^{(i)}(x,l)$ are the shape functions expressed as

$$N^{(i)}(x,l) = A(x,l)S^{(i)}$$
$$+B(x)B^T(0)[N^{(i-1)}(l,l) - A(0,l)S^{(i)}] \tag{12}$$

It is noted that the shape functions are different for each element, and Equation (12) is a recursive equation, where $N(0)$ is obtained from the boundary condition

$$w(0) = N^{(0)}(0,l)\phi \tag{13}$$

Regarding to Equation (5), it is necessary to have the derivative of the shape functions with respect to the element length, which is given as

$$N_l^{(i)}(x,l) = A_l(x,l)S^{(i)}$$
$$+B(x)B^T(0)[N_l^{(i-1)}(l,l) - A_l(0,l)S^{(i)}] \tag{14}$$

Thus, it is convenient to obtain the finite element formulations by substituting Equations (14) to (5).

4 Comparisons of the displacement- and stress-based methods

The displacement-based finite element method approximates a displacement function as a polynomial, and then the approximated displacement is expressed as the product of the shape functions and the nodal variables. Thus, the nodal variables are the displacements at nodes. This method keeps the displacements at inter-element nodes continuous, but the stresses are discontinuous. Thus, the discontinuity produces the errors of approximated solutions. In addition, the displacement-based finite element method uses excessive nodal variables. An Euler-Bernoulli beam usually uses a cubic polynomial to approximate the transverse displacement, so there are four nodal variables. While performing a stiffness matrix, one needs only two nodal variables due to differentiation.

Table 1 compares the displacement- and stress-based finite element methods for a two-node Euler-Bernoulli beam element. Both methods use cubic polynomials to approximate displacements, but the stress-based finite element method produces only one-half of the number of nodal variables. To check the compatibility, the stress finite element method keeps displacements, the first and second derivatives of displacement continuous. To check the boundary conditions, the stress finite element method can satisfy the boundary conditions associated with displacements, the first and second derivatives of displacement.

One considers a cantilever beam problem, and the beam is discretized as two elements, and the approximated displacement of each element is a cubic polynomial. Elements 1 and 2 refer to the element adjacent to the clamped and free ends, respectively. The displacement-based finite element method has isoparametric elements, but the stress-based finite element method produces different shape functions for each element. Table 2 compares the shape functions of both elements for this example. The last column of this table shows that the total number of degrees of freedom for the stress-based finite element method is a half of that for the displacement-based finite element method.

5 Numerical simulations

The sliding beam problem does not have an exact solution. One intends to verify the stress-based finite element method first, so the problem is simplified as a cantilever beam. Then, the sliding beam problem is solved the stress-based finite element method introduced in Sections 2 and 3. The dynamic responses are compared with literature.

5.1 Verification of the stress-based finite element method

A cantilever beam is solved by the stress-based finite element method, and the error analysis is performed. The parameters of the beam are specified as:

$L = 10$ (m), $EI = 1.4 \times 10^4$ (N-m^2), $A = 1.2$ (kg/m) where L is the length of the beam, E is the Youngs modulus, I is the second moment of area, ρ is the mass per unit volume, and A is the area of the cross section.

In this example, the beam is initially deflected as its first mode shape, so the exact solution can be obtained. One uses the displacement- and stress-based finite element methods to solve this problem, and the error analysis is listed in Table 3. Based on the comparison of the same number of elements, the displacement-based finite element method provides slightly smaller errors. However, if the errors are compared based on the same number of degrees of freedom, the stress-based finite element method provides much smaller errors.

5.2 Dynamic simulations of sliding beams using the displacement- and stress-based finite element methods

Two prescribed motion profiles for the dynamic simulations of a sliding beam are expressed as [19] Motion profile 1:

$$L(t) = L_1 + vt + \frac{1}{2}at^2 \tag{15}$$

Motion profile 2:

$$L(t) = L_2 + \frac{c}{\tau}[t - \frac{\tau}{2\pi}\sin(\frac{2\pi}{\tau}t) \tag{16}$$

where $L_1 = 0.5250$ m, $v = -0.1145$ m/sec, $a = 0$ m/sec^2, $L_2 = 0.3500$ m, $c = 0.7000$ m, and $\tau = 1.2000$ sec.

The dynamic simulations of the sliding beam problem is demonstrated by using the displacement- and stress-based finite element methods. The finite element model is solved by using the Newmark direct integration method with a constant time-step size to obtain the transient response of the system. Four simulation cases are demonstrated as follows: Case 1: displacement-based finite element method discretetized as two two-node elements with cubic shape functions to approximate the transverse displacement Case 2: stress-based finite element method discretized as four two-node elements with linear shape functions to approximate the second derivative of displacement Case 3: stress-based finite element method discretized as two three-node elements with quadratic shape functions to approximate the second derivative of displacement Case 4: stress-based finite element method discretized as two two-node elements with cubic shape functions to approximate the second derivative of displacement

For the prescribed motion profile as Equation (15), the number of degrees of freedom for the above four cases is four. Their time responses of the displacement of the beam at the free end, the second derivative of displacement at the clamped end, and the total energy (the sum of kinetic energy and potential energy) are shown in Figures 2, 3 and 4, respectively. Figures 2 and 3 show the consistent responses for the four cases. Examining Figure 4, cases 1, 3 and 4 are consistent, but case 2 does not provide an accurate response. For the prescribed motion profile as Equation (16), the dynamic responses are shown in Figures 5 to 7. Figure 5 shows that the displacement at the free end converges to the same solution obtained by the stress-based finite element method (see cases 2, 3 and 4). To compare with Reference [15], the results are in good agreement.

6 Conclusions

This paper developed the stress-based finite element method to solve the sliding beams. The conventional displacement-based finite element method cannot satisfy the compatibility and the stress boundary conditions. In contrast, the stress-based finite element method does not have these problems. This paper demonstrated the stress-based finite element method for the sliding beams. The method first specifies the assumed stress functions and then integrates them with respect to the axial coordinate. Next, the boundary conditions are applied to determine the integration constants. The most challenging task for this problem is that its spatial domain is changing with time. Thus, it is necessary to develop a variable spatial-domain beam element for the stress-based finite element method. In order to verify the proposed method, the sliding beam problem is reduces to a cantilever beam problem because the exact solution of the sliding beam

Table 1: Comparisons of the displacement- and stress-based methods for a two-node Euler-Bernoulli beam element

Comparison Items	Displacement-based Finite Element Method	Stress-based Finite Element Method
Polynomial degree of the approximated displacement	3	3
Polynomial degree of the approximated second derivative of displacement	1	1
Nodal variables	Displacement and the first derivative of displacement at both ends	Second derivative of displacement at both ends
Compatibility	Displacement and the first derivative	Displacement, the first and second Second derivative of displacement
Boundary conditions satisfied	Displacement and the first derivative of displacement	Displacement, the first and second derivatives of displacement
Number of degrees of freedom	4	2

Table 2: Shape functions of the displacement- and the stress-based finite element methods for a two-element cantilever beam

Finite element methods	Element number		No. of degrees of freedom
	1	2	
Displacement-based finite element method	$\begin{bmatrix} -\frac{6}{l^2}+\frac{12x}{l^3} & -\frac{4}{l}+\frac{5x}{l^2} & \frac{6}{l^2}-\frac{12x}{l^3} & -\frac{2}{l}+\frac{6x}{l^3} \\ -\frac{6x}{l^2}+\frac{3x^2}{l^3} & 1-\frac{4x}{l}+\frac{3x^2}{l^2} & \frac{6x}{l^2}-\frac{6x^2}{l^3} & -\frac{2x}{l}+\frac{3x^2}{l^2} \\ 1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3} & x-\frac{2x^2}{l}+\frac{x^3}{l^2} & \frac{3x^2}{l^2}-\frac{2x^3}{l^3} & -\frac{x^2}{l}+\frac{x^3}{l^2} \end{bmatrix}$		4
Stress-based finite element method	$\begin{bmatrix} 1-\frac{x}{l} & \frac{x}{l} \\ x-\frac{x^2}{2l} & \frac{x^2}{2l} \\ \frac{x^2}{2}-\frac{x^3}{6l} & \frac{x^3}{6l} \end{bmatrix}$	$\begin{bmatrix} 0 & 1-\frac{x}{l} \\ \frac{l}{2} & l-\frac{(l-x)^2}{2l} \\ \frac{l(2l+3x)}{6} & \frac{(l-x)^3}{6l}+lx \end{bmatrix}$	2

Note: x is the axial coordinate, and l is the length of beam.

Table 3: Error comparisons for the displacement- and stress-based finite element methods

Number of elements	Errors of the 1st natural frequency by using		Errors of the total energy by using		Errors of the clamped-end stress by using	
	DBFEM	SBFEM	DBFEM	SBFEM	DBFEM	SBFEM
1	4.754E-3 (DOF=2)	1.465E-2 (DOF=1)	9.932E-3 (DOF=2)	2.966E-2 (DOF=1)	7.545E-2 (DOF=2)	2.067E-2 (DOF=1)
2	4.834E-4 (DOF=4)	1.695E-3 (DOF=2)	1.325E-3 (DOF=4)	3.747E-3 (DOF=2)	1.931E-2 (DOF=4)	1.867E-2 (DOF=2)
3	1.013E-4 (DOF=6)	2.765E-4 (DOF=3)	5.616E-4 (DOF=6)	9.117E-4 (DOF=3)	1.922E-2 (DOF=6)	1.846E-2 (DOF=3)
4	3.271E-5 (DOF=8)	7.602E-5 (DOF=4)	4.246E-4 (DOF=8)	5.111E-4 (DOF=4)	1.910E-2 (DOF=8)	1.824E-2 (DOF=4)

Note: DBFEM is the displacement-based finite element method, SBFEM is the stress-based finite element method, and DOF is the degrees of freedom.

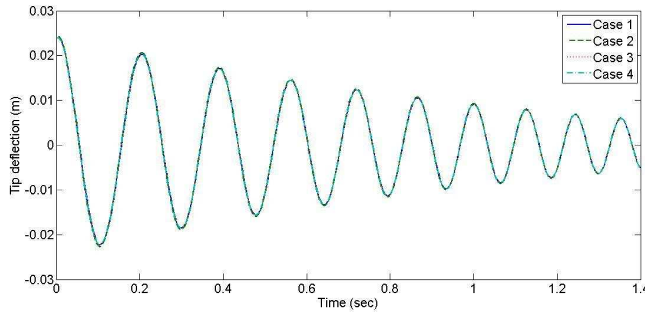

Fig. 2: Time responses of the displacement at the free end based on the motion profile 1

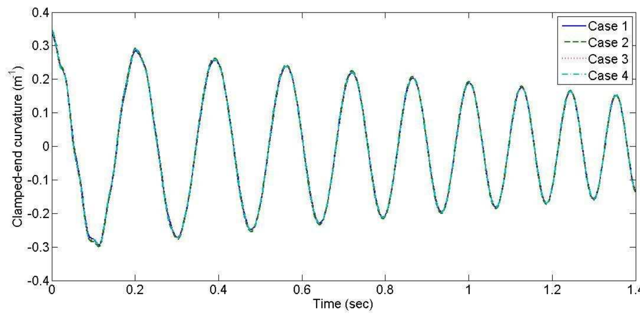

Fig. 3: Time responses of the second derivative of displacement at the clamped end based on the motion profile 1

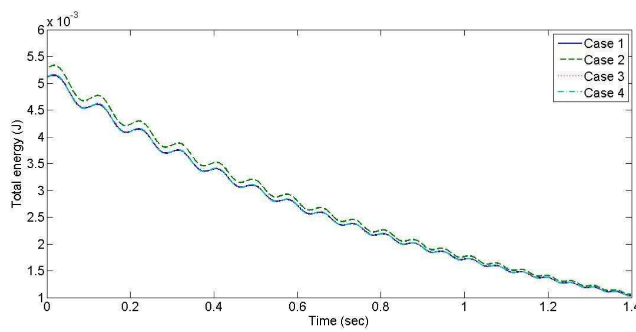

Fig. 4: Time responses of the total energy based on the motion profile 1

problem is not available. Based on the comparison of the same number of degrees o freedom, the errors of the first natural frequency, the total energy, and the clamped-end stress obtained by the stress-based finite element method provides a more accurate solution than those obtained by the displacement-based finite element method. This paper also demonstrates the dynamic simulations of four cases based on two prescribed motion profiles of the sliding beam, and the results obtained by the proposed method are in good agreement with literature.

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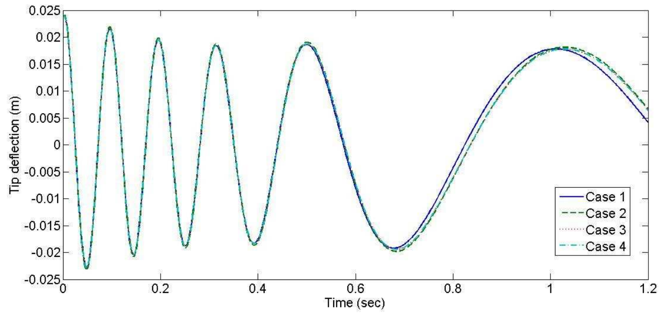

Fig. 5: Time responses of the displacement at the free end based on the motion profile 2

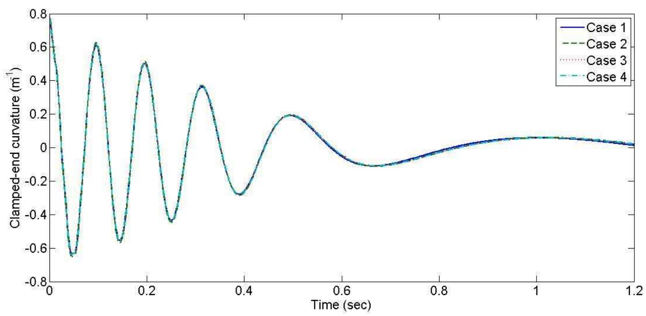

Fig. 6: Time responses of the second derivative of displacement at the clamped end based on the motion profile 2

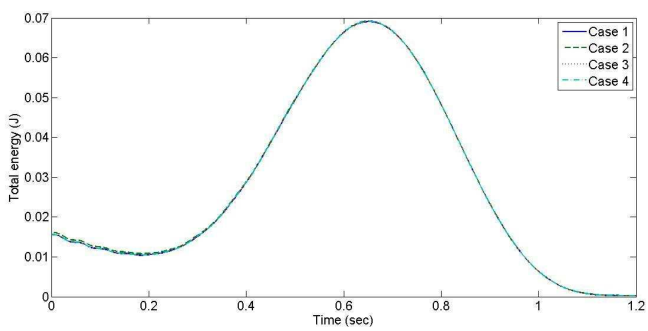

Fig. 7: Time responses of the total energy based on the motion profile 2

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