A Fixed Point Theorem in Generalized Metric Spaces

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Abstract: In this paper, we state and prove a generalization of Ciric fixed point theorem[1] in generalized metric space by using a quasi-contractive map. Result presented in this paper generalize and extend well known fundamental metrical fixed point theorems in the literature (Banach [2], Kannan [3], Nadler [4], Reich [5], etc.) in the setting of generalized metric spaces.

Keywords: Fixed point, G-metric space, Quasi-contraction

1 Introduction and Preliminaries

In 2006, Mustafa and Sims [6] introduced the concept of G-metric spaces to overcome fundamental flaws in Dhae’s theory of generalized metric spaces as follows:

**Definition 1** Let $X$ be a non-empty set, and let $G : X \times X \times X \rightarrow R^+$ be a function satisfying the following axioms: for all $x, y, z, a \in X$,

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $G(x, y, y) > 0$ with $x \neq y$;
(G3) $G(x, y, z) \leq G(x, y, z)$ with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$;
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, z)$;

then the function $G$ is called a Generalized metric or more specifically a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space.

**Definition 2** [6] Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ in $X$, is said to be a $G$-Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer $N$ such that $G(x_n, x_m, x_k) < \epsilon$, for all $n, m, k \geq N$; i.e., if $G(x_n, x_m, x_k) \rightarrow 0$ as $n, m, k \rightarrow \infty$.

**Definition 3** [6] Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ in $X$, is said to be $G$-convergent to a point $x \in X$ if $\lim_{m,n \rightarrow \infty} G(x, x_n, x_m) = 0$, i.e., for each $\epsilon > 0$, there exists a positive integer $N$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

**Definition 4** [6] A $G$-metric space $(X, G)$ is said to be $G$-complete if every $G$-Cauchy sequence in $(X, G)$ is $G$-convergent in $X$.

**Definition 5** [6] A $G$-metric space $(X, G)$ is called a symmetric $G$-metric space if $G(x, y, y) = G(x, y, y)$, for all $x, y \in X$.

Motivated by the work of Mustafa and Sims [6,7], various researchers (see, e.g., [8-10]) have proved number of well known results in G-metric spaces.

2 Main Result

In this section, we introduce quasi-contraction mappings in G-metric spaces as follows:

**Definition 6** A mapping $T : X \rightarrow X$ of a $G$-metric space $X$ into itself is said to be quasi-contraction if there exists a number $q$, $0 \leq q < 1$ such that

$G(Tx, Ty, Ty) \leq q \max\{G(x, y, y), G(x, Tx, Tx), G(y, Ty, Ty), G(x, Ty, Ty), G(y, Tx, Tx)\}$.

**Definition 7** (1) Let $T$ be a mapping of a $G$-metric space $X$ into itself. For $A \subseteq X$, define

(i) $\delta(A) = \sup\{G(a, b, c) : a, b, c \in A\}$ and
(ii) for each $x \in X$,

$O(x, n) = \{x, Tx, T^2x, T^3x, \ldots, T^n x\}, n = 1, 2, 3, \ldots$ and $O(x, \infty) = \{x, Tx, T^2x, T^3x, \ldots\}$.

A space $(X, G)$ is said to be $T$-orbitally complete iff every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in X$ converges in $X$.

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Theorem 1 Let \((X, G)\) be a G-metric space. Suppose that \(T : X \to X\) is a quasi-contraction and \(X\) is \(T\)-orbitally complete. Then we have

(i) \(T\) has a unique fixed point.
(ii) \(\lim_{n \to \infty} T^n x = z\) for all \(x \in X\).
(iii) \(G(T^n x, z, z) = \frac{q^n}{1-q} G(x, T x, T x)\) for all \(x \in X\) and \(n \in \mathbb{N}\).

Proof. For each \(x \in X\) and \(0 \leq i \leq n - 1, 0 \leq j \leq n\), we have
\[
G(T^i x, T^j x, T^j x) = G(TT^{i-1} x, TT^{j-1} x, TT^{j-1} x) \\
\leq q \max \{ G(T^{i-1} x, T^{j-1} x, T^{j-1} x), G(T^{i-1} x, TT^{j-1} x, TT^{j-1} x), G(T^{i-1} x, TT^{j-1} x, T^{j-1} x), G(T^{i-1} x, T^{j-1} x, TT^{j-1} x) \} \\
= q \max \{ G(T^{i-1} x, T^{j-1} x, T^{j-1} x), G(T^{i-1} x, T^{j-1} x, T^{j-1} x), G(T^{i-1} x, T^{j-1} x, T^{j-1} x), G(T^{i-1} x, T^{j-1} x, T^{j-1} x) \}
\]
\[
\leq q \max \{ G(T^{i-1} x, T^{j-1} x, T^{j-1} x), G(T^{i-1} x, T^{j-1} x, T^{j-1} x), G(T^{i-1} x, T^{j-1} x, T^{j-1} x), G(T^{i-1} x, T^{j-1} x, T^{j-1} x) \}
\]
\[
\leq q \delta(O_T(x, n))
\]
where
\[
\delta(O_T(x, n)) = \max \{ G(T^{i} x, T^{j} x, T^{j} x) : 0 \leq i, j \leq n \}.
\]

Since \(0 \leq q < 1\), there exists \(h_n(x) \leq n\) such that
\[
G(x, T^{h_n} x, T^{h_n} x) = \delta(O_T(x, n)).
\]

Then we have
\[
G(x, T^{h_n} x, T^{h_n} x) \leq G(x, T x, T x) + q \delta(O_T(x, n)) = G(x, T x, T x) + q G(x, T^{h_n} x, T^{h_n} x).
\]

It implies that
\[
G(x, T^{h_n} x, T^{h_n} x) \leq \frac{1}{1-q} G(x, T x, T x) ...(1)
\]

For all \(n, m \geq 1\) and \(m < n\), it follows from the quasi-contractive condition of \(T\) and (1) that
\[
G(T^n x, T^m x, T^m x) = G(TTT^{n-1} x, TT^{m-n+1} T^{m-1} x, TT^{m-n+1} T^{m-1} x) \\
\leq q \delta(O_T(T^{m-n+1} x, m - n + 1)) \\
= G(T^{n-1} x, T^{m-n+1} T^{m-1} x, TT^{m-n+1} T^{m-1} x) \\
= q G(T^{n-2} x, T^{m-n+2} T^{m-2} x, TT^{m-n+2} T^{m-2} x) \\
\leq q^2 \delta(O_T(T^{m-2} x, m - n + 2))
\]
\[
\leq q^n \delta(O_T(x, m)) \\
\leq q^n G(x, T x, T x) ...(A)
\]

This gives \(\{T^n x\}\) is a Cauchy sequence in \(X\). Since \(X\) is \(T\)-orbitally complete, there exists \(z\) belongs to \(X\) such that \(\lim_{n \to \infty} T^n x = z\) ....(2)

By using the quasi-contractive condition, we get
\[
G(z, z, T z) = 0 \\
\leq G(z, T^{m-n+1} x, T^{m-n+1} x) \\
+ q \max \{ G(T^n x, z, z), G(T^n x, TT^n x, TT^n x), G(z, z, T z), G(T^n x, T z, T z), G(z, TT^n x, TT^n x) \} ...(3)
\]

Taking limit as \(n \to \infty\) in (3) and using (2), we get
\[
G(z, z, T z) \leq q G(z, z, T z).
\]

Since \(0 \leq q < 1\), we obtain \(G(z, T z, T z) = 0\). This gives, \(T\) has a fixed point \(z \in X\).

To prove uniqueness of fixed point, let \(w\) be another fixed point of \(T\). Then by using quasi-contractive condition on \(T\), we have
\[
G(z, w, w) = G(T z, T w, T w) \\
\leq q \max \{ G(z, z, w), G(z, T z, z), G(w, T w, w)\} \\
G(z, w, w) \leq q G(z, w, w)
\]
a contradiction, hence \(z = w\). This proves uniqueness of fixed point. Also, by taking limit as \(n \to \infty\) in (A), we have
\[
G(T^n x, z, z) = \frac{q^n}{1-q} G(x, T x, T x).
\]

Hence result follows.

3 Conclusion

Result presented in this paper generalize and extend well known fundamental metrical fixed point theorems in the literature (Banach [2], Kannan [3], Nadler [4], Reich [5], etc.) in the setting of generalized metric spaces.

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