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5-Cyclic Graphs with Minimum Degree Distance

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Abstract: In this paper, we find some characterizations of connected 5-cyclic graphs in terms of the degree sequence with respect to minimum degree distance.

Keywords: Degree, distance, connected, 5-cyclic

1 Introduction

Let $G \in \mathscr{G}_n^5$ represent the family of the connected graphs having 5 cycles with n vertices. Any graph in $\mathscr{G}_n^5$ will contain five linearly independent cycles having at least $n+4$ edges. If x,y are any two vertices from the graph, then $d(x,y)$ represents the geodesic path between the two vertices x and y. In a connected graph, we define the degree distance of the vertex $x \in V(G)$ in a manner that $D'(x) = D(x)d(x)$ where the factor $D(x) = \sum_{x,y \in V(G)} d(x,y)$. In this consequence, we define the degree distance of the graph G as:

$$D'(G) = \sum_{x,y \in V(G)} D'(x) = \sum_{x,y \in V(G)} d(x)D(x)$$
$$= \frac{1}{2} \sum_{x,y \in V(G)} d(x,y)(d(x)+d(y))$$

In [1], Moon has characterize the trees by using the concept of degree sequence of trees. Tomescu used the same concept of degree sequence of graphs and characterized the uni-cyclic and bi-cyclic graphs [4]. Zhu characterized the tri-cyclic graphs by using the degree sequence of the connected graphs [7]. Rahim *et al.* continued the same concept of the degree distance of connected graphs and characterized the 4-cyclic connected graphs by their degree sequence [8]. Motivated from these, we will characterize the 5-cyclic connected graphs with their different degree sequence orders.
Keeping in view the degree distances of the cyclic graph, we will show the minimal graph having 5-cycles. We will introduce some new 5-cyclic graphs having different degree sequences. In this paper, we will determine all the extremal 5-cyclic graphs achieving the minimum degree distance.

2 Properties Of Degree Distance of 5-Cyclic Graph

Lemma 1.*Let if $G \in \mathscr{G}_n^5$ then:*
(i) if $n = 6$, then the unique extremal graphs with minimum degree distance is isomorphic to $K_4 + \alpha + \beta$. Where α and β are the vertices of degree 2,
(ii) The minimum degree distance for $G \in \mathscr{G}_n^5$ is $min(D'(G)) = 126$.

Proof: If G is graph from the family of a 5-cyclic connected graphs then:
(i) the unique extremal graph from the family of the connected graphs with 5-cycles must be defined on at least 6 vertices by adding two vertices of degree 2 to the complete graph K_4 i.e. $K_4 + \alpha + \beta$. The new vertices α and β share a common vertex. We see that graph will have the minimum degree distance having 5-cycles. The extremal graph of such order figure is depicted below.

(ii) First of all, we will find the degree distance of each of the vertex and then by applying the summation over all the degree distances of the vertex. After an easy calculation, we get the degree distance of the graph i.e. $D'(K_4 + \alpha + \beta) = 126$.

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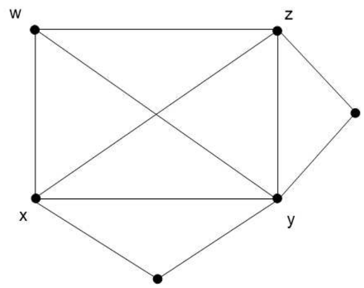

Fig. 1: Unique extremal Graph with 5-cycles $K_4+\alpha+\beta$

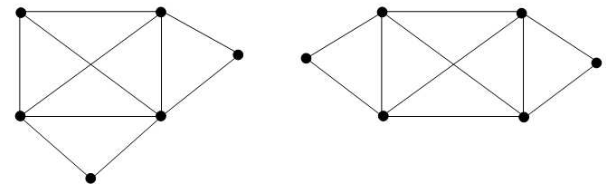

Fig. 2: $K_4+\alpha+\beta$

Lemma 2.*Let the order of graph be $n \geq 6$ and let $n-1 \geq d_1 \geq d_2 \geq d_3 \geq d_4 \geq \ldots \geq d_{n-1} \geq d_n \geq 1$ be the degrees of the which are to be from the set of natural number, respective vertices of connected graph from $G \in \mathscr{G}_n^5$ then:*
(i) $\sum_{i=1}^{n} d(v_i) = 2(n+4) = 2n+8$,
(ii) $d_i \geq 2$, this will hold at least for the six of the vertices of the graph.

Proof: Let G be a graph from the family of connected Cyclic graph i.e $G \in \mathscr{G}_n^5$. By the definition of the 5-cyclic graph, it must contain 5 cycles.
(i) By the definition of the hand shaking lemma, we say that $2|E| = \sum\limits_{x \in V(G)} d(v)$

$$\sum d(v) = d(w)+d(x)+d(y)+d(z)+d(\alpha)+d(\beta)$$
$$\sum d(v) = 3+4+5+4+2+2 = 20$$

So generally, we can say by the definition of the 5-cyclic graph that $\sum_{v_i \in V(G)} d(v_i) = 2(n+4) = 2n+8$. K_4 with added two vertices α and β of degree 2 is the only unique 5-cyclic extremal graph having the minimal degree distance, which is required.
(ii) As by the definition of the 5 cyclic graph, we see that the extremal graph having the minimum degree distance is on six vertices and each vertex is greater than or equal to two. So, we can say that for each $G \in \mathscr{G}_n^5$ at least six of the vertices must be of degree greater than or equal to 2 i.e $d_i \geq 2$ where $i = 1,2,\ldots,6$, which is our second required condition of the lemma.

($\Leftarrow$) Now we will discuss the case when order of the graph is 6 i.e $n=6$ for $G \in \mathscr{G}_n^5$. We see form the extremal graph $K_4+\alpha+\beta$ that for $n=6$. We have sum of the degrees as $d_1+d_2+d_3+d_4+d_5+d_6 = 20$. Now, we suppose that $d_6 > 3$ then $d_1+d_2+d_3+d_4+d_5+d_6 > 20$ which is contradiction to our 5-cyclic connected graph result that $\sum d(v_i) = 2n+8$. So we say that $d_6 \leq 3$.

*Case 1.*If $d_6 = 3$ then we calculate that $d_1+d_2+d_3+d_4+d_5 = 17$. The degree sequence for such graph will be of the form $d_1 = 5, d_2 = 4, d_3 = d_4 = 3, d_5 = 2$ i.e $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5$. The graph of such a degree sequence with $d_6 = 3$ is shown in Fig b.

*Case 2.*Now if we consider for 5-cyclic graph that $d_6 < 3$ then we say that $d_6 = 2$ and the degree sequence for such a graph is $d_1 = d_2 = d_3 = d_4 = 4, d_5 = d_6 = 2$ or $d_1 = d_2 = 3, d_3 = 4, d_4 = 5, d_5 = d_6 = 2$. This degree sequence will have a unique realization in $\mathscr{G}_n^5$. The graphs with such degree sequence is isomorphic to K_4 with two added vertices as shown in Fig 2.

Let us consider different cases for order of graphs greater than equal to six i.e $n \geq 6$ for all $G \in \mathscr{G}_n^5$. Let the result be true and hold for $k \leq n$.

Case 3. Consider if $d_n > 1$ then $d_n = 2$ because for $d_n > 2$ we see that $d_1+d_2+d_3+\ldots+d_n > 2n+8$. For $d_n = 2$ and if we consider n to be sufficiently large then by solving the degree sequence equation $d_1+d_2+d_3+\ldots+d_n = 2n+8$ we will have following possible graphs with its degree sequence form as given.

- If $d_1 = 10$, $d_2 = d_3 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to graph G_1.
- If $d_1 = 9$, $d_2 = 3, d_3 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to G_2, G_3 and G_4 graph.
- If $d_1 = 8, d_2 = d_3 = 3, d_4 = d_5 = \ldots = d_n = 2$ and $d_1 = 8$, $d_2 = 4, d_3 = d_4 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to G_5 and G_6 graph.
- If $d_1 = 7$, $d_2 = d_3 = d_4 = 3, d_5 = d_6 = \ldots = d_n = 2$ or $d_1 = 7$, $d_2 = 5, d_3 = d_4 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to $G_7, G_8, G_9, G_{10}, G_{11}$ and G_{12} graph.
- If $d_1 = 6$, $d_2 = d_3 = d_4 = d_5 = 3, d_6 = \ldots = d_n = 2$ or $d_1 = 6$, $d_2 = d_3 = 4$, $d_4 = d_5 = \ldots = d_n = 2$ or $d_1 = 6$, $d_2 = 5, d_3 = 3$, $d_4 = d_5 = \ldots = d_n = 2$ or $d_1 = 6$, $d_2 = d_3 = d_4 = d_5 = 3, d_6 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to $G_{13}, G_{14}, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}, G_{20}$ and G_{21} graph.
- If $d_1 = 5$, $d_2 = 4, d_3 = d_4 = d_5 = 3, d_6 = \ldots = d_n = 2$ or $d_1 = d_2 = 5, d_3 = d_4 = 3, d_5 = \ldots = d_n = 2$ or $d_1 = d_2 = 5, d_3 = 4, d_4 = d_5 = \ldots = d_n = 2$ or $d_1 = 5, d_2 = d_3 = 4, d_4 = 3, d_5 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to $G_{22}, G_{23}, G_{24}, G_{25}$ and G_{26} graph.
- If $d_1 = d_2 = d_3 = d_4 = 4, d_5 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to G_{27} graph.

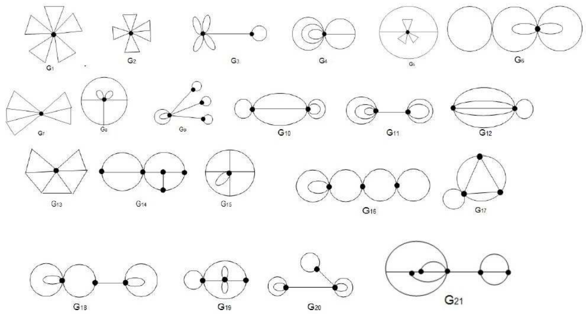

Fig. 3: All Possible 5-cyclic graphs with $d_n > 1$

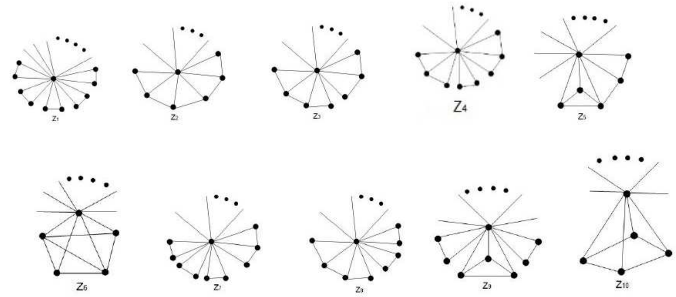

Fig. 4: All Possible 5-cyclic graphs with $d_n = 1$

• If $d_1 = d_2 = d_3 = 4, d_4 = d_5 = 3, d_6 = \ldots = d_n = 2$ or $d_1 = d_2 = 4, d_3 = d_4 = d_5 = d_6 = 3, d_7 = \ldots = d_n = 2$ or $d_1 = d_2 = 4, d_3 = d_4 = d_5 = 3 = d_6 = 3, d_7 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to G_{28}, G_{29} and G_{30} graph.
• If $d_1 = d_2 = \ldots = d_8 = 3, d_9 = \ldots = d_n = 2$ then the unique resultant graph with such degree sequence is isomorphic to G_{31} graph.

Case 4. Now if we consider the case that $d_n = 1$ then we will have the following the following subcases:

Now, if we consider that $d_1 = n-1$ then, we will have the following possibilities:
• If $d_1 = n-1$, $d_2 = d_3 = \ldots = d_{11} = 2$ and $d_{12} = \ldots = d_n = 1$then the unique resultant graph with such degree sequence is isomorphic to Z_1 graph.
• If $d_1 = n-1$, $d_2 = d_3 = d_4 = d_5 = 3, d_6 = d_7 = 2, d_8 = \ldots = d_n = 1$then the unique resultant graph with such degree sequence is isomorphic to Z_2 graph.
• If $d_1 = n-1$,$d_2 = d_3 = d_4 = 3, d_5 = d_6 = d_7 = d_8 = 2, d_9 = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_3 graph.
• If $d_1 = n-1$, $d_2 = d_3 = 3, d_4 = d_5 = \ldots = d_9 = 2, d_{10} = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_4 graph.
• If $d_1 = n-1$, $d_2 = 4, d_3 = d_4 = d_5 = 3, d_6 = 2, d_7 = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_5 graph.
• If $d_1 = n-1$, $d_2 = d_3 = 4, d_5 = d_6 = 3, d_7 = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_6 graph.
• If $d_1 = n-1, d_2 = 3, d_3 = \ldots = d_{10} = 2, d_{11} = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_7 graph.
• If $d_1 = n-1$, $d_2 = d_3 = 3, d_4 = d_5 = \ldots = d_9 = 2, d_{10} = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_8 graph.
• If $d_1 = n-1$, $d_2 = 4, d_3 = d_4 = d_5 = 3, d_6 = 2d_7 = \ldots = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_9 graph.
• If $d_1 = n-1$, $d_2 = d_3 = d_4 = d_5 = 3, d_7 = \ldots = \ldots = d_n = 1$ then the unique resultant graph with such degree sequence is isomorphic to Z_{10} graph. • Now, we suppose for the case that if $d_1 \leq n-2$ and $d_n = 1$. Then for the index i we say that $1 \leq i \leq n-1$. Now if we have $d_i \leq 2$ then we get the result that $\sum d_i \leq 2n-1$ which is contradiction to our hypothesis that if $G \in \mathscr{G}_n^5$ it must have the sum of degrees in a form that $\sum d_i = 2n+8$. Now we can use another approach to the solution by finding a maximal index j such that $n-1 \geq j-1 \geq 1$ and $d_j \geq 3$ and $d_{j+1} \leq 2$ and the degree sequence of the required graph to be in form $d_1 \geq d_2 \geq d_3 \geq d_4 \ldots \geq d_{j-2} \geq d_{j-1} \geq d_j \geq d_{j+1} \geq \ldots \geq d_n \geq 1$. Now considering the sequence at least six of the members of the defined sequence $d_1, d_2, \ldots, d_{j-1}, d_j, d_{j+1}, \ldots, d_n, 1$ must be greater than 2 for which we have already defined that $d_1 \leq n-2$. Already we know that for 5-cyclic connected graph $G \in \mathscr{G}_n^5$. It must have the sum of degrees in a form that $\sum d_i = 2n+8$. As we considered the case that $d_1 \leq n-2$ and $d_n = 1$. So we will get the degree sum as

$$\sum_{i=1}^{n-1} d_i = d_1 + d_2 + \ldots + d_{n-1} = 2(n-1) + 8 = 2n+6$$

Now if we apply the principal of Induction hypothesis, we will notice that there exist a graph $G \in \mathscr{G}_n^5$ having the degree sequence as mentioned earlier. Further, if we add new vertex to the graph in such a way that the defined additional vertex is bonded to an edge of the vertex with degree d_{j-1}. As a result, we will obtain a new graph which will have 5 connected cycles i.e. new graph will be from the family of the 5-cyclic connected graphs. The newly formed graph with 5-cycles will have the degree sequence of the form $d_1 \geq d_2 \geq \ldots \geq d_n = 1$. So with this final result Lemma 6.2 comes to an end.
Let us consider a vertex v where $v \in V(G)$ and $G \in \mathscr{G}_n^5$. Let us take the degree of vertex v be $d(v) = k$ then by using the result of the Tomescu [4]– [5] we can say that $D(v \geq 2n-k-2)$. For the equality let y another vertex from the set of vertices of G and $d(v,y) \leq 2$ then

$D(v) = 2n - k - 2$. On other hand the degree distance of the graph will be

$$D'(G) = \sum_{v\in V(G)} d(v)D(v) \geq \sum_{k=1}^{n-1} kx_k(2n-k-2)$$

where x_i represent multiplicities of degree i i.e multiplicity of degree i and $1 \leq i \leq n-1$.
Now by denoting as in Tomescu [4]– [5] we can write

$$F(x_1, x_2, x_3, x_4, ..., x_{n-2}, x_{n-1}) = \sum_{k=1}^{n-1} kx_k(2n-k-2).$$
□

Now to investigate minimum of $F(x_1, x_2, x_3, x_4, ..., x_{n-2}, x_{n-1})$ over the set of natural numbers because the degrees of the vertices are from the set of natural numbers such that $x_1, x_2, ..., x_{n-1} \geq 0$ which must satisfy the conditions of the Lemma 5.2. We can define the following corollary by all the conditions as discussed:

Corollary 1.*Let if order of G is $n \geq 6$, then we can say that the $x_1, x_2, x_3, x_4, ..., x_{n-2}, x_{n-1} \geq 0$ are the multiplicities of the degree of the vertices of the graph $G \in \mathscr{G}_n^5$ iff:*
$(i) \sum_{i=1}^{n-1} x_1 = n$
$(ii) \sum_{i=1}^{n-1} ix_1 = 2n+8$
(iii) *also* $x_1 \leq n-6$.

Proof: Let us consider the set of vectors in form $(x_1, x_2, x_3, x_4, ..., x_{n-2}, x_{n-1})$, where $x_1, x_2, x_3, x_4, .$
$.., x_{n-2}, x_{n-1}$ are the multiplicities or natural numbers such that be denoted by Δ and must satisfy the conditions (i), (ii) and (iii) of the corollary 6.1. Let us consider the index in the form $m \geq 2, p > 0, m+p \leq n-2, x_m \geq 1, x_p \geq 1$ and also define a transformation t_1 and t_2 for these indices.

$$t_1(x_1, x_2, x_3, x_4, ..., x_{n-2}, x_{n-1}) = (x'_1, x'_2, x'_3, x'_4, ..., x'_{n-2}, x'_{n-1}) = (x_1, x_2, x_3, x_4, ..., x_{m-1}+1, x_m-1, ..., x_{m+p}-1, x_{m+p+1}+1, ..., x_{n-2}, x_{n-1}),$$

where

$$x'_i = x_i \text{ but } i \notin \{m-1, m, m+p, m+p+1\}$$

and
$x'_{m-1} = x_{m-1}+1, x'_m = x_m - 1, x'_{m+p} = x_{m+p} - 1,$
$x'_{m+p+1} = x_{m+p+1} + 1.$
Now we will consider the 2nd transformation t_2 which will act on the set of the vector represented by Δ in a following way. Let m be an index in such a way that $m \geq 2$ and the multiplicity $x_m \geq 2$. So we define further that:

$$t_2(x_1, x_2, x_3, x_4, ..., x_{n-2}, x_{n-1}) = (x'_1, x'_2, x'_3, x'_4, ..., x'_{n-2}, x'_{n-1}) = (x_1, x_2, x_3, x_4, ..., x_{m-1}+1, x_m-2, ..., x_{m+1}+1, ..., x_{n-2}, x_{n-1})$$

where we say that $x'_i = x_i$ but $i \notin \{m-1, m, m+1\}$ and also $x'_{m-1} = x_{m-1}+1, x'_m = x_m - 2, x'_{m+1} = x_{m+1} + 1.$
□

3 Conclusion

Let if $G \in \mathscr{G}_n^5$ then:
(i) if $n = 6$, then the unique extremal graphs with minimum degree distance is isomorphic to $K_4 + \alpha + \beta$. Where α and β are the vertices of degree 2,
(ii) The minimum degree distance for $G \in \mathscr{G}_n^5$ is $min(D'(G)) = 126$.

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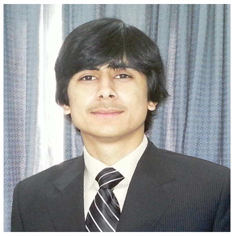

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