Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110121

On The Distribution of The Ratio of Two Hyper-Erlang Random Variables

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Received: 13 Feb. 2016, Revised: 15 Oct. 2016, Accepted: 19 Oct. 2016
Published online: 1 Jan. 2017

Abstract: The distribution of ratio of two random variables has been studied by several authors especially when the two random variables are independent and come from the same family. In this paper, the exact distribution of the ratio of two independent Hyper-Erlang distribution is derived. However, closed expressions of the probability density, cumulative distribution function, reliability function, hazard function, moment generating function and the r^{th} moment are found for this ratio distribution and proved to be a linear combination of the Generalized-F distribution. Moreover, the particular case, the ratio of two independent Hyper-Exponential distribution is examined.

Keywords: Ratio Distribution, Hyper-Erlang Distribution, Erlang Distribution, Generalized-F Distribution, Log-logistic Distribution, Probability Density Function, Survivor function, Moment Generating Function

1 Introduction

The distributions of ratio of random variables are of interest in many areas of the sciences, engineering, physics, number theory, order statistics, economics, biology, genetics, medicine, hydrology, psychology, classification, and ranking and selection, see [1],[2], [3] and [4]. Examples include safety factor in engineering, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and Mendelian inheritance ratios in genetics, see [1] and [2]. Also ratio distribution involving two Gaussian random variables are used in computing error and outage probabilities, see [5]. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels, see [6] and [7]. An important example of ratios of random variables is the stress strength model in the context of reliability. It describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $Pr(X < Y)$ is a measure of component reliability see [6] and [7].

The ratio distribution X/Y have been studied by several authors especially when X and Y are independent random variables and come from the same family. For a historical review, see the papers by Marsaglia [8] and Korhonen and Narula [9] for the normal family, Press [10] for Student's t family, Basu and Lochner [11] for the Weibull family, Hawkins and Han [12] for the non-central chi-squared family, Provost [13] for the gamma family, Pham-Gia [14] for the beta family, Nadarajah and Gupta [15] for the Logistic family, Nadarajah and Kotz [16] for the Frèchet family, Ali, Pal, and Woo [1] for the inverted gamma family, Nadarajah [17] for Laplace family, and [7] for the Generalized-F family.

The distribution X/Y, when X and Y are two independent Hyper-Erlang distributions, was not examined by any author. In this paper, we examine this distribution and find a closed expression of the probability density function (PDF). We showed that the PDF of this distribution is a linear combination of the Generalized-F distribution. As a consequence, the ratio distribution of two independent Hyper-Erlang distributions is obtained from the known Generalized-F distribution. Thus, the expressions of the cumulative distribution function

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(CDF), reliability function, hazard function, moment generating function and moments of order r for X/Y are determined. Next, we apply our results to a particular case of X/Y, when X and Y are two independent Hyper-Exponential distribution. Eventually, we illustrate our work in an example showing some graphs.

2 Some Preliminaries

2.1 The Hyper-Erlang distribution

The Hyper-Erlang distribution is the mixture of m mutually independent Erlang distributions or parallel m-phase Erlang distribution weighted with the probabilities p_i and each i^{th} Erlang stage E_i has the shape parameter k_i and rate parameter α_i, for $1 \le i \le m$, written as $E_{\alpha_i,k_i} \sim E(\alpha_i,k_i)$. Let X be a Hyper-Erlang distribution, then we write $X \sim H_m(\overrightarrow{p},\overrightarrow{\alpha},\overrightarrow{k})$, where $\overrightarrow{p} = (p_1,p_2,...,p_m) \in [0,1]^m$ are the weighted probabilities with $\sum_{i=1}^{m} p_i = 1$, $\overrightarrow{\alpha} = (\alpha_1,\alpha_2,...,\alpha_m) \in \mathbb{R}_+^m$ are rate parameters, and $\overrightarrow{k} = (k_1,k_2,...,k_m) \in \mathbb{N}^m$ are the shape parameters. The PDF of X is given as

$$f_X(t) = \sum_{i=1}^{m} p_i f_{E_i}(t) \tag{1}$$

where $f_{E_i}(t) = \frac{(\alpha_i t)^{k_i-1}\alpha_i e^{-\alpha_i t}}{(k_i-1)!} I_{(0,\infty)}(t)$ is the PDF of the Erlang distribution $E_{\alpha_i,k_i} \sim E(\alpha_i,k_i)$. The cumulative distribution function of X is $F_X(x) = \sum_{i=1}^{m} p_i F_{E_i}(x) = 1 - \sum_{i=1}^{m}\sum_{j=1}^{k_i-1} \frac{p_i(\alpha_i x)^j e^{-\alpha_i x}}{(j-1)!}$, the moment generating function is $\Phi_X(t) = \sum_{i=1}^{m} p_i \Phi_{E_i}(t) = \sum_{i=1}^{m} \frac{p_i \alpha_i^m}{(\alpha_i - t)^m}$, The moment of order r is $E[X^r] = \sum_{i=1}^{m} p_i E[E^r_{\alpha_i,k_i}] = \sum_{i=1}^{m} \frac{p_i \Gamma(r+k_i)}{\alpha_i^r \Gamma(k_i)}$. Then we have $E[X] = \sum_{i=1}^{m} \frac{p_i k_i}{\alpha_i}$, $E[X^2] = \sum_{i=1}^{m} \frac{p_i k_i (k_i+1)}{\alpha_i^2}$, see [18], [19], and [20].

The particular case of this distribution is when $\overrightarrow{k} = (1,1,...,1)$. This case is the Hyper-Exponential distribution where the m-phases are exponential distribution, then $E_{\alpha_i,1} \sim E(\alpha_i,1) = Exp(\alpha_i)$, the exponential distribution with parameter α_i, $1 \le i \le m$. The PDF here is

$$f_{X_m}(t) = \sum_{i=1}^{m} p_i \alpha_i e^{-\alpha_i t} I_{(0,\infty)}(t) \tag{2}$$

This particular case can be written as $X \sim H_m(\overrightarrow{p},\overrightarrow{\alpha},\overrightarrow{1}) = H_m(\overrightarrow{p},\overrightarrow{\alpha})$.

2.2 Generalized-F distribution

Let X be a random variable that has the Generalized-F distribution or called the generalized beta-prime distribution with three positive parameters v_1, v_2 and γ. The PDF of X is $f(t,v_1,v_2,\gamma) = \frac{\gamma^{v_2} t^{v_1-1}}{B(v_1,v_2)(t+\gamma)^{v_1+v_2}}$, $t \ge 0$, where

$$B(v_1,v_2) = \int_0^1 t^{v_1-1}(1-t)^{v_2-1} dt \tag{3}$$

is the usual Beta function, see [7]. Taking $\gamma = \frac{\beta}{\alpha}$, where $\alpha > 0$ and $\beta > 0$, we can write the PDF as

$$f(t,v_1,v_2,\frac{\beta}{\alpha}) = \frac{\left(\frac{\beta}{\alpha}\right)^{v_2} t^{v_1-1}}{B(v_1,v_2)\left(t+\frac{\beta}{\alpha}\right)^{v_1+v_2}} \tag{4}$$

The Generalized-F distribution is related to the F distribution with two parameters, where γX is a F distribution with 2α and 2β degrees of freedom, [7] and [21]. Thus, the distribution of Generalized-F distribution can be obtained from this relation, see [21], [22] and [23]. As a result we have the cumulative distribution function of X is

$$F(x) = I_{\frac{x}{x+\gamma}}(v_1,v_2), \tag{5}$$

where $I_x(a,b) = \frac{B(x,a,b)}{B(a,b)}$ is the regularized incomplete beta function and $B(x,a,b) = \int_0^x t^{a-1}(1-t)^{b-1} dt$ is the incomplete beta function. The moment generating function of X is

$$\Phi(t) = \frac{\Gamma(v_1+v_2)}{\Gamma(v_2)} U(v_1, 1-v_2, -\gamma t), \tag{6}$$

where $U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-zt} t^{a-1}(1+t)^{b-a-1} dt$ is the confluent hypergeometric function of the second kind. Moment of order k of X is

$$E[X^k] = \frac{\gamma^k \Gamma(v_1+k)\Gamma(v_2-k)}{\Gamma(v_1)\Gamma(v_2)} \text{ when } k < v_2 \tag{7}$$

else diverges, thus the expectation is $E[X] = \frac{\gamma v_2}{(v_2-1)}$, when $v_2 > 1$ and $Var[X] = \frac{\gamma^2 2 v_1 (v_1+v_2-1)}{v_2^2 (v_2-1)^2 (v_2-2)}$ where $v_2 > 2$.

The particular case of the Generalized-F distribution is when $v_1 = v_2 = 1$, which is the log-logistic distribution with parameter $1/\gamma$ and 1, we write

$$X \sim GF(1,1,\gamma) = loglogistic(\frac{1}{\gamma},1) \tag{8}$$

when $v_1 = v_2 = 1$. In this case we have the PDF of X is

$$f(t) = \frac{\gamma}{(t+\gamma)^2} \tag{9}$$

if $t > 0$ and $f(t) = 0$ if $t \le 0$, and CDF is

$$F(x) = \frac{x}{x+\gamma}. \tag{10}$$

if $x > 0$ and $f(x) = 0$ if $x \le 0$.

3 Ratio of Hyper-Erlang Distribution

In this section, we examine the ratio of the Hyper-Erlang distribution. A closed expression of the PDF is derived and written as a linear combination of the Generalized-F distribution. As a consequence, the expressions CDF, reliability function, hazard function, moment generating function and moments of order r for this distribution are determined.

We suppose that X and Y are two independent Hyper-Erlang distribution. Then we take $X \sim H_m(\overrightarrow{p}, \overrightarrow{\alpha}, \overrightarrow{k})$, $\overrightarrow{p} = (p_1, p_2, ..., p_m) \in [0,1]^m$ $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_m) \in \mathbb{R}_+^m$, and $\overrightarrow{k} = (k_1, k_2, ..., k_m) \in \mathbb{N}^m$ and $Y \sim H_n(\overrightarrow{q}, \overrightarrow{\beta}, \overrightarrow{l})$, $\overrightarrow{q} = (q_1, q_2, ..., q_n) \in [0,1]^n$, $\overrightarrow{\beta} = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}_+^n$, and $\overrightarrow{l} = (l_1, l_2, ..., l_n) \in \mathbb{N}^n$. From Eq (1) the PDF of X and Y are

$$f_X(t) = \sum_{i=1}^{m} p_i f_{E_{\alpha_i,k_i}}(t) \text{ and } f_Y(t) = \sum_{j=1}^{n} q_j f_{E_{\beta_j,l_j}}(t) \quad (11)$$

respectively, where $E_{\alpha_i,k_i} \sim E(\alpha_i, k_i)$, $1 \le i \le m$, and $E_{\beta_j,l_j} \sim E(\beta_j, l_j)$, $1 \le j \le n$.

Theorem 1.*Let X and Y be two independent Hyper-Erlang distributed according to (11). Then the PDF of X/Y is given by*

$$f_{X/Y}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j f_{W_{i,j}}(t), \quad (12)$$

where $f_{W_{i,j}}(t)$ is the PDF of the Generalized-F distribution $W_{i,j} \sim GF(k_i, l_j, \frac{\beta_j}{\alpha_i})$.

*Proof.*The PDF of ratio distribution X/Y is given by $f_{X/Y}(t) = \int_0^\infty y f_X(yt) f_Y(y) dy$ for $t \ge 0$. However the PDF of X and Y, are given in Eq (11), Thus we obtain that $f_{X/Y}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \int_0^\infty y \left(f_{E_{\alpha_i,k_i}}(yt) f_{E_{\beta_j,l_j}}(y) \right) dy$. The product in this integral is a multiplication of the PDF of two Erlang distribution corresponding to their parameters which is given as $f_{E_{\alpha_i,k_i}}(yt) f_{E_{\beta_j,l_j}}(y) = \frac{\left(\frac{\beta_j}{\alpha_i}\right)^{l_j} t^{k_i-1}}{(k_i-1)!(l_j-1)!} y^{k_i+l_j-2} e^{-y(t+\frac{\beta_j}{\alpha_i})}$ for $t \ge 0$ and $t < 0$ gives $f_{Y/X}(t) = 0$. Then, for $t \ge 0$,

$$f_{X/Y}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \frac{\left(\frac{\beta_j}{\alpha_i}\right)^{l_j} t^{k_i-1}}{(k_i-1)!(l_j-1)!} \int_0^\infty y^{k_i+l_j-1} e^{-y(t+\frac{\beta_j}{\alpha_i})} dy$$

To evaluate the above integral, we use the integral $\int_0^\infty x^a e^{-xb} dx = \frac{a!}{b^{a+1}}$ for $b > 0$ and any positive integer a. Thus $\int_0^\infty y^{k_i+l_j-1} e^{-y(t+\frac{\beta_j}{\alpha_i})} dy = \frac{(k_i+l_j-1)!}{(t+\frac{\beta_j}{\alpha_i})^{k_i+l_j}}$. We obtain that

$$f_{X/Y}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \frac{\left(\frac{\beta_j}{\alpha_i}\right)^{l_j} (k_i+l_j-1)! t^{k_i-1}}{(k_i-1)!\,(l_j-1)!(t+\frac{\beta_j}{\alpha_i})^{k_i+l_j}}$$

However, $\frac{(k_i-1)!(l_j-1)!}{(k_i+l_j-1)!}$ is the particular case of the beta function $B(k_i, l_j)$ in Eq (3), where k_i and l_j are integers. Therefore, we obtain that

$$f_{X/Y}(t) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j \frac{\left(\frac{\beta_j}{\alpha_i}\right)^{l_j} t^{k_i-1}}{B(k_i, l_j)(t+\frac{\beta_j}{\alpha_i})^{k_i+l_j}}$$

Referring to Eq (4), the expression in the above summation the PDF of the Generalized-F distribution, $GF(k_i, l_j, \frac{\beta_j}{\alpha_i})$. Writing $W_{i,j} \sim GF(k_i, l_j, \frac{\beta_j}{\alpha_i})$, we obtain the result.

Theorem 1 showed that the PDF of the ratio distribution of two independent Hyper-Erlang distribution is a linear combination of the Generalized-F distribution. From this linear form, we can find other related functions for X/Y, such as the CDF, reliability function, hazard function, moment generating function and moment of order r. As a consequence of Theorem 1, we prove in the next corollaries that these related functions are also a linear combination of the those of the Generalized-F distribution.

Corollary 1.*Let X and Y be two independent Hyper-Erlang distributed according to (11). Then the CDF of X/Y is*

$$F_{X/Y}(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j I_{\frac{x}{x+\frac{\beta_j}{\alpha_i}}}(k_i, l_j)$$

where $I_x(a,b)$ is the regularized incomplete beta function.

In the next corollaries, we give the expression of the reliability function and hazard function of X/Y.

Corollary 2.*Let X and Y be two independent Hyper-Erlang distributed according to (11). Then the reliability function of X/Y is*

$$R_{X/Y}(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j I_{\frac{\gamma_{ij}}{x+\gamma_{ij}}}(k_i, l_j)$$

and the hazard function of X/Y is

$$h_{X/Y}(t) = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j f_{W_{i,j}}(t)}{\sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j I_{\frac{\gamma_{ij}}{x+\gamma_{ij}}}(k_i, l_j)}$$

where $I_x(a,b)$ is the regularized incomplete beta function and $\gamma_{ij} = \frac{\beta_j}{\alpha_i}$, $1 \le i \le m$ and $1 \le j \le n$.

*Proof.*The reliability function of X/Y is related to the CDF as $R_{X/Y}(x) = 1 - F_{X/Y}(x)$. We use the expression of $F_{X/Y}(x)$ in Corollary 1 as

$F_{X/Y}(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_jI_{\frac{x}{x+\frac{\beta_j}{\alpha_i}}}(k_i,l_j)$. However, from [24], we have $I_{\frac{x}{x+\gamma_{ij}}}(k_i,l_j)=1-I_{1-\frac{x}{x+\gamma_{ij}}}(l_j,k_i)=1-I_{\frac{\gamma_{ij}}{x+\gamma_{ij}}}(l_j,k_i)$. Then we have $F_{X/Y}(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_j-\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_jI_{\frac{\gamma_{ij}}{x+\gamma_{ij}}}(l_j,k_i)=1-\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_jI_{\frac{\gamma_{ij}}{x+\gamma_{ij}}}(k_i,l_j)$. Thus, the reliability is $R_{X/Y}(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_jI_{\frac{\gamma_{ij}}{x+\gamma_{ij}}}(k_i,l_j)$. Moreover, the hazard function is $f_{X/Y}(t)/R_{X/Y}(t)$. Substituting the expressions of $f_{X/Y}(t)$ in (12)and $R_{X/Y}(t)$,we obtain the result.

Corollary 3.*Let X and Y be two independent Hyper-Erlang distributed according to (11). Then the moment generating function of X/Y is*

$$\Phi_{X/Y}(t)=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_j\Phi_{W_{i,j}}(t) \tag{13}$$

where $\Phi_{W_{i,j}}(t)=\frac{\Gamma(k_i+l_j)}{\Gamma(l_j)}U\left(k_i,1-l_j,-\frac{\beta_jt}{\alpha_i}\right)$ and $U(a,b,z)$ is the confluent hypergeometric function of the second kind.

Proposition 1.*Let X and Y be two independent Hyper-Erlang distributed according to (11). Then the moment of $Z=X/Y$ of order r is*

$$E[Z^r]=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_jE[W_{i,j}^r]=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_j\beta_j^r\Gamma(k_i+r)\Gamma(l_j-r)}{\alpha_i^r\Gamma(k_i)\Gamma(l_j)}$$

for $r<\min\{l_j\}$, $1\le j\le n$ and $W_{i,j}\sim GF(k_i,l_j,\frac{\beta_j}{\alpha_i})$.

*Proof.*From Eq (13) the moment generating function of Z is $\Phi_Z(t)=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_j\Phi_{W_{i,j}}(t)$ and the r^{th} derivative is given as $\frac{d^r\Phi_Z(t)}{dt^r}=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_j\frac{d^r\Phi_{W_{i,j}}(t)}{dt^r}$. Now, since the moment of order r of Z is the r^{th} derivative of $\Phi_Z(t)$ at $t=0$, this gives that $E[Z^r]=\sum_{i=1}^{m}\sum_{j=1}^{n}p_iq_jE[W_{i,j}^r]$, where $E[W_{i,j}^r]$ is the moment of order r of the Generalized-F distribution $W_{i,j}\sim GF(k_i,l_j,\frac{\beta_j}{\alpha_i})$.

On the other hand, Eq (7) gives that

$$E[W_{i,j}^r]=\frac{\beta_j^r\Gamma(k_i+r)\Gamma(l_j-r)}{\alpha_i^r\Gamma(k_i)\Gamma(l_j)}$$

when $r<\min\{l_j\}$, $1\le j\le n$. Thus, we obtain the required results.

From proposition 1, we conclude some particular moments. The expectation, for $r=1$, $E[Z]=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_j\beta_jk_i}{\alpha_i(l_j-1)}$ for $l_j\ne1$, $1\le j\le n$, also for $r=2$, $E[Z^2]=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_j\beta_j^2k_i(k_i-1)}{\alpha_i^2(l_j-1)(l_j-2)}$ for $l_j\ne1,2$ and $1\le j\le n$.

4 Ratio of Hyper-Exponential Distribution

Next, we consider the particular case of the Hyper-Erlang distribution, the Hyper-Exponential distribution. Let X and Y be two independent Hyper-Exponential random variables each distribution of different stages. Thus, $X\sim H_m(\overrightarrow{p},\overrightarrow{\alpha})$, $\overrightarrow{p}=(p_1,p_2,\dots,p_m)\in[0,1]^m$, $\sum_{i=1}^{m}p_i=1$ and $\overrightarrow{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)\in\mathbb{R}_+^m$, and $Y\sim H_n(\overrightarrow{q},\overrightarrow{\beta})$, $\overrightarrow{q}=(q_1,q_2,\dots,q_n)\in\mathbb{R}_+^m$, $\sum_{j=1}^{n}q_j=1$, and $\overrightarrow{\beta}=(\beta_1,\beta_2,\dots,\beta_n)\in[0,1]^m$. Then from Eq (2) the PDF of X and Y are

$$f_X(t)=\sum_{i=1}^{m}p_i\alpha_ie^{-\alpha_it} \text{ and } f_Y(t)=\sum_{j=1}^{n}q_j\beta_je^{-\beta_jt} \tag{14}$$

We note again that in this case $X\sim H_m(\overrightarrow{p},\overrightarrow{\alpha},\overrightarrow{k})=H_m(\overrightarrow{p},\overrightarrow{\alpha})$ where $\overrightarrow{k}=(1,1,\dots,1)$ and $Y\sim H_n(\overrightarrow{q},\overrightarrow{\beta},\overrightarrow{l})=H_n(\overrightarrow{q},\overrightarrow{\beta})$ where $\overrightarrow{l}=(1,1,\dots,1)$. Then referring to our results in section 3 and by taking the particular cases when $\overrightarrow{k}=(1,1,\dots,1)$ and $\overrightarrow{l}=(1,1,\dots,1)$, we obtain the results of the ratio of two independent Hyper-Exponential distribution. We state them in the following corollaries.

Corollary 4.*Let $X\sim H_m(\overrightarrow{p},\overrightarrow{\alpha})$ and $Y\sim H_n(\overrightarrow{q},\overrightarrow{\beta})$ be two independent Hyper-Exponential distribution. Then the PDF of X/Y is*

$$f_{X/Y}(t)=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_j\frac{\beta_p}{\alpha_i}}{\left(t+\frac{\beta_p}{\alpha_i}\right)^2}$$

for $t>0$ and $f_{X/Y}(t)=0$, for $t\le0$. The CDF is

$$F_{X/Y}(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_jx}{x+\frac{\beta_j}{\alpha_i}}$$

for $x>0$ and $F_{X/Y}(x)=0$, for $x\le0$. The reliability function is

$$R_{X/Y}(t)=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_j\left(\frac{\beta_j}{\alpha_i}\right)}{t+\frac{\beta_j}{\alpha_i}}$$

for $t>0$ and $R_{X/Y}(t)=0$, for $t\le0$. The hazard function is

$$h_{X/Y}(t)=\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_j\frac{\beta_j}{\alpha_i}}{\left(t+\frac{\beta_p}{\alpha_i}\right)^2}}{\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{p_iq_j\left(\frac{\beta_j}{\alpha_i}\right)}{t+\frac{\beta_j}{\alpha_i}}}$$

for $t > 0$ *and* $h_{X/Y}(t) = 0$, *for* $t \leq 0$.

Proof. The proof is directly obtained by taking the particular case of the ratio of two independent Hyper-Erlang distribution given in Theorem 1 and Corollaries 1 and 2. In this particular case we have from Equation 8, $W_{i,j} \sim GF(1,1,\frac{\beta_j}{\alpha_i}) = loglogistic(\frac{\alpha_i}{\beta_j},1)$ and knowing that PDF and CDF of $W_{i,j}$ are given in Eq (9) and Eq (10)as $f(t) = \frac{\frac{\beta_j}{\alpha_i}}{\left(t+\frac{\beta_j}{\alpha_i}\right)^2}$ and $F(x) = \frac{x}{x+\frac{\beta_j}{\alpha_i}}$ respectively. Thus, the results are obtained.

4.1 Application

Next, we give an example to illustrate our work. Let X and Y be two independent Hyper-Erlang distribution with $X \sim H_3(\overrightarrow{p},\overrightarrow{\alpha},\overrightarrow{k})$, $\overrightarrow{p} = (0.5,0.2,0.3)$, $\overrightarrow{\alpha} = (1,3,5)$, and $\overrightarrow{k} = (2,4,3)$ and $Y \sim H_4(\overrightarrow{q},\overrightarrow{\beta},\overrightarrow{l})$, $\overrightarrow{q} = (0.2,0.1,0.5,0.2)$, $\overrightarrow{\beta} = (1,4,6,2)$, and $\overrightarrow{l} = (3,5,2,3)$. By applying Theorem 1, we obtain that the PDF of X/Y is

$$f_{X/Y}(t) = \frac{1.2t}{(1+t)^5} + \frac{9.6t}{(2+t)^5} + \frac{1536t}{(4+t)^7} + \frac{54t}{(6+t)^4} + \frac{194.4t^3}{(1+3t)^7} + \frac{1555.2t^3}{(2+3t)^7} + \frac{464486t^3}{(4+3t)^9} + \frac{5832t^3}{(6+3t)^6} + \frac{225t^2}{(1+5t)^6} + \frac{1800t^2}{(2+5t)^6} + \frac{403200t^2}{(4+5t)^8} + \frac{8100t^2}{(6+5t)^5}$$

for $t > 0$ and $f_{X/Y}(t) = 0$ for $t \leq 0$. Figure 1 shows the curve of $f_{X/Y}(t)$ and Figures 2,3 and 4 show the CDF, reliability and hazard function of X/Y respectively, obtained in Corollaries 1 and 2.

5 Conclusion

The closed expressions of the PDF, CDF, reliability function, hazard function, moment generating function and the r^{th} moment of the ratio of two independent Hyper-Erlang distribution are given. The expressions are proved to be a linear combination of the Generalized-F distribution. Next, we observe the particular case, the ratio of two independent Hyper-Exponential distribution and expressions are again established in a particular and more simple form. Eventually, we illustrate our results in an example, by finding the ratio distribution.

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