Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/060307

Global Domination Integrity of Graphs

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Received: 19 Sep. 2016, Revised: 12 Jul. 2017, Accepted: 22 Jul. 2017
Published online: 1 Sep. 2017

Abstract: R. Sundareswaran and V. Swaminathan [11], introduced the concept of the domination integrity of a graph. It is a useful measure of vulnerability and it is defined as $DI(G) = \min\{|S| + m(G-S)$, where S is a dominating set and $m(G-S)$ is the order of a maximum component of $G-S\}$. In this paper we introduce the concept of global domination integrity of a graph G, and define it as $GDI(G) = \min\{|S| + m(G-S)$, where S is a global dominating set and $m(G-S)$ is the order of a maximum component of $G-S\}$. The global domination integrity of some graphs is obtained. The relations between global domination integrity and other parameters are determined.

Keywords: Integrity, Domination integrity, Global dominating set, Global domination integrity

1 Introduction

The stability of a network (computer, communication, or transportation) composed of nodes (processing) and links (communication or transportation) is of prime importance to network designers. As the network begins losing links or nodes, eventually it loses effectiveness. Communication networks are designed such that they are not easily disrupted under external attack and moreover, they can easily be reconstructed if they are disrupted [8]. These desirable properties of networks can be measured by various parameters such as connectivity and edge-connectivity. However, these parameters do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have recently been introduced in an attempt to cope with this. These include connectivity and edge-connectivity [3], integrity and edge-integrity [1,2], tenacity and edge-tenacity [4,8]. All graphs considered here are finite, undirected without loops or multiple edges. As usual p and q denote the number of vertices and edges of a graph G. Any undefined term or notation in this paper can be found in Harary [6]. The degree of a vertex v in a graph G denoted by $degv$ is the number of edges of G incident with v. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G), (\delta(G))$. A vertex of degree one is called a pendant vertex. A complete subgraph or clique is an induced subgraph such that there is an edge between each pair of vertices in the subgraph. The clique number $w(G)$ is the order of the largest complete subgraph of G. For a vertex $v \in V$, the open neighborhood of v in G, denoted by $N(v)$, is the set of all vertices that are adjacent to v. A friendship graph F_n is a graph which consists of n triangles with a common vertex. The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$ and $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j : 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\}$, [5].

The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G [6].

Definition 1.1.[7] A set D of vertices in a graph G is a dominating set of G if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set.

Barefoot et al. [2], have introduced the concept of integrity, which is defined as follows.
The integrity of a finite graph G is $I(G) = \min\{|S| + m(G-S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of the largest component.

Sultan et al. [10], have introduced the concept of hub-integrity of a graph as a new measure of vulnerability. The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \min\{|S| + m(G-S)\}$, where S is a hub set and $m(G-S)$ is the order of a maximum component of $G-S$.

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R. Sundareswaran and V. Swaminathan [11], introduced the concept of domination integrity of a graph G, which is defined as follows.

Definition 1.2. $DI(G) = \min\{|S| + m(G-S)$, where S is a dominating set and $m(G-S)$ is the order of a maximum component of $G-S\}$.

Definition 1.3. [9] A dominating set D of G is a global dominating set $gd-set$ of G if D is also a dominating set of the complement $\overline{G}$ of G. The global domination number γ_g of G is the minimum cardinality of $gd-set$.

In a network, a minimum global dominating set of nodes provides a link with the rest of the nodes. If D_g is a minimum global dominating set and if the order of the largest component of $G-D_g$ is small, then the removal of D_g results in a chaos in the network because not only the decision making process is paralyzed but also the communication between the remaining members is minimized. So, we introduce the concept of global domination integrity of a graph as another measure of vulnerability of a graph.

The following results are needed to prove the main results.

Proposition 1.1. *[9] For a (p,q) graph G without isolated vertices, $\frac{(2q-p(p-3))}{2} \leq \gamma_g \leq p-\beta+1$.*

Theorem 1.1. *[10] For any graph G, $HI(G) = \gamma(G)$ if and only if $G = \overline{K}_p$.*

Theorem 1.2. *[2] The integrity of the path P_p is $\lceil 2\sqrt{p+1}\rceil - 2$.*

Proposition 1.2. *[9] For any graph G,*

1. $\gamma_g(G) = \gamma_g(\overline{G})$,
2. $\gamma(G) \leq \gamma_g(G)$,
3. $\frac{\gamma(G)+\gamma(\overline{G})}{2} \leq \gamma_g(G) \leq \gamma(G)+\gamma(\overline{G})$.

2 Global domination integrity of graphs

Definition 2.1. The global domination integrity of a graph G is defined as $GDI(G) = \min\{|S| + m(G-S)$, where S is a global dominating set and $m(G-S)$ is the order of a maximum component of $G-S\}$. Explicitly, $GDI(G) \geq I(G)$ for any graph G. Also, the definition shows that $GDI(G) \geq DI(G)$, and $GDI(G) \geq \gamma_g$.

Definition 2.2. A GDI-set of G is any subset S of $V(G)$ for which $GDI(G) = |S| + m(G-S)$.

Proposition 2.1.

(a) For any complete graph K_p, $GDI(K_p) = p$.
(b) For any path P_p,

$$GDI(P_p) = \begin{cases} 2, \text{ if } p = 2\,; \\ 3, \text{ if } p = 3,4\,; \\ 4, \text{ if } p = 5,6,7\,; \\ \lceil \frac{p}{3} \rceil + 2, \text{ if } p \geq 8. \end{cases}$$

(c) For any cycle C_p, $p \geq 4$,

$$GDI(C_p) = \begin{cases} 4, \text{ if } p = 4,5\,; \\ \lceil \frac{p}{3} \rceil + 2, \text{ if } p \geq 6. \end{cases}$$

(d) For the star $K_{1,p-1}$, $p \geq 3$, $GDI(K_{1,p-1}) = 3$.
(e) For the double star $S_{n,m}$,

$$GDI(S_{n,m}) = 3.$$

(f) For the complete bipartite graph $K_{n,m}$,

$$GDI(K_{n,m}) = min\{n,m\} + 2.$$

(g) For the wheel graph $W_{1,p-1}$, $p \geq 4$,

$$GDI(W_{1,p-1}) = \begin{cases} 4, \text{ if } p = 5\,; \\ \lceil \frac{p-1}{3} \rceil + 3, \text{ if } p \geq 6. \end{cases}$$

Theorem 2.1. *For any graph G, $GDI(G) = \gamma(G)$ if and only if $G = \overline{K_p}$.*

Proof. If $GDI(G) = \gamma(G)$, then $|S| + m(G-S) = \gamma(G)$, we have the following cases:
Case 1: If $\gamma(G) = 1$, then $|S| + m(G-S) = 1$, it follows that $|S| = 1$. So $G \cong K_1 = \overline{K_1}$.
Case 2: If $\gamma(G) = 2$, then $|S| + m(G-S) = 2$. But $|S| = 1$ and $m(G-S) = 1$, is impossible, since there does not exist graph satisfying these values. Then $|S| = 2$ and $m(G-S) = 0$. So, $G \cong \overline{K_2}$. In general, $|S| + m(G-S) = \gamma(G)$ if $|S| = \gamma(G)$ and $m(G-S) = 0$. In addition $|S| = \gamma(G)$ only in $\overline{K_p}$. Thus $G = \overline{K_p}$. Converse is obvious. □

Theorem 2.2. *For any graph G, $GDI(G) = \gamma_g(G)$ if and only if $G = \overline{K_p}$.*

Proof. The proof is similar to the proof of Theorem (2.1). □

Corollary 2.1. *For any graph G, $GDI(G) = \gamma_g(G) = \gamma(G)$ if and only if $G = \overline{K_p}$.*

Observation 2.1 *For any graph G, $GDI(G) \geq \frac{\gamma(G)+\gamma(\overline{G})}{2}$.*

Proof. By Proposition (1.2), the proof follows. □

Observation 2.2 *For any graph G, $GDI(G) \geq \gamma_g(G) - \gamma(\overline{G})$.*

Proof. Since $GDI(G) \geq \gamma_g$ and by Proposition (1.2), $\gamma_g \geq \gamma$ and $\gamma(G) + \gamma(\overline{G} \geq \gamma_g$, hence $GDI(G) \geq \gamma_g(G) - \gamma(\overline{G})$. □

Observation 2.3

1. *$GDI(G) = 2$ if and only if $G \cong K_2$ or $G \cong \overline{K_2}$.*
2. *$GDI(G) = 3$ if and only if $G \cong K_3$ or $G \cong \overline{K_3}$ or $G \cong K_{1,p-1}$ or $G \cong S_{n,m}$.*

Proof. (1) $GDI(G) = 2$, means that $|S| + m(G - S) = 2$, and the last equation is achieved only if $|S| = 2$ and hence $G \cong K_2$ *or* $G \cong \overline{K_2}$.
(2) The proof is similar to that of (1). □

Proposition 2.2. *For any graph G of order p, if* $GDI(G) = p$, *then* $diam(G) \leq 2$.

Proof. Let G be a graph of order p, such that $GDI(G) = p$. Assume, on the contrary, that $diam(G) \geq 3$. Then, G contains a path P_4. Therefore, $GDI(G) \leq p - 1$ a contradiction. Hence $diam(G) \leq 2$. □

Proposition 2.3. *(1) For any connected graph G,* $1 \leq GDI(G) \leq p$. *The complete graph* K_p *achieves the upper bound and lower bound is achieved by* K_1.
(2) $GDI(G) \leq q + 1$, *the equality holds if* $G \cong K_2$ *or* P_2.
(3) For any connected graph G, $1 \leq GDI(\overline{G}) \leq p$. *The complete graph* K_p *achieves the upper bound and lower bound is achieved by* K_1.

Theorem 2.3. *For any graph G,*

1. $2 \leq GDI(G) + GDI(\overline{G}) \leq 2p$.
2. $1 \leq GDI(G).GDI(\overline{G}) \leq p^2$.

Proof. The proof follows from Proposition (2.3). □

Lemma 2.1. *Let T be a tree. Then* $GDI(T) = p$ *if and only if either* $T \cong P_2$ *or* $T \cong P_3$.

Proof. If T is a tree with $GDI(T) = p$, then by Proposition (2.2), $diam(T) \leq 2$. If $diam(T) = 2$, then T is $K_{1,p-1}$. But $GDI(K_{1,p-1}) = 3$. Since $diam(T) = 2, p \geq 3$, so, $T = K_{1,2} = P_3$, and if $diam(T) = 1$, then T is P_2. Conversely, Let T be either P_2 or P_3. Then the proof follows from Proposition (2.1). □

Theorem 2.4. *For any graph G,* $GDI(G) = HI(G) = \gamma(G)$ *if and only if* $G \cong \overline{K_p}$.

Proof. By Theorem (2.1) and Theorem (1.1) , the proof follows. □

Theorem 2.5. *For any graph G, if* $diam(G) \geq 5$, *then* $GDI(G) \leq \lceil \frac{p}{2} \rceil + 1$.

Proof. Let G be a graph with $diam(G) \geq 5$. $GDI(G) > \lceil \frac{p}{2} \rceil + 1$, then $diam(G) < 5$ a contradiction. Hence the result holds. □

Theorem 2.6. *Let* $K_{1,p-1}$ *be a star with* $p \geq 3$, *and let* G_s *be a spider graph which is constructed by subdividing each edge once in* $K_{1,p-1}$ *as in Figure 1. Then* $GDI(G_s) = \Delta(G_s) + 2$.

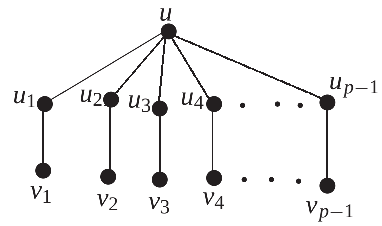

Figure 1: Spider graph

Proof. Let $V(G_s) = \{u, u_1, u_2, u_3, ..., u_{p-1}, v_1, v_2, v_3, ..., v_{p-1}\}$. We choose the vertex u and all vertices which are adjacent with it as a global dominating set of G, this means that $S = \{u, u_1, u_2, u_3, ..., u_{p-1}\}$, $|N[u]| = |\{u, u_1, u_2, u_3, ..., u_{p-1}\}| = \Delta(G_s) + 1$. Then $|S| = \Delta(G_s) + 1$, and $m(G_s - S) = 1$, this implies that $GDI(G_s) \leq |S| + m(G_s - S) = \Delta(G_s) + 2$. Also we can choose $S = \{u, v_1, v_2, v_3, ..., v_{p-1}\}$ as a global dominating set of G_s, then $|S| = \Delta(G_s) + 1$, and $m(G_s - S) = 1$. Then $GDI(G_s) \leq |S| + m(G_s - S) = \Delta(G_s) + 2$. Clearly, there does not exist a global dominating set of G_s such that $GDI(G_s) < |S| + m(G_s - S) = \Delta(G_s) + 2$. □

Theorem 2.7. *For any connected graph G of order* $p \geq 3, GDI(K_{1,p-1}) \leq GDI(G) \leq GDI(K_p)$.

Remark 2.1. If H is a subgraph of G, then not necessarily $GDI(H) \leq GDI(G)$, for example $G \cong P_5$ and $H \cong 3P_1 \cup P_2, GDI(G) = 4$ and $GDI(H) = 5$.

Theorem 2.8. *For any disconnected graph G,* $GDI(G) = DI(G)$.

Proof. Since $\gamma_g(G) \geq \gamma(G)$, $GDI(G) \geq DI(G)$. Let $G \cong G_1 \cup G_2 \cup ... \cup G_r$ and suppose $S = \{F_1, F_2, ..., F_r\}$, a family of dominating sets of $G_1, G_2, ..., G_r$. Without loss of generality, suppose the set F_1 is a dominating set of G_1, F_2 is a dominating set of G_2, ..., and F_r is a dominating set of G_r. Since any vertex $f \in F_1$ is adjacent to all vertices in $G_2, G_3, ..., G_r$ in $\overline{G}$, and any vertex $f \in F_2$ is adjacent to all vertices in $G_1, G_3, ..., G_r$ in $\overline{G}$,..., and any vertex $f \in F_r$ is adjacent to all vertices in $G_1, G_2, G_3, ..., G_{r-1}$ in $\overline{G}$. Therefore, S is a dominating set of $\overline{G}$. Thus, S is a global dominating set of G. So $DI(G) = GDI(G)$. □

Corollary 2.2. *For any graph G,* $GDI(G) \geq \gamma(G)$.

Proof. Since $GDI(G) \geq \gamma_g(G)$, from Proposition (1.2), we get the result. □

Proposition 2.4. *If* $w(G) = p - 1$, *then* $GDI(G) = p$.

Proof. Since $w(G) = p - 1$, G has a complete graph of order $p - 1$, so by Proposition (2.1), the proof follows. □

Proposition 2.5. $GDI(\overline{K_{n,m}}) = n + 1, n \geq m$.

Proof. Let $v_1, v_2, ..., v_n, u_1, u_2, ..., u_m$ be the vertices of $\overline{K_{n,m}}$. We claim that $S = \{v_1, u_1\}$, a global dominating set of $\overline{K_{n,m}}$. Since $\overline{K_{n,m}} \cong K_n \cup K_m$, $N(v_1) = \{v_2, v_3, ..., v_n\}$ in K_n and $N(u_1) = \{u_2, u_3, ..., u_m\}$ in K_m, then $\{v_1, u_1\}$ is a dominating set of $K_n \cup K_m$. Also since $N(v_1) = \{u_1, u_2, u_3, ..., u_m\}$ in $\overline{K_n \cup K_m}$ and since $N(u_1) = \{v_1, v_2, v_3, ..., v_n\}$ in $\overline{K_n \cup K_m}$, $\{v_1, u_1\}$ is a dominating set of $\overline{K_n \cup K_m}$. Therefore, S is a global dominating set of $\overline{K_{n,m}}$ and $m(\overline{K_{n,m}} - S) = n - 1$. Then

$$GDI(\overline{K_{n,m}}) \leq |S| + m(\overline{K_{n,m}} - S) = n + 1. \quad (2.1)$$

Since $GDI(K_n) = n$, it is clear that there does not exist any set S_1 other than S such that

$$|S_1| + m((K_n \cup K_m) - S_1) < |S| + m((K_n \cup K_m) - S). \quad (2.2)$$

Hence, from (2.1) and (2.2), $GDI(\overline{K_{n,m}}) = n + 1$. □

The next result is an immediate consequence of Proposition (2.1) and Proposition (2.5).

Corollary 2.3. *$GDI(K_{n,m}) = GDI(\overline{K_{n,m}})$ if and only if $n = m + 1$.*

Remark 2.2.

1. For a path P_p, $GDI(P_p) = GDI(\overline{P_p})$, if and only if $p = 3, 4, 5$.
2. For a cycle C_p, $GDI(C_p) = GDI(\overline{C_p})$,
3. For a complete K_p, $GDI(K_p) = GDI(\overline{K_p})$,
4. For a star $K_{1,p-1}, p \geq 4$, $GDI(K_{1,p-1}) \neq GDI(\overline{K_{1,p-1}})$.

Mycielski's construction

Let G be a graph with $V(G) = \{v_1, v_2, .., v_p\}$, and let $U = \{u_1, u_2, ..., u_p\} \cup \{x\}$. A graph $\mu(G)$ is obtained by adding the vertices of U to G such that u_i is a adjacent to all of the vertices of $N(v_i)$, and x is adjacent to all of the vertices of $U = \{u_1, u_2, ..., u_p\}$.

Theorem 2.9. *For $p \geq 2$,*

$$GDI(\mu(P_p)) = \begin{cases} 4, \text{ if } p = 2\,; \\ p + 1, \text{ if } p \geq 3. \end{cases}$$

Proof. Let $V = \{v_1, v_2, ..., v_p\}$, be the vertex set of P_p, and let $V(\mu(P_p)) = \{v_1, v_2, ..., v_p\} \cup \{v'_1, v'_2, ..., v'_p\} \cup \{x\}$ and $|V(\mu(P_p))| = 2p + 1$. The following cases are considered:
Case 1: For $p = 2$, since $\mu(P_2) \cong C_5$, from proposition (2.1), $GDI(\mu(P_2)) = 4$.
Case 2: For $p = 3$, consider $S = \{v_2, v'_2, x\}$, a global dominating set of $\mu(P_3)$ such that $|S| = 3$ and $m(\mu(P_3) - S) = 1$. This implies that $GDI(\mu(P_3)) \leq |S| + m(\mu(P_3) - S) = 4$.
Case 3: For $p \geq 4$, consider $S = \{v_2, v_3, ..., v_{p-1}\} \cup \{x\}$, a global dominating set of $\mu(P_p)$ such that $|S| = p - 1$ and $m(\mu(P_p) - S) = 2$. Thus

$$GDI(\mu(P_p)) \leq |S| + m(\mu(P_p) - S) = p - 1 + 2 = p + 1. \quad (2.3)$$

Now we show that the number $|S| + m(\mu(P_p) - S)$ is minimum. If S_1 is any global dominating set that is different from set S such that $m(\mu(P_p) - S_1) = 1$, then $|S_1| \geq p + 1$ and hence $|S_1| + m(\mu(P_p) - S_1) \geq p + 2$, also if $m(\mu(P_p) - S_1) \geq 2$, it is clear that $|S_1| + m(\mu(P_p) - S_1) \geq |S| + m(\mu(P_p) - S)$. Hence, for any global dominating set S_1,

$$|S_1| + m(\mu(P_p) - S_1) \geq p + 1. \quad (2.4)$$

From (2.3) and (2.4), $GDI(\mu(P_p)) = p + 1$. □

Theorem 2.10.

$$GDI(\mu(C_p)) = \begin{cases} p + 2, \text{ if } p = 3, 4, 5\,; \\ \frac{2p}{3} + 4, \text{ if } p \equiv 0 \ (mod\ 3) \text{ and } p \geq 6; \\ \lceil \frac{2p}{3} \rceil + 4, \text{ if } p \equiv 1, 2 \ (mod\ 3) \text{ and } p \geq 6. \end{cases}$$

Proof. Let $V(\mu(C_p)) = \{v_1, v_2, ..., v_p\} \cup \{v'_1, v'_2, ..., v'_p\} \cup \{x\}$ and $|V(\mu(C_p))| = 2p + 1$. To prove this result, the following cases are considered:
Case 1: For $p = 3, 4, 5$. If we remove all vertices of C_p and the vertex x, we get a total disconnected graph so that order of the largest component of $\mu(C_p)$ is 1, this implies that $GDI(\mu(C_p)) = p + 2$.
Case 2: For $p \geq 6$. We consider a global dominating set of $\mu(C_p)$ as follows:

- When $p \equiv 0 \ (mod\ 3), p = 3k$ for some integer $k \geq 2$. Consider $S = \{v_1, v_2, v_4, v_5, v_7, v_8, .., v_{p-4}, v_{p-2}, v_{p-1}\} \cup \{x\}$, then $|S| = \frac{2p}{3} + 1$ and $m(\mu(C_p) - S) = 3$. This implies that

$$GDI(\mu(C_p)) \leq |S| + m(\mu(C_p) - S) = \frac{2p}{3} + 4. \quad (2.5)$$

To discuss the minimality of $|S| + m(\mu(C_p) - S)$, we consider any set S_1 as global dominating set different from S such that $m(\mu(C_p) - S_1) = 2$, then $|S_1| \geq p + 1$. Thus $|S_1| + m(\mu(C_p) - S_1) \geq p + 3$, if $m(\mu(C_p) - S_1) = 1$, then $|S_1| \geq p + 1$. Thus $|S_1| + m(\mu(C_p) - S_1) \geq p + 2$. Finally, suppose that $m(\mu(C_p) - S_1) \geq 3$, it is clear that $|S_1| + m(\mu(C_p) - S_1) \geq |S| + m(\mu(C_p) - S)$. Hence, for any global dominating set S_1,

$$|S_1| + m(\mu(C_p) - S_1) \geq \frac{2p}{3} + 4. \quad (2.6)$$

Therefore, from (2.5) and (2.6), $GDI(\mu(C_p)) = \frac{2p}{3} + 4$.

- When $p \equiv 1 \ (mod\ 3), p = 3k + 1$ for some integer $k \geq 2$. Consider $S = \{u_1, u_2, u_4, u_5, u_7, u_8, ..., u_{p-3}, u_{p-2}, u_p\} \cup \{x\}$, and $|S| = \lceil \frac{2p}{3} \rceil + 1, m(\mu(C_p) - S) = 3$. Therefore,

$$GDI((\mu(C_p)) \leq |S| + m(\mu(C_p) - S) = \lceil \frac{2p}{3} \rceil + 4. \quad (2.7)$$

To show that the number $|S| + m(\mu(C_p) - S)$ is minimum, let us consider any global dominating set S_1 of $\mu(C_p)$

different from S such that $m(\mu(C_p)-S_1)=2$, then $|S_1|\geq p+1$. Thus $|S_1|+m(\mu(C_p)-S_1)\geq p+3$, also, if $m(\mu(C_p)-S_1)=1$, then $|S_1|\geq p+1$, so that $|S_1|+m(\mu(C_p)-S_1)\geq p+2$. If $m(\mu(C_p)-S_1)\geq 3$, it is easy to show that $|S_1|+m(\mu(C_p)-S_1)\geq |S|+m(\mu(C_p)-S)$. Hence, for any global dominating set S_1,

$$|S_1|+m(\mu(C_p)-S_1)\geq \lceil\frac{2p}{3}\rceil+4. \quad (2.8)$$

From (2.7) and (2.8), we have $GDI(\mu(C_p))=\lceil\frac{2p}{3}\rceil+4$.

- When $p\equiv 2\ (mod\ 3), p=3k+2$ for some integer $k\geq 2$. Consider $S=\{u_1,u_2,u_4,u_5,u_7,u_8,..,u_{p-3},u_{p-1},u_p\}\cup\{x\}$, then $|S|=\lceil\frac{2p}{3}\rceil+1$ and $m(\mu(C_p)-S)=3$. This implies that

$$GDI(\mu(C_p))\leq |S|+m(\mu(C_p)-S)=\lceil\frac{2p}{3}\rceil+4. \quad (2.9)$$

The proof follows similar to that of above Case. Hence, $GDI(\mu(C_p))=\lceil\frac{2p}{3}\rceil+4$. This completes the proof. □

Theorem 2.11. $\gamma_g(\mu(K_{1,p-1}))=3$.

Proof. Let $V(\mu(K_{1,p-1}))=\{v_0,v_1,v_2,...,v_{p-1}\}\cup\{v'_0,v'_1,v'_2,...,v'_{p-1}\}\cup\{x\}$. Then $|V(\mu(K_{1,p-1}))|=2p+1$. Figure 2 shows the graph $\mu(K_{1,p-1})$. Since in $\mu(K_{1,p-1})$, $deg(v_0)=2p-2, deg(v'_0)=p$ and $deg(x)=p$, we choose v_0,v'_0,x, as global dominating set of $\mu(K_{1,p-1})$, this means that $S=\{v_0,v'_0,x\}$. To prove that the set S is minimum, we consider $S_1=\{v_0,v'_0\}$, we see that the set S_1 is a dominating set of $\mu(K_{1,p-1})$, but it is not dominating set of $\overline{\mu(K_{1,p-1})}$, since the vertices $\{v_1,v_2,v_3,...,v_{p-1}\}$ is not dominated by v_0,v'_0. Thus S_1 is not a global dominating set of $\mu(K_{1,p-1})$. If we consider $S_1=\{v_0,x\}$, then S_1 is a domination set of $\mu(K_{1,p-1})$. But $\{v'_1,v'_2,...,v'_{p-1}\}$ is not dominated by the vertices $\{v_0,x\}$ in $\overline{\mu(K_{1,p-1})}$, so S_1 is not a global dominating set of $\mu(K_{1,p-1})$. Finally if $S_1=\{v'_0,x\}$, clearly, S_1 is not dominating set of $\mu(K_{1,p-1})$, therefore, it is not global dominating set of $\mu(K_{1,p-1})$. Hence, $\gamma_g=3$. □

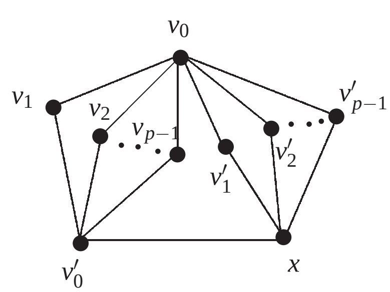

Figure 2: $\mu(K_{1,p-1})$

Theorem 2.12. $GDI(\mu(K_{1,p-1}))=4$.

Proof. Let $V(\mu(K_{1,p-1}))=\{v_0,v_1,v_2,...,v_{p-1}\}\cup\{v'_0,v'_1,v'_2,...,v'_{p-1}\}\cup\{x\}$. Then $|V(\mu(K_{1,p-1}))|=2p+1$. Consider $S=\{v_0,v'_0,x\}$, a global dominating set of $\mu(K_{1,p-1})$ such that $|S|=3$ and $m(\mu(K_{1,p-1}))=1$. This implies that $GDI(\mu(K_{1,p-1}))\leq |S|+m(\mu(K_{1,p-1})-S)=4$.
Clearly, $|S|$ and $m(\mu(K_{1,p-1})-S)$ are minimum, since $|S|$ is a γ_g-set and $m(\mu(K_{1,p-1}))=1$ is also the minimum, this completes the proof. □

Theorem 2.13. $GDI(\mu(\overline{K_p}))=p+2$.

Proof. Let $V(\overline{K_p})=\{v_1,v_2,...,v_p\}$ and $V(\mu(\overline{K_p}))=\{v_1,v_2,...,v_p\}\cup\{v'_1,v'_2,...,v'_p\}\cup\{x\}$. Since $\mu(\overline{K_p})=\overline{K_p}\cup K_{1,p}$, we choose all vertices of $\overline{K_p}$ and central vertex x of $K_{1,p}$ as global dominating set, this means that $S=\{v_1,v_2,...,v_p,x\}$, since $N(x)=\{v_1,v_2,...,v_p\}$, and $N(v_i)=\{v'_1,v'_2,...,v'_p\}, 1\leq i\leq p$ in $\overline{\mu(\overline{K_p})}$, then S is dominating set of $\overline{\mu(\overline{K_p})}$, so S is global dominating set of $\mu(\overline{K_p})$. Thus $|S|=p+1$ and $m(\mu(\overline{K_p})-S)=1$. Then $GDI(\mu(\overline{K_p}))=p+2$. Clearly, we can not remove any vertex of S, since all vertices $v_1,v_2,...,v_p$ are an isolated vertices and the vertex x is adjacent to $v'_1,v'_2,...,v'_p$, hence the result. □

Proposition 2.6. *If G is one of the following graphs: $K_{1,3},P_4,P_5,C_3,C_4$, or C_3+e, then $GDI(G)=q$.*

Observation 2.4 *If G is one of the graphs: $K_{n,m}$, where $n\leq m$ and $m-n=4$, $S_{2,2},P_p,p=12,14,16$ or $C_p,p=12,14,16$, then $GDI(G)=\frac{|V(G)|}{2}$.*

Remark 2.3. If $G\cong K_p,P_3$, *or* C_4 , then $GDI(G)=p$.

Proposition 2.7. *For any graph G without isolated vertices, $GDI(G)\geq\frac{(2q-p(p-3))}{2}$.*

Proof. Since $GDI(G)\geq\gamma_g(G)$, we get the result by Proposition (1.1). □

Definition 2.3. [12] The switching of a vertex v of G means removing all the edges incident to v and adding edges joining v to every vertex which are not adjacent to v in G. We denote the resultant graph by G_{psv}.

Theorem 2.14. *If C_{psv} is a graph obtained by switching of a vertex in cycle C_p, then*

$$GDI(C_{psv})=\begin{cases}3, & if\ p=3,4\ ;\\ 4, & if\ p=5\ ;\\ 5, & if\ p=6;\\ \lceil 2\sqrt{p-4}\rceil+1, & if\ p\geq 7.\end{cases}$$

Proof. Let $v_1,v_2,...,v_p$ be the vertices of C_p, and C_{psv} denote the graph obtained by switching of a vertex v of C_p. Let the switched vertex be v_1. We have the following cases:

Case 1: $p=3$. Since $C_{3sv} \cong K_1 \cup K_2$, then it is clear that $GDI(C_{3sv})=3$.
Case 2: $p=4$. Since $C_{4sv} \cong K_{1,3}$ and from Proposition (2.1), $GDI(C_{4sv})=3$.
Case 3: $p=5$. Consider $S=\{v_1,v_3,v_4\}$, a global dominating set of C_{5sv}, and $m(C_{5sv}-S)=1$. This implies that $GDI(C_{5sv}) \leq |S|+m(C_{5sv}-S)=4$. Now, we show that the number $|S|+m(C_{5sv}-S)$ is minimum. If the vertex v_1 is removed from set S, then v_1 will not be dominated by any vertex of S in $\overline{C_{5sv}}$, so $S_1=\{v_3,v_4\}$ is a dominating set of C_{5sv} but not a dominating set of $\overline{C_{5sv}}$. So, $S_1=\{v_3,v_4\}$ is not a global dominating set of C_{psv}. Hence, S is a minimum. On the other hand, if we suppose that any set S_1 other than S such that $m(C_{5sv}-S_1) \geq 2$, then $|S_1| \geq 3$, hence $|S_1|+m(C_{5sv}-S_1) \geq 5$. Then, for any $S_1, |S_1|+m(C_{5sv}-S_1) > |S|+m(C_{5sv}-S)$. Then $GDI(C_{5sv})=4$.
Case 4: $p=6$. Consider $S=\{v_1,v_3,v_4,v_5\}$, a global dominating set of C_{6sv}, and $m(C_{6sv}-S)=1$. This implies that $GDI(C_{6sv}) \leq |S|+m(C_{6sv}-S)=5$. The proof follows similar to that of case 3. Then $GDI(C_{6sv})=5$.
Case 5: For $p \geq 7$. Since $N(v_1)=\{v_3,v_4,...,v_{p-1}\}$, $N(v_2)=\{v_3\}$ and $N(v_p)=\{v_{p-1}\}$, hence the set $S_1=\{v_1,v_3,v_{p-1}\}$ is a dominating set of C_{psv}. Since $N(v_1)=\{v_2,v_p\}$ and $N(v_3,v_{p-1})=\{v_2,v_3,v_4,...,v_p\}$ in $\overline{C_{psv}}$, then $S_1=\{v_1,v_3,v_{p-1}\}$ is a dominating set of $\overline{C_{psv}}$. Thus S_1 is a global dominating set of C_{psv} and $C_{psv}-S_1=P_{p-5}\cup 2K_1$, so $m(C_{psv}-S_1)=p-5$. Let $S_2=\{v_i/v_i \in I - \ set\ of\ P_{p-5}\}$. Take $V_1=\{v_i/v_i \in I - \ set\ of\ P_{p-5}\}$ and $|V_1|=|S_2|$. Consider $S=S_1\cup S_2$, then S is a global dominating set of C_{psv} (as $S_1 \subseteq S$), $|S_1|+|S_2|=|S_1|+|V_1|$ and $C_{psv}-S=2K_1\cup P_{p-5}-V_1$, hence, $m(C_{psv}-S)=m(2K_1\cup P_{p-5}-V_1)$. By Theorem (1.2),

$$\begin{aligned}|S|+m(C_{psv}-S) &= |S_1|+|V_1|+m((2K_1\cup P_{p-5})-V_1)\\ &= |S_1|+I(2K_1\cup P_{p-5})\\ &= 3+\lceil 2\sqrt{p-4}\rceil-2=\lceil 2\sqrt{p-4}\rceil+1.\end{aligned}$$

Then
$$GDI(C_{psv})=\lceil 2\sqrt{p-4}\rceil+1. \tag{2.10}$$

We show that the minimality of $|S|+m(C_{psv}-S)$. If S_3 is any global dominating set of C_{psv} which containing one vertex from S_1, namely v_1, and does not containing any vertex of S_2 such that $|S_3|=k<p$. Then

$$|S_3|+m(C_{psv}-S_3)=k+p-k=p>|S|+m(C_{psv}-S). \tag{2.11}$$

For more details, put $S_3=\{v_1,v_2,v_p\}$, a global dominating set such that S is γ_g- *set* of $C_{psv}, |S_3|=3$ and $m(C_{psv}-S_3)=p-3$, then

$$|S_3|+m(C_{psv}-S_3)=3+p-3=p>|S|+m(C_{psv}-S). \tag{2.12}$$

Consider S_5 be another global dominating set of C_{psv} such that $S_5=S_2\cup S_4$, where $S_4=\{v_2,v_p\}$. Then $m(C_{psv}-S_5)=p-|S_2|-2$. Therefore,

$$\begin{aligned}|S_5|+m(C_{psv}-S_5) &=\\ &= |S_2|+2+p-|S_2|-2=p\\ &> |S|+m(C_{psv}-S).\end{aligned} \tag{2.13}$$

Hence, from (2.10), (2.11), (2.12) and (2.13), $GDI(C_{psv})=\lceil 2\sqrt{p-4}\rceil+1$. □

Theorem 2.15. *If K_{psv} is a graph obtained by switching of a vertex in K_p, then $GDI(K_{psv})=p$.*

Proof. Let $v_1,v_2,...,v_p$ be the vertices of K_p, and K_{psv} denotes the graph obtained by switching of a vertex v of K_p. Let the switched vertex be v_1. Since $K_{psv}\cong K_1\cup K_{p-1}$, then from Proposition (2.1), $GDI(K_{psv})=GDI(K_1\cup K_{p-1})=p$. □

Theorem 2.16. *If F_{nsv} is a graph obtained by switching of a vertex in F_n, then*

$$GDI(F_{nsv})=\begin{cases} n+2, & \text{if the switched vetex is } v\,;\\ 5, & \text{if the switched vetex is } v_1\,.\end{cases}$$

Proof. Let $v,v_1,v_2,...,v_{2n}$ be the vertices of F_n, and F_{nsv} denotes the graph obtained by switching of a vertex v of F_n. We have the two cases:
Case 1: Let the switched vertex be v as in the Figure 3. Since $F_{nsv}\cong K_1\cup nK_2$, consider $S=\{v,v_1,v_3,v_5,v_7,...,v_{2n-1}\}$, a global dominating set of F_{nsv} and $m(F_{nsv}-S)=1$. Therefore,

$$GDI(F_{nsv})\leq |S|+m(F_{nsv}-S)=n+2. \tag{2.14}$$

We discuss the minimality of $|S|+m(F_{nsv}-S)$. To do it, we must take into consideration of minimality of both $|S|$ and $m(F_{nsv}-S)$. So if we remove the vertex v from the set S, then there does not exist a vertex in S dominating v, and one can note that each v_i and v_{i+1} are adjacent and removal of v_i from set S, leaves v_{i+1} not dominated by any vertex of S, therefore, $|S|=n+1$. While in $\overline{F_{nsv}}$, the vertex v is enough to be a global dominating set since $N(v)=\{v_1,v_2,...,v_{2n}\}$, hence S is γ_g- set of F_{nsv}. It remains to show that if S_1 is any global dominating set different from S, then $|S_1|+m(F_{nsv}-S_1)\geq n+2$. Since $m(F_{nsv}-S_1)=1$, it is clear that S is minimum. If we consider $m(F_{nsv}-S_1)=0$, then $|S_1|=2n+1$ and $|S_1|+m(F_{nsv}-S_1)=2n+1>|S|+m(F_{nsv}-S)$, also if $m(F_{nsv}-S_1)\geq 1$, then $|S_1|+m(F_{nsv}-S_1)\geq |S|+m(F_{nsv}-S)$. Hence, for any global dominating set S_1,

$$|S_1|+m(F_{nsv}-S_1)\geq n+2. \tag{2.15}$$

Then from (2.14) and (2.15), $GDI(F_{nsv})=n+2$.
Case 2: Let the switched vertex be v_1 as in the Figure 3. Since $N(v)=\{v_2,v_3,...,v_{2n}\}$, we consider $S=\{v,v_1,v_2\}$, a global dominating set of F_{nsv_1} and $m(F_{nsv_1}-S)=2$. Therefore

$$GDI(F_{nsv_1})\leq |S|+m(F_{nsv_1}-S)=5. \tag{2.16}$$

To show that the number $|S| + m(F_{nsv_1} - S)$ is minimum, we must show that $|S|$ is minimum and $m(F_{nsv_1} - S)$ is also minimum. Since v_2 is adjacent to the vertices $\{v_1, v_3, v_4, ..., v_{2n}\}$ in $\overline{F_{nsv_1}}$, if v_2 is removed from set S, then $\{v_3, v_4, ..., v_{2n}\}$, will not be dominated by any vertex in $\overline{F_{nsv_1}}$. Then $S' = \{v, v_1\}$ is not dominating set of $\overline{F_{nsv_1}}$, so S' is not a global dominating set of F_{nsv_1}. Thus S is minimum. For $m(F_{nsv_1} - S)$, if S_1 is any global dominating set other than S, then $|S_1| + m(F_{nsv_1} - S_1) \geq 2$. If $m(F_{nsv_1} - S_1) = 1$, then $|S_1| \geq 2n$, so $|S_1| + m(F_{nv_1} - S_1) \geq 2n + 1$. If $m(F_{nv_1} - S_1) \geq 2$, it is easy $|S_1| + m(F_{nv_1} - S_1) \geq 5$. Then for any global dominating set S_1,

$$|S_1| + m(F_{nsv_1} - S_1) \geq 5. \qquad (2.17)$$

Therefore, from (2.16), and (2.17), $GDI(F_{nsv_1}) = 5$. □

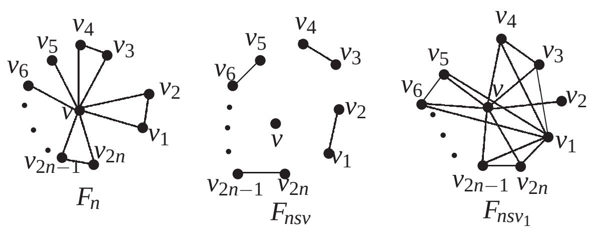

Figure 3: F_n, F_{nsv}, F_{nsv_1}

3 Conclusion

In this paper, we introduced the concept of global domination integrity of graphs, we have obtained the bounds and some properties for global domination integrity of graphs. Relation between global domination integrity and some parameters are established. The global domination integrity of several other families of graphs is an open problem.

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