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m-Normal Cone Metric Spaces

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Abstract: In this paper, by using the concept of Archimedean property, m-normal cone metric space is introduced and some fixed point theorems of contractive mappings satisfying weak contractive conditions on m-normal cone metric space are proved. An example illustrating this concept is given.

Keywords: Archimedean property, Cone metric spaces, m-normal cone metric space, Fixed point, Contractive mappings, Ordered Banach spaces.

1 Introduction

The field of fixed point theory has attracted the attentions of many mathematicians for over nine decades. In 1922, S. Banach [5] established a famous fundamental fixed point theorem (called Banach contraction principle). In fixed point theory, the contractive conditions on underlying mappings play a crucial role in finding solutions of fixed point problems. Banach contraction principle is widely known as a strong source of metric fixed point theory. The principle is applicable in several branches of mathematics. For instance, it has been used to study the existence of solutions of linear and nonlinear integral equations, systems of linear equations as well as to prove the convergence of algorithms in computational mathematics. In respect of its importance, Banach contraction principle has been extended in different directions (see [1,2,4,8,9]).

In 2007, Huang and Zhang [9], replaced the real numbers by ordering Banach space and introduced the concept of cone metric spaces thereby establishing some fixed point theorems for contractive type mappings in normal cone metric spaces. They also discussed some properties of convergence of sequences in the new space. In the same article, completeness notion of the introduced spaces was discussed. Rezapour and Hamlbarani [13] modified the results of [9] for the case of cone metric spaces in absence of the normality condition. Cho and Bae [7] extended the notion of Hausdorff distance to cone metric spaces and thereby extending the work of [11] by replacing the metric space which was the domain of a multivalued mapping with a normal cone in a complete cone metric space. Ismat *et al* [4] introduced the notion of topological vector space valued cone metric space and obtained some common fixed point results. Azam and Mehmood [2] worked on multivalued fixed point theorems in topological vector space valued-cone metric spaces and consequently improved the results of [4,7,9]. In a related work, Akbar *et al* [1] introduced the notion of cone rectangular metric space and prove Banach contraction mapping principle in cone rectangular metric space setting. Similar results are contained in [3,4,8,12].

In this article, we define the notion of m-normal cone metric space and thereafter prove some fixed point theorems of contractive mappings satisfying weak contractive conditions on m-normal cone metric space. Also, an example of a complete cone metric space is given. This work improves many results in the literature. In particular, it extends the ideas of [9].
Throughout this work, the set of real and natural numbers will always be denoted by $\mathscr{R}$ and $\mathscr{N}$ respectively.

2 Preliminaries

In this section, relevant concepts needed in the main results are presented. The definition of cone metric spaces and related concepts from [9] are given as follows:

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Let E be a real Banach space and P a nonempty subset of E. P is said to be a *cone* if and only if it satisfies the following conditions:

(i) P is closed and $P \neq \{0\}$;
(ii) $a,b \in \mathscr{R}, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$;
(iii) $P \cap (-P) = \{0\}$, where $-P = \{-x : x \in P\}$.

For any given cone $P \subseteq E$, we can define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ if $x \leq y$ but $x \neq y$ and $x \ll y$ to mean $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|. \tag{1}$$

The least positive number K satisfying (1) above is called the *normal constant* of P. The cone P is called *regular* if every increasing sequence which is bounded above is convergent.

Definition 1.[9] *Let X be a nonempty set. Suppose there is a cone P in E such that the mapping $d : X \times X \longrightarrow E$ satisfies*

(i) $d(x,y) > 0$ and $d(x,y) = 0 \Longleftrightarrow x = y, \forall x,y \in X$;
(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(iii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y \in X$.

Then d is called a *cone metric* on X and (X,d) is called a *cone metric space*.
In the remaining part of this paper, we shall simply write X to mean (X,d). It will be stated when used otherwise.

Definition 2.[9] *Let X be a cone metric space and $\{x_n\}$ a sequence in X with $x \in X$. Then*

(i) if for every $c \in E$ with $0 \ll c$, there exists a natural number n_0 such that for all $n \geq n_0$, $d(x_n,x) \ll c$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ converges to x.
(ii) if for every $c \in E$ with $0 \ll c$ there exists an $n_0 \in \mathscr{N}$ such that for all $n,m \geq n_0, d(x_n,x_m) \ll c$, then $\{x_n\}$ is called a *Cauchy sequence* in X. If every Cauchy sequence in X converges to a point in X, then X is called a *complete cone metric space*.

Definition 3.[6] *Let X be a non-empty set. A point $x \in X$ is said to be a fixed point of the self-mapping $T : X \longrightarrow X$ if $Tx = x$. For any other point $x^* \in X$ with $Tx^* = x^*$, if $x = x^*$, then T is said to have a unique fixed point in X.*

Definition 4.[6] *Two positive reals x and y are said to have Archimedean property if there exists an integer $m > 0$ such that $y < mx$.*

3 Results and Discussion

In what follows, by extending the notion of Archimedean property of the natural numbers to arbitrary elements in a Banach space E, the concept of *reflective Archimedean property* of a Banach space is introduced. Consequently, the idea of *m-normal cone* is also established.

We must mention here that the weakness of this work in its present form is our inability to give a non-trivial example of an m-normal cone metric space in which the cone satisfies the reflective Archimedean property. Hence, via the proposed definitions, only theoretical concept is presented hoping to fill-up missing gaps in subsequent article as soon as possible. For an example of a cone metric space , see [9].

Definition 5. *Two positive vectors x and y in a Banach space E are said to have a reflective Archimedean property if there exists an integer m such that $y < mx$ if and only if $y \leq m^{\lambda}x, 0 < \lambda < 1$, where λ is called a bi-conditional index.*
The least positive integer m satisfying the above inequality is called the Archimedean constant for P.

Definition 6. *A cone P in a Banach space E is called m-normal if it possesses a reflective Archimedean property.*

Lemma 1. *Let X be a cone metric space and P an m-normal cone. Then the sequence $\{x_n\}$ in X converges to x if and only if $d(x_n,x) \longrightarrow 0, (n \longrightarrow \infty)$.*

Proof. Suppose $\{x_n\}$ converges to x. This means for every $c \in E$ with $0 \ll c$, there exists a natural number n_0 such that for all $n > n_0$,

$$d(x_n,x) \ll c. \tag{2}$$

Let $\varepsilon > 0$ be given and $cm < \varepsilon$ for any $m > 0$. Since P is m-normal, then (2) implies $d(x_n,x) \ll c \leq cm < \varepsilon$. This means $d(x_n,x) \leq c^{\lambda}m < \varepsilon, \forall n > n_0, 0 < \lambda < 1$. Hence, $d(x_n,x) \longrightarrow 0, (n \longrightarrow \infty)$.
Conversely, let $d(x_n,x) \longrightarrow 0, (n \longrightarrow \infty)$. For $c \in E$, let $0 \ll c$ be given. Then $c - 0 \in intP$. That is, $c - x = \lim x_n \in intP$. Therefore, $c - d(x_n,x) \in intP$. This shows that $d(x_n,x) \ll c$.

Lemma 2. *Let X be a cone metric space and P an m-normal cone. A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if $d(x_n,x_m) \longrightarrow 0, (n,m \longrightarrow \infty)$.*

Proof. Suppose $\{x_n\}$ is a Cauchy sequence in X. Let $\varepsilon > 0$ be given. Choose $c \in E$ with $0 \ll c$ and $cm < \varepsilon, m > 0$. Then there exists an $n_0 \in \mathscr{N}$ such that for all $n,m > n_0$,

$$d(x_n,x_m) \ll c < cm. \tag{3}$$

Lemma 3. *Let X be a complete cone metric space, P an m-normal cone and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n,x) \longrightarrow 0,(n,m \longrightarrow \infty)$.*

Proof. Suppose that $\{x_n\}$ is a Cauchy sequence. This means for every $c \in E$ with $0 \ll c$, there exists an $n_0 \in \mathcal{N}$ such that for all $n,m > n_0$,

$$d(x_n,x_m) \ll c. \tag{4}$$

Let $\varepsilon > 0$ be given and $cm < \varepsilon$ for any $m > 0$. Since P is m-normal, then (4) implies $d(x_n,x_m) \leq c^\lambda m, 0 < \lambda < 1$. That is , $d(x_n,x_m) < \varepsilon, \forall n,m > n_0$. This shows that $d(x_n,x_m) \longrightarrow 0,(n,m \longrightarrow \infty)$.
Conversely, let $d(x_n,x_m) \longrightarrow 0,(n,m \longrightarrow \infty)$. For $c \in E$, let $0 \ll c$ be given. Then $c - 0 \in intP \Longrightarrow c - d(x_n,x_m) \in intP$. This means $d(x_n,x_m) \ll c$. Hence, $\{x_n\}$ is a Cauchy sequence.

Lemma 4. *Let X be a cone metric space and P an m-normal cone. Let $\{x_n\}$ and $\{y_n\}$ be any two sequences in X such that $x_n \longrightarrow x$ and $y_n \longrightarrow y,(n \longrightarrow \infty)$.Then $d(x_n,y_n) \longrightarrow d(x,y),(n \longrightarrow \infty)$.*

Proof. Suppose $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively. This means for any $c \in E$ with $0 \ll c$, there exist $n_1,n_2 \in \mathcal{N}$ such that for all $n > n_1,n_2$,

$$d(x_n,x) \ll \frac{c}{2} \tag{5}$$

$$d(y_n,y) \ll \frac{c}{2} \tag{6}$$

Let $cm < \varepsilon$ for any $\varepsilon,m > 0$. For $n \geq max(n_1,n_2)$, we see that

$$d(x_n,y_n) \leq d(x_n,x)+d(x,y)+d(y,y_n) \tag{7}$$

$$d(x,y) \leq d(x,x_n)+d(x_n,y_n)+d(y_n,y) \tag{8}$$

From (7) and (8), it follows that

$$\begin{aligned} |d(x_n,y_n)-d(x,y)| &\leq d(x_n,x)+d(y_n,y) \\ &\leq c < mc < \varepsilon. \end{aligned} \tag{9}$$

Since P is m-normal, then (9) implies $|d(x_n,y_n)-d(x,y)| \leq m^\lambda c < \varepsilon$. Since ε is arbitrary, it follows that $d(x_n,y_n) \longrightarrow d(x,y),(n \longrightarrow \infty)$.

Theorem 1. *Let X be a complete cone metric space, P an m-normal cone . Suppose the mapping $T : X \longrightarrow X$ is a contraction , then T has a unique fixed point in X.*

Proof. Since T is a contraction, this means that there exists an $\alpha \in (0,1)$ such that for all $x,y \in X$, $d(Tx,Ty) \leq \alpha d(x,y)$.
Choose any point $x_0 \in X$ and let a sequence $\{x_n\}$ in X be defined by

$$x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \cdots, x_{n+1} = Tx_n, \cdots$$

We shall show that the sequence $\{x_n\}$ is a cauchy sequence. For each positive integer n, we have

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \leq \alpha d(x_{n-1},x_n) \\ &\leq \alpha^2 d(x_{n-2},x_{n-1}) \leq \cdots \leq \alpha^n d(x_0,x_1). \end{aligned}$$

For any $n > m$, we have

$$\begin{aligned} d(x_n,x_m) &\leq d(x_m,x_{m+1})+d(x_{m+1},x_{m+2})+\cdots+d(x_{n-1},x_n) \\ &\leq (\alpha^m+\alpha^{m+1}+\cdots+\alpha^{n-1})d(x_0,x_1) \\ &\leq \frac{\alpha^m}{1-\alpha} d(x_0,x_1) \leq a^\lambda d(x_0,x_1) \longrightarrow 0(n,m \longrightarrow \infty). \left[a = \frac{\alpha^m}{1-\alpha}\right] \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $x_n \longrightarrow x$, for some $x \in X$. Now,

$$\begin{aligned} d(Tx,x) &\leq d(Tx,Tx_n)+d(Tx_n,x) \\ &\leq \alpha d(x,x_n)+d(x_{n+1},x) \end{aligned} \tag{10}$$

Since P is m-normal, (10) implies

$$d(Tx,x) \leq \alpha^\lambda d(x,x_n)+d(x_{n+1},x) \longrightarrow 0,(n \longrightarrow \infty)\cdot$$

That is $Tx = x$, showing that x is a fixed point of T.
Assume x^* is another fixed point of T. Then $d(x^*,x) = d(Tx^*,Tx) \leq \alpha d(x^*,x)$, This implies $d(x^*,x) \leq \alpha^\lambda d(x^*,x)$. That is $(1-\alpha^\lambda)d(x^*,x) \leq 0$; proving that $x = x^*$.

Theorem 2. *Let X be a complete cone metric space and P an m-normal cone . For $c \in E$, with $0 \ll c$, $x_0 \in X$ and the set of open ball $B_r(x_0) = \{x \in X : d(x_0,x) < r\}$. Suppose the mapping $T : X \longrightarrow X$ satisfies the contractive condition*

$$d(Tx,Ty) \leq \alpha^\lambda d(x,y), \forall x,y \in B_r(x_0)\cdot$$

where $0 \leq \alpha < 1, 0 < \lambda < 1$ and $d(Tx_0,x_0) < (1-\alpha^\lambda)r$. Then T has a unique fixed point in $B_r(x_0)$.

Proof. For $\alpha = 0$, the result holds trivially. Suppose $\alpha^\lambda \neq 0$. It suffices to only prove that $B_r(x_0)$ is complete and $Tx \in B_r(x_0)$. Let $\{x_n\}$ be a Cauchy sequence in $B_r(x_0)$. Then $\{x_n\}$ is also a Cauchy sequence in X. Since X is complete , $x_n \longrightarrow x$, for some $x \in X$. Hence,

$$d(x_0,x) \leq d(x_0,x_n)+d(x_n,x) < d(x_n,x)+r = r,(n \longrightarrow \infty)\cdot$$

This shows that $x \in B_r(x_0)$. Therefore, $B_r(x_0)$ is complete. Now,

$$\begin{aligned} d(x_0,Tx) &\leq d(x_0,Tx_0)+d(Tx_0,Tx) \\ &< (1-\alpha^\lambda)r+\alpha^\lambda d(x_0,x) = r. \end{aligned}$$

Therefore, $Tx \in B_r(x_0)$.

Theorem 3. *Let X be a complete cone metric space and P an m-normal cone. Suppose the mapping $T : X \longrightarrow X$ satisfies the contractive condition*

$$d(Tx,Ty) < \alpha\left[d(Tx,x)+d(Ty,y)+\rho d(x,Ty), \forall x,y \in X\right],$$

where $0<\alpha<1, 0\leq\rho<1$. Then T has a unique fixed point in X.

Proof. Suppose $\rho\neq 0$. Choose any point $x_0\in X$ and let a sequence of points of X be defined by

$$x_1=Tx_0, x_2=Tx_1, x_3=Tx_2, \cdots, x_{n+1}=Tx_n, \cdots$$

We shall show that $\{x_n\}$ is a Cauchy sequence. Now,

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &< \alpha\left[d(Tx_n,x_n)+d(Tx_{n-1},x_{n-1})+\rho d(x_n,Tx_{n-1})\right] \\ &= \alpha\left[d(x_{n+1},x_n)+d(x_n,x_{n-1})\right] \\ &< \frac{\alpha}{1-\alpha}d(x_n,x_{n-1}). \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} d(x_{n+2},x_{n+1}) &= d(Tx_{n+1},Tx_n) \\ &< \alpha\left[d(Tx_{n+1},x_{n+1})+d(Tx_n,x_n)+\rho d(x_{n+1},Tx_n)\right] \\ &= \alpha\left[d(x_{n+2},x_{n+1})+d(x_{n+1},x_n)\right] \\ &< \left(\frac{\alpha}{1-\alpha}\right)^2 d(x_n,x_{n-1}). \end{aligned}$$

Continuing in this fashion, we see that

$$\begin{aligned} d(x_{n+1},x_n) &\leq \cdots \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x_1,x_0) \\ &= t^n d(x_1,x_0), where \quad t=\frac{\alpha}{1-\alpha}. \end{aligned}$$

For $n>m$, we get

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m+1})+d(x_{m+1},x_{m+2})+\cdots+d(x_{n-1},x_n) \\ &\leq \left(t^m+t^{m+1}+\cdots+t^{n-1}\right)d(x_1,x_0) \\ &< \frac{t^m}{1-t}d(x_1,x_0). \end{aligned} \tag{12}$$

Since P is an m-normal cone, we see that

$$d(x_m,x_n)\leq\frac{t^{m\lambda}}{(1-t)^{\lambda}}d(x_1,x_0)\longrightarrow 0, (n,m\longrightarrow\infty).$$

Therefore, by Lemma 2, $\{x_n\}$ is a Cauchy sequence. Since X is complete, then $x_n\longrightarrow x, (n\longrightarrow\infty)$ for some $x\in X$.

Since T satisfies a contractive condition, then it is continuous.
Consequently,

$$\begin{aligned} x=\lim x_n &\Rightarrow Tx=T(\lim x_n) \\ &\Rightarrow \lim Tx_n=\lim x_{n+1}=x. \end{aligned}$$

This shows that x is a fixed point of T. Now suppose x^* is another fixed point of T. Then

$$\begin{aligned} d(x,x^*) &= d(Tx,Tx^*) \\ &\leq \alpha\left[d(Tx,x)+d(Tx^*,x^*)+\rho d(x,Tx^*)\right] \\ &= \alpha\rho d(x,x^*) \end{aligned}$$

This implies $d(x,x^*)=0$. Hence $x=x^*$.
By putting $\rho=0$ in Theorem 1, yields the following corollary as an extension of Kannan contraction mapping.

Corollary 1.(Also,see [9]) *Let X be a complete cone metric space and P an m-normal cone. Suppose the mapping $T:X\longrightarrow X$ satisfies the contractive condition*

$$d(Tx,Ty)<\alpha\left[d(Tx,x)+d(Ty,y), \forall x,y\in X\right],$$

where $0<\alpha<1$. Then T has a unique fixed point in X.

Theorem 4.*Let X be a complete cone metric space and P an m-normal cone. Suppose the mapping $T:X\longrightarrow X$ satisfies the contractive condition*

$$d(Tx,Ty)<\alpha\left[d(Tx,y)+d(Ty,x)+\rho d(x,y)\right]\forall x,y\in X,$$

where $\alpha\in(0,1), \rho\in[0,1)$, with $m+\rho<m-\alpha, m>0$. Then T has a unique fixed point in X.

Proof. For $\rho=0$, the idea is a replica of Theorem 4 of [9]. So, suppose $\rho\neq 0$. Then choose $x_0\in X$ and let a sequence $\{x_n\}$ in X be defined as

$$x_1=Tx_0, x_2=Tx_1, x_3=Tx_2, \cdots x_{n+1}=Tx_n, \cdots$$

We want to show that $\{x_n\}$ is a Cauchy sequence. For this, consider

$$\begin{aligned} d(x_{n+1},x_n) &= d(Tx_n,Tx_{n-1}) \\ &< \alpha\left[d(Tx_n,x_{n-1})+d(Tx_{n-1},x_n)+\rho d(x_n,x_{n-1})\right] \\ &< \alpha\left[d(x_{n+1},x_{n-1})+d(x_n,x_{n-1})+\rho d(x_n,x_{n-1})\right] \end{aligned}$$

This implies

$$d(x_{n+1},x_n)\leq\alpha\left(\frac{1+\rho}{1-\alpha}\right)d(x_n,x_{n-1}). \tag{13}$$

On the above steps, using (13), we have $d(x_{n+2},x_{n+1})\leq\alpha^2\left(\frac{1+\rho}{1-\alpha}\right)^2 d(x_n,x_{n-1})$. Inductively, it follows that

$$d(x_{n+1},x_n)<\cdots<t^n d(x_1,x_0), where \quad t=\alpha\left(\frac{1+\rho}{1-\alpha}\right)<1\cdot$$

For $n>m$, we have

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n-1})+d(x_{n-1},x_{n-2})+\cdots+d(x_{m+1},x_m) \\ &\leq \left(t^{n-1}+t^{n-2}+\cdots+t^m\right)d(x_1,x_0) \\ &< \left(\frac{t^m}{1-t}\right)d(x_1,x_0). \end{aligned} \tag{14}$$

Since P is m-normal, we see that $d(x_n,x_m)\leq\frac{t^{m\lambda}}{(1-t)^{\lambda}}d(x_1,x_0)\longrightarrow 0, (n,m\longrightarrow\infty)$. Hence, by Lemma 2, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $x_n\longrightarrow x$, for some $x\in X$. Now, consider

$$\begin{aligned} d(Tx,x) &\leq d(Tx_n,Tx)+d(Tx_n,x) \\ &< \alpha\left[d(Tx_n,x)+d(Tx,x_n)+\rho d(x_n,x)\right]+d(Tx_n,x) \\ &= \alpha\left[d(x_{n+1},x)+d(Tx,x_n)+\rho d(x_n,x)\right]+d(x_{n+1},x) \end{aligned} \tag{15}$$

Letting $n \longrightarrow \infty$ and using the m-normality of P, we see that

$$d(Tx,x) \leq \frac{1}{(1-\alpha)^{\lambda}}\left[\alpha d(x_{n+1},x)+\alpha\rho d(x_n,x)+d(x_{n+1},x)\right] \longrightarrow 0, (n \longrightarrow \infty).$$

Hence, $Tx = x$.
Assume x^* is another fixed point of T. Then

$$\begin{aligned} d(x,x^*) &= d(Tx,Tx^*) \\ &< \alpha\left[d(Tx,x^*)+d(Tx^*,x)+\rho d(x,x^*)\right] \\ &= \alpha\left[2+\rho\right]d(x,x^*). \end{aligned}$$

Clearly, $1-\alpha(2+\rho)>0$, for $\rho \neq 0$. Hence, $d(x,x^*) \leq 0$, by m-normality of P. Therefore, $x = x^*$.This proves that the fixed point of T is unique.

By setting $\rho = 0$ in Theorem 4, yields the following corollary as an extended Chatterjea contraction mapping.

Corollary 2.(Also, see [9]) *Let X be a complete cone metric space and P an m-normal cone. Suppose the mapping*
$T : X \longrightarrow X$ satisfies the contractive condition

$$d(Tx,Ty) < \alpha\left[d(Tx,y)+d(Ty,x)\right] \forall x,y \in X,$$

where $\alpha \in (0,1)$. Then T has a unique fixed point in X.

The following examples are gotten from the idea of [9].

*Example 1.*Let $E = \mathscr{R}^2$, the Euclidean plane and $P = \{(x,y) : x,y \geq 0\}$ an m-normal cone in E. Let $X = \{(x,0) \in \mathscr{R}^2 : 0 \leq x \leq 1\} \bigcup \{(0,x) \in \mathscr{R}^2 : 0 \leq x \leq 1\}$.
Let the mapping $T : X \times X \longrightarrow E$ be defined by

$$d\left((x,0),(y,0)\right) = \left(\frac{b}{a}|x-y|, |x-y|\right), a,b \in \mathscr{N}, a < b,$$

$$d\left((0,x),(0,y)\right) = \left(|x-y|, \frac{b}{a}|x-y|\right), a,b \in \mathscr{N}, a < b.$$

For $x = y = 0$ or $x = y = 1$, the above construction is trivial. So, let $x \neq y$. Since $E = \mathscr{R}^2$ and every finite dimensional subspace of a normed space is complete, it follows that X is a complete cone metric space. X is an m-normal complete metric space since the cone P is m-normal. For any two fixed natural numbers a,b with $a < b$, the result is obvious by using the ideas of [9].

4 Conclusion

Huang and Zhang [9] came up with the idea of cone metric spaces and proved important results concerning the existence of fixed points for such contractions in the said spaces. We continued this investigations and introduced the concept of m-normal cone metric spaces. This idea particularly extends the work of [9] from cone metric space to m-normal cone metric space. In general, the field of metric fixed point theory is also improved. In this direction, allowing the metric space under discussion to be a classical one, then Theorem 1 gives the famous Banach fixed point theorem; for $\rho = 0$, we have the Kannan contraction mapping in Theorem 1; and putting $\rho = 0$, gives the famous Chatterjea contraction mapping in Theorem 4.

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