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Harmonious Coloring of Middle Graph of Quadrilateral Snake Graphs

Akhlak Mansuri[1,*] *and R. S. Chandel*[2]

[1]Department of Mathematics, Govt. Girls College, Mandsaur (Madhya Pradesh), India
[2]Department of Mathematics, Govt. Geetanjali Girls College, Bhopal (Madhya Pradesh), India

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Abstract: The harmonious coloring is the kind of proper vertex coloring in which each edge has different color pair and the smallest number of colors required for such coloring is called harmonious chromatic number and it is denoted by $\chi_H(G)$. In this article, we study about the harmonious coloring and obtain the harmonious chromatic number of the middle graph of quadrilateral and alternate quadrilateral snake graphs.

Keywords: harmonious coloring, harmonious chromatic number, middle graph, quadrilateral snake graphs and alternate quadrilateral snake graphs

1 Introduction

The harmonious coloring [6,7,9,18,19] is the kind of proper vertex coloring in which each edge of a graph G has different color pair that is no two edges share the same color pair. The harmonious chromatic number $\chi_H(G)$, is the smallest number of colors required for harmonious coloring. The middle graph of a given graph G is formed by subdividing each edge exactly once and connecting these newly obtained vertices of adjacent edges of G. A quadrilateral snake graph Q_n is obtained from a path $u_1,u_2,\ldots,u_n$ by joining u_i and u_{i+1} to new vertices v_i and w_i respectively and adding edges v_iw_i for $(1 \le i \le n-1)$ such that every edge of a path is replaced by a cycle C_4. We take the following definitions from [1, 3,4,9,11,12,13,14,15,16,17]: quadrilateral snake graph, double quadrilateral snake graph, triple quadrilateral snake graph, alternate quadrilateral snake graph, double alternate quadrilateral snake graph and triple alternate quadrilateral snake graph. In this paper we obtain the harmonious chromatic number of middle graph of these above mentioned snake graphs as well as k-quadrilateral snake graph (consist of k quadrilateral snake graphs with a common path) and k-alternate quadrilateral snake graph (consist of k alternate quadrilateral snake graphs alternatively with a common path).The harmonious coloring has variety of application in communication networks like transportation networks, computer networks, airway network system, satellite navigation system, radio navigation system etc.

2 Definitions

Definition 1.A quadrilateral snake Q_n is obtained from a path $u_1,u_2,\ldots,u_n$ by joining u_i and u_{i+1} to new vertices v_i and w_i respectively and adding edges v_iw_i for $(1 \le i \le n-1)$. That is every edge of a path is replaced by a cycle C_4.

Definition 2.A double quadrilateral snake $D(Q_n)$ consists of two quadrilateral snakes that have a common path. That is, a double quadrilateral snake is obtained from a path $u_1,u_2,\ldots,u_n$ by joining u_i and u_{i+1} to a new vertex v_i, x_i and w_i, y_i and then joining v_i and w_i, x_i and y_i for for $(1 \le i \le n-1)$.

Definition 3.A triple quadrilateral snake $T(Q_n)$ is obtained from a path $u_1,u_2,\ldots,u_n$ by joining u_i and u_{i+1} to a new vertex v_i, x_i, p_i and w_i, y_i, q_i and then joining v_i and w_i, x_i and y_i, p_i and q_i for $(1 \le i \le n-1)$.

Definition 4.A alternate quadrilateral snake AQ_n is obtained from a path $u_1,u_2,\ldots,u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i and w_i respectively and

* Corresponding author e-mail: akhlaakmansuri@gmail.com

adding edges $v_i w_i$ for $(1 \le i \le n-1)$. That is every alternate edge of a path is replaced by a cycle C_4.

Definition 5.A double alternate quadrilateral snake $D(AQ_n)$ is obtained from a path $u_1, u_2, ..., u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i and w_i, y_i respectively and then joining v_i and w_i, x_i and y_i for $(1 \le i \le n-1)$.

Definition 6.A triple alternate quadrilateral snake $T(AQ_n)$ is obtained from a path $u_1, u_2, ..., u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i, p_i and w_i, y_i, q_i respectively and then joining v_i and w_i, x_i and y_i, p_i and q_i for $(1 \le i \le n-1)$.

Throughout the paper we consider n as the number of vertices of the path P_n.

3 Harmonious Chromatic Number of $M(Q_n)$, $M(DQ_n)$, $M(TQ_n)$ and $M(kQ_n)$

Theorem 3.1. For the middle graph of quadrilateral snake graph Q_n, the harmonious chromatic number, $\chi_H(M(Q_n)) = 5n-2, n \ge 2$.

Proof. Let Q_n as the quadrilateral snake graph and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. For obtaining middle graph, we subdivide each edge $u_i u_{i+1}$, $u_i v_i$, $u_i w_i$ and $v_i w_i$ $(1 \le i \le n-1)$ of Q_n by the vertices e_i, e'_i, e''_i and e'''_i $(1 \le i \le n-1)$ of Q_n in the middle graph of Q_n, therefore $V(M(Q_n)) = \{u_i : 1 \le i \le n\} \cup \{v_i, w_i, e_i, e'_i, e''_i, e'''_i : 1 \le i \le n-1\}$. Now coloring the vertices of $M(Q_n)$ as follows: define $c : V(M(Q_n)) \longrightarrow \{1,2,3,...,5n-2\}$ where $n \ge 2$ by $c(e'_i) = 2i-1$ for $(1 \le i \le n-1)$, $c(e'''_i) = 2i$, $c(e''_i) = 2n-2+i$, $c(e_i) = 3n-3+i$, $c(v_i) = 4n-3$, $c(w_i) = 4n-2$ for $(1 \le i \le n-1)$ and $c(u_i) = 4n-2+i$ for $(1 \le i \le n)$. **Claim 1**: c is proper; from above each

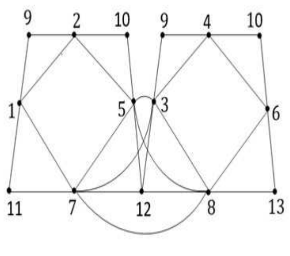

Fig. 1: $M(Q_3)$ with harmonious coloring, $\chi_H(M(Q_3)) = 13$.

$c(e_i)$, $c(e'_i)$ $c(e''_i)$, $c(e'''_i)$ and its neighbors are assigned by different colors that is $c(e_i) \ne c(e'_i) \ne c(e''_i) \ne c(e'''_i)$, although $c(v_i) = c(v_{i+1})$ and $c(w_i) = c(w_{i+1})$, but these vertices are at least at a distance 2. Therefore, it is proper. **Claim 2**: c is harmonious; it is obvious that no two edges share the same color pair and the same colored vertices are at least at a distance 3. Therefore, it is harmonious. **Claim 3**: c is minimum; all the vertices are colored by $5n-2$ colors, if we repeat (assign) any color on any vertex from these $5n-2$ colors, color pairs will be repeated which leads to contradict the harmonious coloring, therefore it is minimum. Hence $\chi_H(M(Q_n)) = 5n-2$. Fig. 1 shows the middle graph of Q_n with harmonious coloring.

Theorem 3.2. For the middle graph of double quadrilateral snake graph DQ_n, the harmonious chromatic number, $\chi_H(M(DQ_n)) = 8n-5, n \ge 2$

Proof. Let TQ_n as the double quadrilateral snake graph and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. Now we obtain the middle graph as discussed in Theorem 1, therefore $V(M(DQ_n)) = \{u_i : 1 \le i \le n\} \cup \{v_i, w_i, x_i, y_i, e'_i, e''_i, e_i, l'_i, l''_i, m'_i, m''_i : 1 \le i \le n-1\}$. Now coloring the vertices of $M(DQ_n)$ as follows: define $c : V(M(DQ_n)) \longrightarrow \{1,2,3,...,8n-5\}$ where $n \ge 2$ by $c(l'_i) = 2i-1$, $c(l''_i) = 2i$, $c(m'_i) = 2n-3+2i$, $c(m''_i) = 2n-2+2i$, $c(e_i) = 4n-4+i$, $c(e'_i) = 5n-5+i$, $c(e''_i) = 6n-6+i$, $c(v_i) = 7n-6 = c(x_i), c(w_i) = 7n-5 = c(y_i)$ for $(1 \le i \le n-1)$ and $c(u_i) = 7n-5+i$ for $(1 \le i \le n)$. To prove c is harmonious and minimum, it can be proved as described in Theorem 3.1. Fig. 2 shows the harmonious coloring for $M(DQ_3)$.

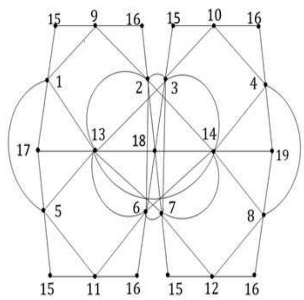

Fig. 2: $M(DQ_3)$ with harmonious coloring, $\chi_H(M(DQ_3)) = 19$.

Theorem 3.3. For the middle graph of triple quadrilateral snake graph TQ_n, the harmonious chromatic number, $\chi_H(M(TQ_n)) = 11n-8, n \ge 2$.

Proof. Let TQ_n as the triple quadrilateral snake graph and P_n as the path graph with n vertices $u_1, u_2, ..., u_n$. Now we obtain the middle graph as discussed in Theorem 1, therefore $V(M(TQ_n)) = \{u_i : 1 \le i \le n\} \cup \{v_i, w_i, x_i, y_i, p_i, q_i, e_i, e'_i, e''_i, e'''_i, l'_i, l''_i, m'_i, m''_i, z'_i, z''_i : 1 \le i \le n-1\}$. Now coloring the vertices of $M(TQ_n)$ as follows: define $c : V(M(TQ_n)) \longrightarrow \{1,2,3,...,11n-8\}$ where $n \ge 2$ by $c(l'_i) = 2i-1$, $c(l''_i) = 2i$, $c(m'_i) = 2n+2i-3$, $c(m''_i) = 2n-2+2i$, $c(z'_i) = 4n+2i-5$, $c(z''_i) = 4n-4+2i$,

$c(e_i) = 6n-6+i$, $c(e'_i) = 7n-7+i$, $c(e''_i) = 8n-8+i$, $c(e'''_i) = 9n-9+i$, $c(v_i) = c(x_i) = c(p_i) = 10n-9$, $c(w_i) = c(y_i) = c(q_i) = 10n-8$ for $(1 \le i \le n-1)$ and $c(u_i) = 10n-8+i$ for $(1 \le i \le n)$. To prove c is harmonious and minimum, it can be proved as described in Theorem 1. Fig. 3 shows the harmonious coloring for $M(TQ_3)$.

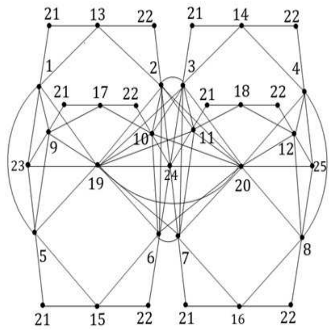

Fig. 3: $M(TQ_3)$ with harmonious coloring, $\chi_H(M(TQ_3)) = 25$.

4 Harmonious Chromatic Number of Middle Graph of k-Quadrilateral Snake Graph

Theorem 4.1. For the middle graph of k-quadrilateral snake graph kQ_n, the harmonious chromatic number, $\chi_H(M(kQ_n)) = (3k+2)n-(3k-1)$, $n \ge 2$.

Proof. From theorems 3.1, 3.2 and 3.3, it is easy to conclude that the harmonious chromatic number of middle graph of k-quadrilateral snake graph is $(3k+2)n-(3k-1)$.

5 Harmonious Chromatic Number of $M(AQ_n)$, $M(DAQ_n)$, $M(TAQ_n)$ and $M(k(AQ_n))$

Theorem 5.1. For middle graph of alternate quadrilateral snake graph AQ_n, the harmonious chromatic number, $\chi_H(M(AQ_n)) = \frac{7n}{2}+1$, n is even and ≥ 4.

Proof. Let AQ_n as the alternate quadrilateral snake graph and P_n as the path graph with n vertices $u_1, u_2, \ldots, u_n$. Now we obtain the middle graph as discussed in Theorem 1, therefore $V(M(AQ_n)) = \{u_i : 1 \le i \le n\} \cup \{v_i, w_i, e'_i, l'_i, l''_i : (1 \le i \le \frac{n}{2})\} \cup \{e_i : (1 \le j \le n-1)\}$. Now coloring the vertices of $M(AQ_n)$ as follows: define $c : V(M(AQ_n)) \longrightarrow \{1,2,3,\ldots,\frac{7n}{2}+1\}$ where $n \ge 4$ by $c(l'_i) = 2i-1$, $c(l''_i) = 2i$, $c(e_i) = \frac{3n}{2}+i$ for $(1 \le i \le n-1)$, $c(u_i) = \frac{5n}{2}-1+i$ for $(1 \le i \le n)$, $c(e'_i) = n+i$, $c(v_i) = \frac{7n}{2}$, $c(w_i) = \frac{7n}{2}+1$ for $(1 \le i \le \frac{n}{2})$. To prove c is harmonious and minimum, proceed theorem 3.1. Fig. 4 shows the middle graph of AQ_n with harmonious coloring.

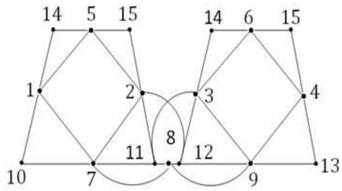

Fig. 4: $M(AQ_4)$ with harmonious coloring, $\chi_H(M(AQ_4)) = 15$.

Theorem 5.2. For the middle graph of double alternate quadrilateral snake graph $D(AQ_n)$, the harmonious chromatic number, $\chi_H M(D(AQ_n)) = 5n+1$, n is even and ≥ 4.

Proof. Let $D(AQ_n)$ as the double alternate quadrilateral snake graph and P_n as the path graph with n vertices $u_1, u_2, \ldots, u_n$. Now we obtain the middle graph as discussed in Theorem 1, therefore $V(M(D(AQ_n))) = \{u_i : 1 \le i \le n\} \cup \{v_i, w_i, x_i, y_i, e'_i, e''_i, l'_i, l''_i, m'_i, m''_i : (1 \le i \le \frac{n}{2})\} \cup \{e_i : (1 \le i \le n-1)\}$. Now coloring the vertices of $M(D(AQ_n))$ as follows: define $c : V(M(D(AQ_n))) \longrightarrow \{1,2,3,\ldots,5n+1\}$ where $n \ge 4$ by $c(l'_i) = 2i-1$, $c(l''_i) = 2i$, $c(m'_i) = n+2i-1$, $c(m''_i) = n+2i$, $c(e'_i) = 4n-1+i$, $c(e''_i) = \frac{9n}{2}-1+i$, $c(v_i) = c(x_i) = 5n$, $c(w_i) = c(y_i) = 5n+1$ for $(1 \le i \le \frac{n}{2})$, $c(e_i) = 2n+i$ for $(1 \le i \le n-1)$ and $c(u_i) = 3n-1+i$ for $(1 \le i \le n)$. To prove c is harmonious and minimum, proceed Theorem 3.1. Fig. 5 shows the middle graph of $D(AQ_4)$ with harmonious coloring.

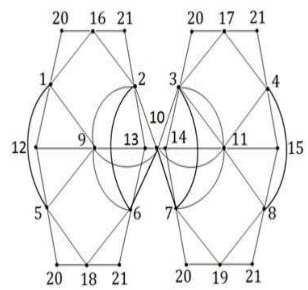

Fig. 5: $M(D(AQ_4))$ with harmonious coloring, $\chi_H(M(D(AQ_4))) = 21$.

Theorem 5.3. For the middle graph of triple alternate quadrilateral snake graph $T(AQ_n)$, the harmonious chromatic number, $\chi_H(M(T(AQ_n))) = \frac{13n}{2}+1$, n is even and ≥ 4.

Proof. Let $T(AQ_n)$ as the triple alternate quadrilateral snake graph and P_n as the path graph with n vertices

$u_1, u_2, ..., u_n$. Now we obtain the middle graph as discussed in Theorem 3.1, therefore $V(M(T(AQ_n))) = \{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i, x_i, y_i, p_i, q_i, e_i, e'_i, e''_i, e'''_i, l_i, l'_i, l''_i, m_i, m'_i, m''_i : (1 \leq i \leq \frac{n}{2})\}$. Now coloring the vertices of $M(T(AQ_n))$ as follows: define $c : V(M(T(AQ_n))) \longrightarrow \{1, 2, 3, ..., \frac{13n}{2} + 1\}$ where $n \geq 4$ by $c(l'_i) = 2i - 1$, $c(l''_i) = 2i$, $c(m'_i) = n + 2i - 1$, $c(m''_i) = n + 2i$, $c(l_i) = 2n + 2i - 1$, $c(m_i) = 2n + 2i$ for $(1 \leq i \leq \frac{n}{2})$. $c(e'_i) = 3n + i$ for $(1 \leq i \leq n - 1)$, $c(e'''_i) = 4n + i - 1$, $c(e_i) = \frac{9n}{2} - 1 + i$, $c(e''_i) = 5n - 1 + i$ for $(1 \leq i \leq \frac{n}{2})$. $c(u_i) = \frac{11n}{2} - 1 + i$ for $(1 \leq i \leq n)$ and $c(v_i) = c(x_i) = c(p_i) = \frac{13n}{2}$, $c(w_i) = c(y_i) = c(q_i) = \frac{13n}{2} + 1$ for $(1 \leq i \leq \frac{n}{2})$. To prove c is harmonious and minimum, it can be proved as described in Theorem 3.1. Fig. 6 shows the middle graph of $T(AQ_4)$ with harmonious coloring.

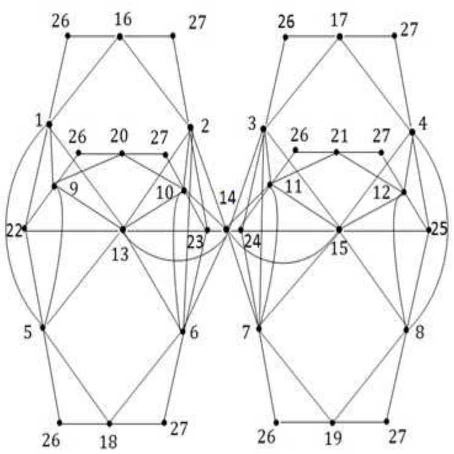

Fig. 6: $M(T(AQ_4))$ with harmonious coloring, $\chi_H(M(T(AQ_4))) = 27$.

6 Harmonious Chromatic Number of Middle Graph of k-Alternate Quadrilateral Snake Graph

Theorem 6.1. For the middle graph of k-alternate quadrilateral snake graph kAQ_n, the harmonious chromatic number, $\chi_H(M(kAQ_n)) = (\frac{3k}{2} + 2)n + 1, n \geq 4$.
Proof. From theorems 5.1, 5.2 and 5.3 it is easy to conclude that the harmonious chromatic number of middle graph of k-quadrilateral snake graph is $(\frac{3k}{2} + 2)n + 1$.

7 Conclusion

In this paper we obtained the harmonious chromatic number of the middle graph of quadrilateral and alternate quadrilateral snake graphs, that is $\chi_H M(Q_n) = 5n - 4$, $\chi_H M(DQ_n) = 8n - 7$, $\chi_H M(TQ_n) = 11n - 10$, $\chi_H M(kQ_n) = (3k + 2)n - (3k + 1)$, $\chi_H M(AQ_n) = \frac{7n}{2} - 1$, $\chi_H M(D(AQ_n)) = 5n - 1$, $\chi_H M(T(AQ_n)) = \frac{13n}{2} - 1$.$\chi_H M(T(AQ_n)) = (\frac{3k}{2} + 2)n - 1$.

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