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An Efficient Analytical Method for Solving Singular Initial Value Problems of Nonlinear Systems

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Abstract: The aim of the present analysis is to apply a relatively recent method, the residual-power series method (RPSM), in order to obtain efficient analytical numerical solutions for a class of nonlinear systems of initial value problems with finitely many singularities. The solution methodology provided the analytical solutions in terms of a rapidly convergent series with easily computable components. This novel approach possesses main advantage as compared to other exiting methods; it reproduces exact form when the solution is polynomial without linearization or perturbation; it can be applied without any limitation on the nature of the problem, type of classification, and the number of mesh points. Numerical experiments are discussed quantitatively to illustrate the theoretical statements and to show potentiality, superiority, and applicability of the proposed technique for solving such nonlinear singular system of differential equations. The results demonstrate reliability and efficiency of the technique developed.

Keywords: Residual-power series method, Nonlinear systems of differential equations, Singular initial value problems, Numerical analytical solutions

1 Introduction

The analysis of nonlinear system of singular initial value problems (IVPs) with interpretation of its numerous classical applications in chemistry, mechanics, physics and astrophysics including the thermal behaviors of a spherical cloud of gas, isothermal gas spheres, theories of thermionic currents, electro-hydrodynamics, and stellar stability structure, etc. [1,2,3,4,5,6,7], needs a constantly growing use of suitable method of mathematical models to expand the ability to translate mathematical equations into concrete conclusions concerning the phenomenological analysis, compute the best solution, and describe its evolution in time and space. This configuration gives us a strong motivation to search for methods in order to solve these systems with consideration the difficulties produced by singularities. Unfortunately, investigation about system of singular IVPs is scarce especially discussion on finding solution. Indeed, in most cases, these systems are almost impossible to be solved analytically, so they would be attacked using various approximate and numerical methods with great interest by several authors. Therefore, this class of singular systems takes a central seat in the mathematical modeling literature.

The residual-power series method (RPSM) is a numerical as well as analytical method for solving many types of systems of ordinary and partial differential equations. The method provides the solution in terms of convergent power series with easily computable components, whereas the analytical approximate solution should be constructed in the form of a polynomial [8,9,10]. In addition to all, the RPSM is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders and suited for the linear problems whilst the RPSM is an alternative procedure for obtaining analytic Taylor series solution of systems of singular IVPs. By concept of residual error, we get a series solution in practice as well a truncated series solution. This series solution does not

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exhibit the real behaviors of the problem but gives a good approximation to the true solution in the given region.

The attention in this paper is to obtain a symbolic approximate RPS solution for system of singular IVPs of the following form:

$$y_1'(x)+\frac{1}{p_1(x)}\sum_{j=1}^{n}\left(a_{1,,j}f_{1,j}\left(y_j(x)\right)\right)+F_1\left(x,\vec{y}(x)\right)=0,$$
$$y_2'(x)+\frac{1}{p_2(x)}\sum_{j=1}^{n}\left(a_{2,,j}f_{2,j}\left(y_j(x)\right)\right)+F_2\left(x,\vec{y}(x)\right)=0,$$
$$\vdots$$
$$y_n'(x)+\frac{1}{p_n(x)}\sum_{j=1}^{n}\left(a_{n,j}f_{n,j}\left(y_j(x)\right)\right)+F_n\left(x,\vec{y}(x)\right)=0, \tag{1}$$

subject to the initial conditions

$$\vec{y}\,(x_0)=\vec{\alpha}, \tag{2}$$

where $\vec{y}(x) = (y_1(x), y_2(x), ..., y_n(x))$, $x \in [x_0, x_0+b]$, $x_0, b \in \mathbb{R}$ with $b > 0$, $f_{i,j} : \mathbb{R} \to \mathbb{R}$, $F_i : [x_0, x_0+b] \times \mathbb{R}^n \to \mathbb{R}$, $i, j = 1, 2, ..., n$, are linear or nonlinear analytical functions, $p_i(x)$, $i = 1, 2, ..., n$, are also analytical functions on the given interval, $\vec{\alpha} = (\alpha_1(x), \alpha_2(x), ..., \alpha_n(x))$ such that $\alpha_i \in \mathbb{R}$ for $i = 1, 2, ..., n$, and $y_i(x), i = 1, 2, ..., n$, are unknown functions of independent variable x on $[x_0, x_0+b]$ to be determined. Consequently, we assume that $y_i(x)$ are analytical functions and the singular system (1) and (2) has a unique analytic solution on the given interval.

In literature, there are only few researches dealing with the approximate solutions for systems of singular IVPs; For instance, the authors in [11] have developed the Adomian decomposition method (ADM) to get approximate solutions for singular linear system of transistor circuits. In [12], the authors have provided the reproducing kernel Hilbert space method (RKHSM) to further investigation about numerical solutions for singular second-order initial/boundary value problems (IBVPs). Also in [13], the authors have introduced the variational iteration method (VIM) to solve singular perturbation IVPs with delays conditions. In contrast, the existence theorem of solutions of IVPs for nonlinear singular discrete systems have been established in [14] by monotone iterative technique combined with the method of upper and lower solutions. For a comprehensive introduction in this field, we refer to [15,16,17,18,19,20,21] in order to know more details about singular IVPs including their history, applications, method of solutions, and so forth. But on the other aspects as well, the applications of other versions of series solutions to linear and nonlinear problems can be found in [22,23,24,25], and for numerical solvability of different categories of differential equations one can consult the reference [26,27,28,29,30,31,32].

Generally, the solutions of such systems of singular IVPs that are obtained using an existed analytical methods are usually very difficult to be exist in this manner, so it is required to select an effective suitable numerical method to deal with. In this paper, we are interested in the series solutions of strongly linear and nonlinear singular systems based on the RPSM. This method is efficient and easy to use for solving nonlinear systems of singular IVPs without linearization, perturbation, or discretization. Furthermore, the proposed method has the following characteristics. Firstly, the RPSM obtains Taylor expansion of the solution as well as the exact solution is available whenever the solution is polynomial. Moreover, the solutions and all their derivatives are applicable for each arbitrary point in the given interval. Secondly, it does not require any modification while switching from first order up to higher order; as a result, the RPSM can be applied directly to the given problem by choosing an appropriate value for the initial guess approximation. Thirdly, it is not effected by computation round off errors and one is not faced with necessity of large computer memory. Finally, it needs small computational requirements with high precision and less time.

The outline of this paper is organized as follows. In the next section, we present basic facts, notations, formulation and preliminary results related to the RPSM for system of singular IVPs. In Section 3, The validity together with capability of the modified technique is verified through illustrative examples. The approximate solutions are found in closed form of a convergent series with easily computable components, which are coincides with exact solutions. This article ends in Section 4 with some concluding remarks.

2 Formulation of solution for system of singular IVPs

In this section, we employ a new technique based on the RPSM to find out a series solution for system of singular IVPs associated with a class of initial conditions. First, we begin with formulate and analyze of the proposed method in relation to solve such singular systems. Afterwards, a convergence theorem is presented in order to capture the behavior of solutions.

The RPSM consists of expressing a solution of singular system (1) and (2) as a power series expansion about the initial point $x = x_0$. To achieve our goal, we suppose that these solutions take the following form

$$y_i(x) = \sum_{m=0}^{\infty} y_{i,m}(x), i = 1, 2, ..., n,$$

where $y_{i,m}(x) = c_{i,m}(x - x_0)^m$ are terms of approximations.

Obviously, when $m = 0$, $y_{i,0}(x)$ satisfy initial conditions (2) such as

$y_{i,0}(x) = y_i(x_0) = c_{i,0}, i = 1,2,\ldots,n$, where $y_{i,0}(x)$ are called the initial guesses of approximations of $y_i(x), i = 1,2,\ldots,n$. Thus, after choosing these initial guesses, we can calculate the others terms $y_{i,m}(x)$ for $m = 1,2,\ldots$ as well as approximate the solutions $y_i(x)$ for singular system (1) and (2) according to the following *kth* truncated series

$$y_i^k(x) = \sum_{m=0}^{k} c_{i,m}\left(x-x_0\right)^m, i = 1,2,...,n. \quad (3)$$

Prior to applying the RPSM, we have to rewrite the singular system (1) and (2), to facilitate, as follows

$$\begin{aligned} &p_i(x)y_i'(x)+\sum_{j=1}^{n}\left(a_{i,,j}\; f_{i,j}\left(y_j(x)\right)\right) \\ &+p_i(x)F_i\left(x,\vec{y}(x)\right)=0. \end{aligned} \quad (4)$$

As a consequence to substitute the *kth* truncated series $y_i^k(x), i = 1,2,...,n$, in equation (4), we have

$$\begin{aligned} Res_i^k(x) = p_i(x)&\sum_{m=1}^{k} mc_{i,m}\left(x-x_0\right)^{m-1} \\ &+\sum_{j=1}^{n} a_{i,,j}f_{i,j}\left(\sum_{m=0}^{k} c_{i,m}\left(x-x_0\right)^m\right) \\ &+p_i(x)F_i\left(x,\sum_{m=0}^{k}\vec{c_m}\left(x-x_0\right)^m\right), \end{aligned} \quad (5)$$

where $\vec{c_m} = (c_{1,m},c_{2,m},...,c_{n,m}), m = 0,1,...,k$, in which $Res_i^k(x), i = 1,2,...,n$, are called the the *kth* residual functions.

Furthermore, the ∞*th* residual functions $Res_i^\infty(x)$ which is defined as $Res_i^\infty(x) = \lim_{k\to\infty} Res_i^k(x)$, $i = 1,2,...,n$. Obviously that $Res_i^\infty(x) = 0$ for each x in $[x_0, x_0+b]$, which are infinitely differentiable functions at $x = x_0$. Specifically, $\frac{d^{k-1}}{dx^{k-1}}Res_i^\infty(x_0) = \frac{d^{k-1}}{dx^{k-1}}Res_i^k(x_0) = 0$, whereas this relation is a fundamental rule in RPSM and its applications.

Throughout this paper, we will use the following notations of matrices in order to simplify and reduce the computations:

$$A_1=\begin{bmatrix} \zeta_{1,1}^{(1)} & \zeta_{1,2}^{(1)} & \cdots & \zeta_{1,n}^{(1)} \\ \zeta_{2,1}^{(1)} & \zeta_{2,2}^{(1)} & \cdots & \zeta_{2,n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n,1}^{(1)} & \zeta_{2,n}^{(1)} & \cdots & \zeta_{n,n}^{(1)} \end{bmatrix},$$

where $\zeta_{i,j}^{(1)} = a_{i,j}\frac{d}{dc_{j,0}}f_{i,j}(c_{j,0}), i,j = 1,2,...,n,$

$$P_s=\begin{bmatrix} sp_1'(x_0) & 0 & \cdots & 0 \\ 0 & sp_2'(x_0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & sp_n'(x_0) \end{bmatrix},$$

$$C_s=\begin{bmatrix} c_{1,s} \\ c_{2,s} \\ \vdots \\ c_{n,s} \end{bmatrix}, \text{ and } Res_s=\begin{bmatrix} Res_1^s \\ Res_2^s \\ \vdots \\ Res_n^s \end{bmatrix}, \text{ for } s = 1,2,3,....$$

Now, in order to obtain the first approximate solutions $y_i^1(x), i = 1,2,...,n$, we set $k = 1$ and $y_i^1(x) = \sum_{m=0}^{1} c_{i,m}\left(x-x_0\right)^m$. Then, we differentiate both sides of equation (5) with respect to x and substitute $x = x_0$ to get that

$$Res_i^1(x_0) = \left(A_1+P_1\right)C_1 + H_1, \quad (6)$$

where the matrix B_1 is given by

$$H_1=\begin{bmatrix} p_1'(x_0)F_1(x_0,\vec{y}(x_0)) \\ p_2'(x_0)F_2(x_0,\vec{y}(x_0)) \\ \vdots \\ p_n'(x_0)F_n(x_0,\vec{y}(x_0)) \end{bmatrix}.$$

Based on the fact $Res_i^\infty(x_0) = Res_i^1(x_0) = 0$, it is to be noted that equation (6) consists of n linear algebraic equations associated to n variables, which can be solved directly by using Mathcad 14 software package. That is to say,

$$\begin{aligned} C_1 &= -\left(A_1+P_1\right)^{-1}H_1 :\overset{def}{=} \vec{\beta^{(1)}} \\ &= \left(\beta_1^{(1)},\beta_2^{(1)},...,\beta_n^{(1)}\right); \end{aligned}$$

Thus, the first approximation of singular system (1) and (2) can be written as

$$y_i^1(x) = y_i(x_0) + \beta_i^{(1)}\left(x-x_0\right), i = 1,2,...,n.$$

Similarly, in order to obtain the second approximate solutions $y_i^2(x), i = 1,2,...,n$, we set $k = 2$ and $y_i^2(x) = \sum_{m=0}^{2} c_{i,m}\left(x-x_0\right)^m$. Then, we differentiate both sides of equation (5) twice with respect to x and substitute $x = x_0$ to get the following result:

$$\begin{aligned} C_2 &= -\frac{1}{2}\left(A_1+2P_2\right)^{-1}\left(A_2+H_2\right) :\overset{def}{=} \vec{\beta^{(2)}} \\ &= \left(\beta_1^{(2)},\beta_2^{(2)},...,\beta_n^{(2)}\right), \end{aligned}$$

where the matrices A_2 and H_2 are given, respectively, as

$$A_2=\begin{bmatrix} \zeta_{1,1}^{(2)} & \zeta_{1,2}^{(2)} & \cdots & \zeta_{1,n}^{(2)} \\ \zeta_{2,1}^{(2)} & \zeta_{2,2}^{(2)} & \cdots & \zeta_{2,n}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{n,1}^{(2)} & \zeta_{2,n}^{(2)} & \cdots & \zeta_{n,n}^{(2)} \end{bmatrix},$$

where $\zeta_{i,j}^{(2)} = \left(\beta_j^{(1)}\right)^2 a_{i,j} \frac{d^2}{d(c_{j,0})^2} f_{i,j}(c_{j,0})$, $i,j = 1,2,...,n$, and

$$H_2 = \begin{bmatrix} G_{11}(x_0, \overrightarrow{y}(x_0)) + p_1''(x_0) G_{21}(x_0, \overrightarrow{y}(x_0)) \\ G_{12}(x_0, \overrightarrow{y}(x_0)) + p_2''(x_0) G_{22}(x_0, \overrightarrow{y}(x_0)) \\ \vdots \\ G_{1n}(x_0, \overrightarrow{y}(x_0)) + p_n''(x_0) G_{2n}(x_0, \overrightarrow{y}(x_0)) \end{bmatrix},$$

where $G_{1i}(x_0, \overrightarrow{y}(x_0)) = 2p_i'(x_0) \frac{d}{dx_0} F_i(x_0, \overrightarrow{y}(x_0)) + \sum_{j=1}^{n} c_{j,1} \frac{d}{dc_{j,0}} F_i(x_0, \overrightarrow{y}(x_0))$,
$G_{2i}(x_0, \overrightarrow{y}(x_0)) = c_{i,1} - F_i(x_0, \overrightarrow{y}(x_0)), i = 1,2,...,n.$

The above result is valid due to the fact that $\frac{d}{dx} Res_i^2(x_0) = 0, i = 1,2,...,n$. Hence, the second approximate solutions of singular system (1) and (2) can be written as

$$y_i^2(x) = y_i(x_0) + \beta_i^{(1)}(x - x_0) + \beta_i^{(2)}(x - x_0)^2. \quad (7)$$

This procedure can be repeated till the arbitrary order coefficients of the RPS solutions for the singular system (1) and (2) are obtained. Additionally, higher accuracy can be achieved by evaluating more components of the solution, that is to say, one can choose large k in the truncation series (3). Moreover, as a matter of fact, the next theorem shows convergence of the RPSM to capture the behavior of the solutions.

Theorem 1. *Suppose that $y_i(x), i = 1,2,...,n$, are the exact solutions for singular system (1) and (2). Then, the approximate solutions obtained by the RPSM are in fact the Taylor expansion of $y_i(x)$.*

Proof. Let the approximate solutions for singular system (1) and (2) be taken the following form

$$\widetilde{y}_i(x) = c_{i,0} + c_{i,1}(x - x_0) + c_{i,2}(x - x_0)^2 + ..., \quad (8)$$

for $i = 1,2,...,n$. Now, in order to prove the theorem, it is enough to show that the coefficients $c_{i,m}$ in equation (8) will be as follows

$$c_{i,m} = \frac{1}{m!} y_i^{(m)}(x_0), i = 1,2,...,n. \quad (9)$$

for each $m = 0,1,...,$ where $y_i(x), i = 1,2,...,n$, are the exact solutions for singular system (1) and (2). Clear that for $m = 0$, the initial conditions (2) gives

$$c_{i,0} = y_i(x_0), i = 1,2,...,n. \quad (10)$$

Consequently, for $m = 1$, differentiate both sides of equation (4) with respect to x to obtain

$$\begin{aligned} & p_i(x) y_i''(x) + p_i'(x) y_i'(x) \\ & + \sum_{j=1}^{n} a_{i,,j} y_j'(x) \frac{d}{dy_j} f_{i,j}(y_i(x)) \\ & + p_i'(x) F_i(x, \overrightarrow{y}(x)) \\ & + p_i(x) \begin{bmatrix} \frac{\partial}{\partial x} F_i(x, \overrightarrow{y}(x)) - \\ \sum_{j=1}^{n} y_j'(x) \frac{\partial}{\partial y_j} F_i(x, \overrightarrow{y}(x)) \end{bmatrix} = 0, \end{aligned} \quad (11)$$

and then set $x = x_0$ in equation (11) to holds that

$$\begin{aligned} & p_i'(x_0) y_i'(x_0) + \sum_{j=1}^{n} a_{i,,j} y_j'(x_0) \frac{d}{dy_j} f_{i,j}(y_i(x_0)) \\ & + p_i'(x_0) F_i(x, \overrightarrow{y}(x_0)) = 0. \end{aligned} \quad (12)$$

On the other hand, from equations (8) and (10), for $i = 1,2,...,n$, one can get

$$\widetilde{y}_i(x) = y_i(x_0) + c_{i,1}(x - x_0) + c_{i,2}(x - x_0)^2 + ... \quad (13)$$

By substituting equation (13) into equation (4), differentiating both sides of resulting equation with respect to x, and then setting $x = x_0$, it follows that

$$\begin{aligned} & c_{i,1} p_i'(x_0) + \sum_{j=1}^{n} a_{i,,j} c_{j,1} \frac{d}{dy_j} f_{i,j}(y_i(x_0)) \\ & + p_i'(x_0) F_i(x, \overrightarrow{y}(x_0)) = 0. \end{aligned} \quad (14)$$

By comparison equation (12) with equation (14), we can conclude that $c_{i,1} = y_i'(x_0)$. Thus, according to equation (13), the approximate solution of singular system (1) and (2) can be written as

$$\widetilde{y}_i(x) = y_i(x_0) + y_i'(x_0)(x - x_0) + c_{i,2}(x - x_0)^2 + ... \quad (15)$$

Correspondingly, for $m = 2$, differentiating both sides of equation (4) twice with respect to x as well as setting $x = x_0$ yields that

$$\begin{aligned} & 2p_i(x_0) y_i''(x_0) + p_i''(x_0) y_i'(x_0) \\ & + \sum_{j=1}^{n} a_{i,j} \begin{bmatrix} y_j'(x_0) \frac{d}{dy_j} f_{i,j}(y_i(x_0)) \\ + \left(y_j'(x_0)\right)^2 \frac{d^2}{dy_j^2} f_{i,j}(y_i(x_0)) \end{bmatrix} \\ & + 2p_i'(x_0) \begin{bmatrix} \frac{\partial}{\partial x} F_i(x_0, \overrightarrow{y}(x_0)) \\ - \sum_{j=1}^{n} y_j'(x_0) \frac{\partial}{\partial y_j} F_i(x_0, \overrightarrow{y}(x_0)) \end{bmatrix} \\ & + p_i''(x_0) F_i(x_0, \overrightarrow{y}(x_0)) = 0. \end{aligned} \quad (16)$$

In contrast, by substituting (15) into equation (4), differentiating both sides of resulting equation twice with respect to x, and then setting $x = x_0$, it follows that

$$\begin{aligned} & 4c_{i,2} p_i(x_0) + p_i''(x_0) y_i'(x_0) \\ & + \sum_{j=1}^{n} a_{i,j} \begin{bmatrix} 2c_{i,2} \frac{d}{dy_j} f_{i,j}(y_i(x_0)) \\ + \left(y_j'(x_0)\right)^2 \frac{d^2}{dy_j^2} f_{i,j}(y_i(x_0)) \end{bmatrix} \\ & + 2p_i'(x_0) \begin{bmatrix} \frac{\partial}{\partial x} F_i(x_0, \overrightarrow{y}(x_0)) \\ - \sum_{j=1}^{n} y_j'(x_0) \frac{\partial}{\partial y_j} F_i(x_0, \overrightarrow{y}(x_0)) \end{bmatrix} \\ & + p_i''(x_0) F_i(x_0, \overrightarrow{y}(x_0)) = 0. \end{aligned} \quad (17)$$

Again, by comparison equation (16) with equation (17), we can conclude that $c_{i,2} = \frac{1}{2} y_i''(x_0)$. Thus, according to equation (15), the approximate solution of singular system (1) and (2) can be written as

$$\widetilde{y}_i(x) = y_i(x_0) + y_i'(x_0)(x - x_0) + \frac{1}{2} y_i''(x_0)(x - x_0)^2 + ...$$

By continuing in above procedure, it can easily prove that $c_{i,m} = \frac{1}{m!} y_i^{(m)}(x_0), i = 1,2,...,n$, for $m = 3,4,\ldots$. Thus, the proof of the theorem is complete.

Corollary 1. *Let $y_i(x), i = 1, 2, \dots, n$, be a polynomial for some i, then the RPSM will obtain the exact solution.*

Now, it will be convenient to have a notation for the error in the approximation $y_i(x) \approx y_i^k(x)$. Accordingly, let $Rem_i^k(x)$ be the difference between $y_i(x)$ and its *kth* Taylor polynomial that obtained by the RPSM; that is,

$$Rem_i^k(x) = y_i(x) - y_i^k(x) = \sum_{m=k+1}^{\infty} \frac{1}{m!} y_i^{(m)}(x_0)(x-x_0)^m,$$

where the functions $Rem_i^k(x), i = 1, 2, ..., n$, are called the *kth* remainder for the RPS approximation of $y_i(x)$. In fact, it often happens that the remainders $Rem_i^k(x)$ become smaller and smaller, approaching zero, as k gets large.

3 Numerical results and discussion

In this section, we propose a few numerical simulations of specific examples for singular system (1) and (2) to demonstrate the accuracy and applicability of the RPSM. The method provides the solutions in terms of convergent series with easily computable components, improves the convergence of the series solution, provides immediate and visible symbolic terms of analytical solutions, as well as numerical approximate solutions to both linear and nonlinear test problems. Specifically, the solvability of the more complex second-order system of singular IVPs is discussed in last example of our test problems and the results have shown remarkable performance. The method was used in a direct way without using linearization, perturbation or restrictive assumptions. Throughout this paper, all computations are implemented by using Mathcad 14 software package.

Example 1. Consider the following first-order linear system of singular IVP:

$$\begin{aligned} & y_1'(x) + \left(\sin(x) - \tfrac{2}{x\sinh(x)}\right) y_1(x) \\ & + \left(\tfrac{1}{x\sinh(x)} - 3\cos(x)\right) y_2(x) = f_1(x), \\ & y_2'(x) - \ln(x^2+1) y_1(x) + 4\sin(e^x) y_2(x) = f_2(x), \end{aligned} \quad (18)$$

subject to the initial conditions

$$y_1(0) = 0, y_2(0) = 0, \quad (19)$$

where $f_1(x) = ((1+\sin(x))x^2 + 2x)e^x + \frac{2xe^x + \sin(x)}{\sinh(x)} - 3x\cos(x)\sin(x)$, $f_2(x) = (1 + 4x\sin(e^x))\sin(x) + x\cos(x) - x^2 e^x \ln(x^2+1)$, and $x \in [0, 2]$.

In mathematics, an expression is said to be a closed-form expression if it can be expressed analytically in terms of a finite number of certain well-known functions. Typically, these well-known functions are defined to be elementary functions. The aim now is to discover the exact closed-form solution for equations (18) and (19). Let $y_1^0(x) = 0$ and $y_2^0(x) = 0$ be the initial guesses approximations, then the *kth* truncated series about $x_0 = 0$ for equations (18) and (19) is given by

$$y_1^k(x) = \sum_{m=0}^{k} c_{1,m} x^m = c_{1,1}x + c_{1,2}x^2 + ... + c_{1,k}x^k,$$

$$y_2^k(x) = \sum_{m=0}^{k} c_{2,m} x^m = c_{2,1}x + c_{2,2}x^2 + ... + c_{2,k}x^k.$$

Using the RPS algorithm, the values of the coefficients $c_{1,m}, c_{2,m}, m = 1, 2, 3, \dots, k$, can be found by constructing the following *kth* residual functions

$$\begin{aligned} Res_1^k(x) &= x\sinh(x) \sum_{m=1}^{k} m c_{1,m} x^{m-1} + (x\sinh(x) \\ & \sin(x) - 2) \sum_{m=0}^{k} c_{1,m} x^m + (1 - 3x\sinh(x)\cos(x)) \\ & \left(\sum_{m=0}^{k} c_{2,m} x^m - x\sinh(x) f_1(x)\right), \\ Res_2^k(x) &= \sum_{m=1}^{k} m c_{2,m} x^{m-1} - \ln(x^2+1) \sum_{m=0}^{k} c_{1,m} x^m \\ & + 4\sin(e^x) \sum_{m=0}^{k} c_{2,m} x^m - f_2(x). \end{aligned} \quad (20)$$

Therefore, the 1st-order approximations of the RPS solutions according to the residual functions (20) are $y_1^1(x) = 0$ and $y_1^1(x) = 0$. That is, $c_{1,1} = 0$ and $c_{2,1} = 0$ using $\frac{d}{dx} Res_1^1(0) = 0$ and $\frac{d}{dx} Res_2^1(0) = 0$. Similarly, the 2nd-order approximations of the RPS solutions are $y_1^2(x) = x^2$ and $y_2^2(x) = x^2$. That is, $c_{1,2} = 1$ and $c_{2,2} = 1$ using $\frac{d^2}{dx^2} Res_1^2(0) = 0$ and $\frac{d^2}{dx^2} Res_2^2(0) = 0$. Consequently, the 10th-order approximations of the RPS solutions of $y_1(x)$ and $y_2(x)$ for system (18) and (19) according to the facts that $\frac{d^{10}}{dx^{10}} Res_1^{10}(0) = 0$ and $\frac{d^{10}}{dx^{10}} Res_2^{10}(0) = 0$ are given, respectively, as

$$\begin{aligned} y_1^{10}(x) &= x^2 + x^3 + \tfrac{1}{2}x^4 + \tfrac{1}{6}x^5 + \tfrac{1}{24}x^6 + \tfrac{1}{120}x^7 \\ & + \tfrac{1}{720}x^8 + \tfrac{1}{5040}x^9 + \tfrac{1}{40320}x^{10} = \sum_{j=0}^{8} \tfrac{1}{j!} x^{j+2}, \\ y_2^{10}(x) &= x^2 - \tfrac{1}{6}x^4 + \tfrac{1}{120}x^6 - \tfrac{1}{5040}x^8 + \tfrac{1}{362880}x^{10} \\ &= \sum_{j=0}^{4} (-1)^j \tfrac{1}{(2j+1)!} x^{2j+2}. \end{aligned} \quad (21)$$

Correspondingly, the general approximation forms of solutions of system (18) and (19) are given by

$$\begin{aligned} y_1(x) &= \sum_{m=1}^{\infty} c_{1,m}x^m = x^2 \sum_{j=0}^{\infty} \tfrac{1}{j!}x^j = x^2 e^x, \\ y_2(x) &= \sum_{m=1}^{\infty} c_{2,m}x^m = x \sum_{j=0}^{\infty} (-1)^j \tfrac{1}{(2j+1)!}x^{2j+1} \\ &= x\sin(x), \end{aligned} \tag{22}$$

which are compatible with the exact solutions for system (18) and (19).

Incidentally, the concept of the word "accuracy" refers to how closely a computed or measured value agrees with the true value. To show the accuracy of the present method, we report four types of error functions; The first one is called the absolute error that commonly denoted by Ext_i^k and defined as $Ext_i^k(x) = \left|y_i(x) - y_i^k(x)\right|$; The second one is called the relative error that denoted by Rel_i^k and also defined as $Rel_i^k(x) = Ext_i^k(x)/\left|y_i(x)\right|$; The third one is called the consecutive error that denoted by Con_i^k and defined as $Con_i^k(x) = \left|y_i^{k+1}(x) - y_i^k(x)\right|$; Whereas, the last one is called the residual error that denoted by $RES_i^k(x)$ and defined as follows

$$Res_i^k(x) = \left| \begin{array}{l} p_i(x)\frac{d}{dx}y_i^k(x) \\ +\sum_{j=1}^{n} a_{i,j}\left(f_{i,j}\left(y_j^k(x)\right)\right) \\ +p_i(x)F_i\left(x, \overrightarrow{y}^k(x)\right) \end{array} \right|, \tag{23}$$

where $i = 1,2,...,n$, $\overrightarrow{y}^k(x) = \left(y_1^k(x), y_2^k(x), ..., y_n^k(x)\right)$, $x \in [x_0, x_0+b]$ and $y_i^k(x)$, $i = 1,2,...,n$, are the k*th*-order approximations of the exact solution $y_i(x)$, which is obtained by the RPSM.

Let us carry out an error analysis of the proposed method for this example. In Tables 1 and 2, the exact error has been calculated for various x in the interval $[0,2]$ to measure the extent of agreement between the k*th*-order approximation of the RPS solutions when $k = 10, 15, 20$. These two tables illustrates the rapid convergence of the method by increasing the orders of approximation. It can be seen that the exact errors become smaller as the order of solutions increases, that is, as we progress through more iterations. However, the errors indicators confirm the convergence of the method with respect to the k*th*-order of the solutions. As a result, the RPSM provides us with the accurate approximate solutions of singular system (18) and (19).

*Example 2.*Consider the following first-order nonlinear system of singular IVP:

$$\begin{aligned} y_1'(x) - y_2(x) + \tfrac{1}{x}\left(y_1(x)\right)^{-2} y_2(x) &= f_1(x), \\ y_2'(x) - y_1(x) - \tfrac{1}{x}\left(y_2(x)\right)^{-2} y_1(x) &= f_2(x), \end{aligned} \tag{24}$$

subject to the initial conditions

$$y_1(0) = 1, y_2(0) = 0, \tag{25}$$

where $f_1(x) = \frac{1}{x}\sec\left(x^2\right)\tan(x^2) - (1+2x)\sin(x^2)$, $f_2(x) = \frac{1}{x}\csc\left(x^2\right)\cot(x^2) + (2x-1)\cos(x^2)$, and $x \geq 0$.

As we mentioned earlier, assume that the initial guess approximation, which is the 1st-order approximation, has the form $y_1^1(x) = 1$ and $y_2^1(x) = 0$. Then, the 15th truncated series of the RPS solutions of $y_1(x)$ and $y_2(x)$ for singular system (24) and (25). is given by

$$\begin{aligned} y_1^{15}(x) &= 1 - \tfrac{1}{2}x^4 + \tfrac{1}{24}x^8 - \tfrac{1}{720}x^{12} \\ &= \sum_{j=0}^{3} (-1)^j \tfrac{1}{(2j)!}x^{4j}, \\ y_2^{15}(x) &= x^2 - \tfrac{1}{6}x^6 + \tfrac{1}{120}x^{10} - \tfrac{1}{5040}x^{14} \\ &= \sum_{j=0}^{3} (-1)^j \tfrac{1}{(2j+1)!}x^{4j+2}. \end{aligned}$$

Thus, the exact solution of singular system (24) and (25) has a general form that coincides with the exact solution

$$\begin{aligned} y_1(x) &= \sum_{j=0}^{\infty} (-1)^j \tfrac{1}{(2j)!}\left(x^2\right)^{2j} = \cos(x^2), \\ y_2(x) &= \sum_{j=0}^{\infty} (-1)^j \tfrac{1}{(2j+1)!}\left(x^2\right)^{2j+1} = \sin(x^2). \end{aligned} \tag{26}$$

Consequently, to illustrate the convergence of the approximate solutions $y_1^k(x)$ and $y_2^k(x)$ to the exact solutions $y_1(x)$ and $y_2(x)$ with respect to the kth-order of the solutions, we present numerical results of Example 2 graphically by Figures 1 and 2, which show the exact solutions and some iterated approximations $y_1^k(x)$ and $y_2^k(x)$ for $k = 5, 10, 15, 20$, respectively. These graphs reveal that the proposed method is an effective and convenient method for solving nonlinear singular systems with less computational and iteration steps.

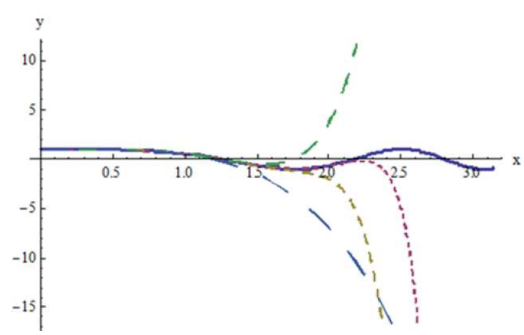

Fig. 1 Plots of the exact solution $y_1(x)$ for Example 3.

However, from the graphical results in Figures 1 and 2, it can be seen that the approximate solutions $y_1^k(x)$ and $y_2^k(x)$ that obtained by using the RPSM match the results

Table 1 Absolute error for 10th, 15th, and 20th-order approximations of $y_1(x)$ for Example 1.

Node	$\text{Ext}_1^{10}(x)$	$\text{Ext}_1^{15}(x)$	$\text{Ext}_1^{20}(x)$
0.0	0	0	0
0.2	5.75858805×10^{-14}	0	0
0.4	1.20381011×10^{-10}	0	0
0.6	1.06320118×10^{-8}	3.33066907×10^{-15}	0
0.8	2.57127877×10^{-7}	3.40616424×10^{-13}	4.44089210×10^{-16}
1.0	3.05861778×10^{-6}	1.22857280×10^{-11}	4.44089210×10^{-16}
1.2	2.32299063×10^{-5}	2.30408581×10^{-10}	1.77635684×10^{-15}
1.4	1.29466624×10^{-4}	2.75367107×10^{-9}	1.06581410×10^{-14}
1.6	5.75356054×10^{-4}	2.36658408×10^{-8}	1.70530257×10^{-13}
1.8	2.15108918×10^{-3}	1.58117871×10^{-7}	2.06412665×10^{-12}
2.0	7.01804652×10^{-3}	8.66165738×10^{-7}	1.91455740×10^{-11}

Table 2 Absolute error for 10th, 15th, and 20th-order approximations of $y_2(x)$ for Example 1.

Node	$\text{Ext}_1^{10}(x)$	$\text{Ext}_1^{15}(x)$	$\text{Ext}_1^{20}(x)$
0.0	0	0	0
0.2	1.04083409×10^{-16}	0	0
0.4	4.19858592×10^{-13}	0	0
0.6	5.44073675×10^{-11}	2.22044605×10^{-16}	0
0.8	1.71452641×10^{-9}	2.14273044×10^{-14}	0
1.0	2.48922799×10^{-8}	7.61946062×10^{-13}	0
1.2	2.21319329×10^{-6}	1.40643053×10^{-11}	4.44089210×10^{-16}
1.4	1.40262258×10^{-6}	1.65358172×10^{-10}	2.22044605×10^{-16}
1.6	6.93722149×10^{-6}	1.39747480×10^{-9}	6.66133815×10^{-16}
1.8	2.83883278×10^{-5}	9.17709908×10^{-9}	7.99360578×10^{-15}
2.0	1.00031710×10^{-4}	4.93879841×10^{-8}	8.14903700×10^{-14}

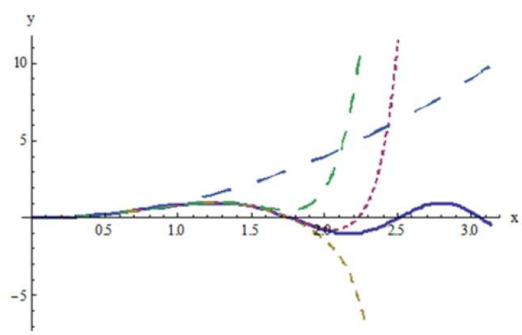

Fig. 2 Plots of the exact solution $y_2(x)$ for Example 3.

of the exact solution $y_1(x)$ and $y_2(x)$ very well, which implies that the errors become smaller as the order of approximate solutions increases, and confirm the convergence of the RPSM with respect to the kth-order of $y_i^k(x), i=1,2$ of the region under consideration.

Example 3. Consider the following second-order nonlinear system of singular IVPs:

$$y_1''(x)=\sin(y_2'(x))-\tfrac{1}{x}y_1(x)-\left(1-\tfrac{1}{x}\right)\cos(x), x\geq 0,$$
$$y_2''(x)=\tfrac{\cos^2(x)}{1+(y_1(x))^2}+\left(\tfrac{1}{\sin(x)}y_1'(x)\right)^2-\csc^2(x), \quad (27)$$

subject to the initial conditions

$$y_1(0)=1, y_1'(0)=0, y_2(0)=0, y_2'(0)=\pi. \quad (28)$$

Using the RPS algorithm, if we select the first two terms as initial guesses of the approximations as $y_1^1(x)=1$, $y_2^1(x)=0$, $y_1^2(x)=0$, $y_2^2(x)=\pi$, then the values of the coefficients $c_{1,m}$ and $c_{2,m}$ for $m=3,4,...,k$, of the kth truncated series (3) can be found by constructing the following kth residual functions, as well as using the fact that

$$\left(\frac{d^{k-1}}{dx^{k-1}}Res_1^k\right)(x_0) = \left(\frac{d^{k-1}}{dx^{k-1}}Res_2^k\right)(x_0) = 0,$$

$$\begin{aligned} Res_1^k(x) &= x\sum_{m=2}^{k} m(m-1)c_{1,m}(x-x_0)^{m-2} \\ &- x\sin\left(\sum_{m=1}^{k} mc_{2,m}(x-x_0)^{m-1}\right) \\ &+ \sum_{m=0}^{k} c_{1,m}(x-x_0)^m + (x-1)\cos(x), \\ Res_2^k(x) &= \begin{pmatrix} \sin^2(x)\sum_{m=2}^{k} m(m-1) \\ c_{2,m}(x-x_0)^{m-2} \\ -\left(\sum_{m=1}^{k} mc_{1,m}(x-x_0)^{m-1}\right)^2 \\ +\sin^2(x)\csc^2(x) \end{pmatrix} \\ &\times\left(1+\left(\sum_{m=0}^{k} c_{1,m}(x-x_0)^m\right)^2\right) \\ &- \sin^2(x)\cos^2(x). \end{aligned} \quad (29)$$

Therefore, the 10th truncated series of the RPS solution of $y_1(x)$ and $y_2(x)$ for singular system (27) and (28) is given as follows

$$\begin{aligned} y_1^{10}(x) &= 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \tfrac{1}{40320}x^8 \\ &- \tfrac{1}{3628800}x^{10} = \sum_{j=0}^{5}(-1)^j\tfrac{1}{(2j)!}x^{2j}, \\ y_2^{10}(x) &= \pi x. \end{aligned}$$

Consequently, the general forms of the approximate solutions of $y_1(x)$ and $y_2(x)$ for singular system (24) and (25) are given, respectively, by

$$y_1(x) = \sum_{m=0}^{k} c_{1,m}x^m = \sum_{j=0}^{\infty}(-1)^j\frac{1}{(2j)!}x^{2j} = \cos(x),$$

and

$$y_2(x) = \sum_{m=0}^{k} c_{2,m}x^m = \pi x,$$

which are coincide with the exact solution, as well as agree with Corollary 1.

Here, our aim is to show how the kth values in the truncation series equation (3) affects the approximate RPS solutions. In this regard, the kth-order approximation $y_1^k(x)$ and $y_2^k(x)$, $x \in [0,1]$, for various k and the exact error are calculated, as well as the error analysis is performed. Furthermore, the maximum and average error functions of $y_1^k(x)$ and $y_2^k(x)$ for singular system (24) and (25) have been listed in Table 3, for $x_i = \frac{i}{10}, i = 0,1,2,\ldots,10$, to illustrates the rapid convergence of the RPSM and to measure the extent of agreements between the kth-order approximate RPS solutions when $k = 5,10,15,20$.

4 Concluding remarks

The goal of the present work was to develop an efficient and accurate method for the solutions of system of singular initial value problems. This goal has been achieved by introducing the residual power series method to solve such classes of singular system. We can conclude that the proposed method is powerful and efficient technique in finding approximate solution for both linear and nonlinear system of singular problems. The proposed algorithm produced a rapidly convergent series with easily computable components using symbolic computation software. There is an important point to make here, the results obtained by the RPS method are very effective and convenient in linear and nonlinear cases with less computational work and time. This confirms our belief that the efficiency of our technique gives it much wider applicability in the future for general classes of linear and nonlinear problems.

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Table 3 The maximum error functions of $y_1^k(x)$,$y_2^k(x)$,$k = 5, 10, 15, 20$, for Example 3.

Description	$k = 5$	$k = 10$	$k = 5$	$k = 20$
$\max\{Ext_1^k(x_i)\}$	1.36436×10^{-3}	2.07625×10^{-9}	4.77396×10^{-14}	1.11022×10^{-16}
$\max\{Ext_2^k(x_i)\}$	0	0	0	0
$\max\{RES_1^k(x_i)\}$	4.03023×10^{-2}	2.73497×10^{-7}	1.12955×10^{-11}	7.99893×10^{-12}
$\max\{RES_2^k(x_i)\}$	8.01106×10^{-5}	1.21799×10^{-10}	2.82828×10^{-12}	7.07071×10^{-13}
$\max\{Rel_1^k(x_i)\}$	2.52518×10^{-3}	3.84276×10^{-9}	8.83572×10^{-14}	2.05483×10^{-16}
$\max\{Rel_2^k(x_i)\}$	0	0	0	0
$\overline{Ext_1^k(x_i)}$	4.51099×10^{-4}	2.58193×10^{-10}	3.63598×10^{-15}	2.11471×10^{-17}
$\overline{Ext_2^k(x_i)}$	0	0	0	0
$\overline{RES_1^k(x_i)}$	4.89750×10^{-3}	1.94374×10^{-8}	2.08501×10^{-12}	4.99000×10^{-12}
$\overline{RES_2^k(x_i)}$	8.13844×10^{-6}	8.42147×10^{-12}	3.05697×10^{-13}	3.05595×10^{-13}
$\overline{Rel_1^k(x_i)}$	2.15903×10^{-4}	2.39813×10^{-10}	4.99000×10^{-15}	3.09419×10^{-17}
$\overline{Rel_2^k(x_i)}$	0	0	0	0

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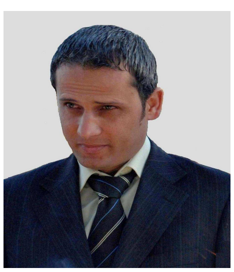

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