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On Sums of Odd and Even Terms of the Fibonacci Sequence

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Abstract: In this paper, we define some properties of sums of Fibonacci numbers. Also we present the sum of $s+1$ consecutive members of Fibonacci sequence and the same thing for even and for odd and their product of adjacent Fibonacci numbers. Mainly, Binet's formula will be used to establish properties of Fibonacci sequence.

Keywords: Fibonacci sequence, Lucas sequence, Binet's formula.

1 Introduction

The amount of literature bears witness to the ubiquity of the Fibonacci numbers and the Lucas numbers. Not only are these numbers popular in expository literature because of their beautiful properties, but also the fact that they `occur in nature' adds to their fascination. The Fibonacci sequence is a source of many nice and interesting identities. The term "Fibonacci numbers" is used to describe the series of numbers generated by the pattern 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144…where each number in the sequence is given by the sum of the previous two terms.

It is well known that the Fibonacci numbers and Lucas numbers are closely related. The term "Lucas numbers" is used to describe the series of numbers generated by the pattern 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 19,…These numbers are of great importance in the study of any subjects such as Algebra, geometry and number theory itself. The sequence of Fibonacci numbers F_n is defined by

$$F_n = F_{n-1} + F_{n-2}\,,\ n \ge 2 \ with\ F_0 = 0,\ F_1 = 1 \qquad (1.1)$$

The sequence of Lucas numbers L_n is defined by

$$L_n = L_{n-1} + L_{n-2}\,,\ n \ge 2 \ with\ L_0 = 2,\ L_1 = 1 \qquad (1.2)$$

In the 19th century, the French mathematician Binet devised two remarkable analytical formulas for the Fibonacci and Lucas numbers [5]. In our case, Binet's formula allows us to express the Fibonacci numbers and Lucas numbers in function of the roots $\Re_1$ & $\Re_2$ of the following characteristic equation, associated to the recurrence relation (1.1) and (1.2):

$$x^2 = x + 1 \qquad (1.3)$$

The Binet's formula for Fibonacci sequence and Lucas sequence is given by

$$F_n = \frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2} \qquad (1.4)$$

$$L_n = \Re_1^n + \Re_2^n \qquad (1.5)$$

In (1.4) & (1.5),

$$\Re_1 = \frac{1+\sqrt{5}}{2} \quad and \quad \Re_2 = \frac{1-\sqrt{5}}{2}$$

Also

$$(\Re_1 + \Re_2) = 1;\ (\Re_1\Re_2) = -1;$$

$$\Re_1 - 1 = \frac{\sqrt{5}-1}{2};\ \Re_2 - 1 = -\left(\frac{\sqrt{5}+1}{2}\right)$$

$$\Re_1 + 1 = \frac{3+\sqrt{5}}{2};\ \Re_2 + 1 = \frac{3-\sqrt{5}}{2};$$

$$\Re_1^2 - 1 = \Re_1;\ \Re_2^2 - 1 = \Re_2$$

In [7], Rajesh and Leversha define some properties of Fibonacci numbers in odd terms. In [10], Zvonko Čerin defines some sums of squares of odd and even terms of Lucas sequence. In [11], Zvonko Čerin improves some results on sums of squares of odd terms of the Fibonacci sequence by Rajesh and Leversha. In [4], H. Belbachir and F. Bencherif recover and extend all result of Zvonko Čerin [9] and Zvonko Čerin and Gianella [13]. In [12], Zvonko

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Čerin and Gianella defines sums of Pell numbers. In [8], Panwar, Rathore and Chawla, present the sum of consecutive members of k-Fibonacci numbers. In this paper we define some properties of sums of Fibonacci numbers for $N_* = \{0,1,2,3,4,...\}$ and N_{**} for the product $N_* \times N_*$.

2 Main Results

In this section, we prove some formulas for sums of a finite number of consecutive terms of the Fibonacci numbers. First we find the formula for the $\sum_{k=o}^{s} F_{v+k}$ when $s \geq 0$ and $v \geq 0$ are integers.

Proposition 1: For $(s,v) \in N_{**}$ the following equality holds:

$$\sum_{k=o}^{s} F_{v+k} = F_{v+s+2} - F_{v+1} \tag{2.1}$$

Proof: By Binet's formula, we have

$$\sum_{k=o}^{s} F_{v+k} = \sum_{k=0}^{s} \frac{\Re_1^{v+k} - \Re_2^{v+k}}{\Re_1 - \Re_2} \tag{2.2}$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{s} \Re_1^{v+k} - \Re_2^{v+k}$$

$$= \frac{1}{\sqrt{5}} \left[\Re_1^{v} \frac{\Re_1^{s+1} - 1}{\Re_1 - 1} - \Re_2^{v} \frac{\Re_2^{s+1} - 1}{\Re_2 - 1} \right]$$

$$= \frac{1}{\sqrt{5}} \left[\frac{\Re_1^{v+s+1} - \Re_1^{v}}{\Re_1 - 1} - \frac{\Re_2^{v+s+1} - \Re_2^{v}}{\Re_2 - 1} \right]$$

$$= \frac{2}{\sqrt{5}} \left[\frac{\Re_1^{v+s+1} - \Re_1^{v}}{\sqrt{5} - 1} + \frac{\Re_2^{v+s+1} - \Re_2^{v}}{\sqrt{5} + 1} \right]$$

$$= \frac{1}{2} \left[\begin{array}{r} \left(\Re_1^{v+s+1} + \Re_2^{v+s+1}\right) + \left(\frac{\Re_1^{v+s+1} - \Re_2^{v+s+1}}{\Re_1 - \Re_2} \right) \\ -(\Re_1^{v} + \Re_2^{v}) - \left(\frac{\Re_1^{v} - \Re_2^{v}}{\Re_1 - \Re_2} \right) \end{array} \right] \tag{2.3}$$

$$= \frac{1}{2} \left(L_{v+s+1} + F_{v+s+1} - L_v - F_v \right)$$

$$\sum_{k=o}^{s} F_{v+k} = F_{v+s+2} - F_{v+1}$$

This completes the proof.

Proposition 2: For $(s,v) \in N_{**}$ the following equality holds:

$$\sum_{k=o}^{s} F_{2v+2k} = \frac{1}{2} \left[L_{2v+2s+2} - F_{2v+2s+2} + F_{2v} - L_{2v} \right] \tag{2.4}$$

Proof: By Binet's formula, we have

$$\sum_{k=o}^{s} F_{2v+2k} = \sum_{k=0}^{s} \frac{\Re_1^{2v+2k} - \Re_2^{2v+2k}}{\Re_1 - \Re_2} \tag{2.5}$$

$$= \frac{1}{\sqrt{5}} \left[\Re_1^{2v} \frac{\Re_1^{2s+2} - 1}{\Re_1^2 - 1} - \Re_2^{2v} \frac{\Re_2^{2s+2} - 1}{\Re_2^2 - 1} \right]$$

Because $\left(\Re_1^2 - 1\right) = \Re_1$ and $\left(\Re_2^2 - 1\right) = \Re_2$, we get

$$= \frac{1}{\sqrt{5}} \left[\frac{\Re_1^{2v+2s+2} - \Re_1^{2v}}{\Re_1} - \frac{\Re_2^{2v+2s+2} - \Re_2^{2v}}{\Re_2} \right]$$

$$= \frac{2}{\sqrt{5}} \left[\frac{\Re_1^{2v+2s+2} - \Re_1^{2v}}{1 + \sqrt{5}} - \frac{\Re_2^{2v+2s+2} - \Re_2^{2v}}{1 - \sqrt{5}} \right]$$

$$= \frac{1}{-2\sqrt{5}} \left[\begin{array}{l} \left(\Re_1^{2v+2s+2} - \Re_2^{2v+2s+2}\right) \\ \quad -\sqrt{5}\left(\Re_1^{2v+2s+2} + \Re_2^{2v+2s+2}\right) \\ \qquad -\left(\Re_1^{2v} - \Re_2^{2v}\right) + \sqrt{5}\left(\Re_1^{2v} + \Re_2^{2v}\right) \end{array} \right]$$

$$= \frac{1}{2} \left[\begin{array}{l} \left(\Re_1^{2v+2s+2} + \Re_2^{2v+2s+2}\right) \\ \quad - \frac{\left(\Re_1^{2v+2s+2} - \Re_2^{2v+2s+2}\right)}{\Re_1 - \Re_2} \\ \quad + \frac{\left(\Re_1^{2v} - \Re_2^{2v}\right)}{\Re_1 - \Re_2} - \left(\Re_1^{2v} + \Re_2^{2v}\right) \end{array} \right] \tag{2.6}$$

$$\sum_{k=o}^{s} F_{2v+2k} = \frac{1}{2} \left[L_{2v+2s+2} - F_{2v+2s+2} + F_{2v} - L_{2v} \right]$$

This completes the proof.

Proposition 3: For $(s,v) \in N_{**}$ the following equality holds:

$$\sum_{k=o}^{s} F_{v+k}^2 = \frac{1}{2} \left[\begin{array}{r} \left(F_{2v+2s+2} - F_{2v}\right) - \frac{1}{5}\left(L_{2v+2s+2} - L_{2v}\right) \\ + \frac{2}{5}\left\{(-1)^{v+s+1} - (-1)^{v}\right\} \end{array} \right] \tag{2.7}$$

Proof: By Binet's formula, we have

$$\sum_{k=o}^{s} F_{v+k}^2 = \sum_{k=0}^{s} \left(\frac{\Re_1^{v+k} - \Re_2^{v+k}}{\Re_1 - \Re_2} \right)^2 \tag{2.8}$$

$$= \frac{1}{5} \sum_{k=0}^{s} \left\{ \Re_1^{2v+2k} + \Re_2^{2v+2k} - 2(\Re_1 \Re_2)^{v+k} \right\}$$

$$=\frac{1}{5}\left[\sum_{k=0}^{s}\Re_1^{2v+2k}+\sum_{k=0}^{s}\Re_2^{2v+2k}-2\sum_{k=0}^{s}(\Re_1\Re_2)^{v+k}\right]$$

$$=\frac{1}{5}\left[\begin{array}{l}\Re_1^{2v}\dfrac{\Re_1^{2s+2}-1}{\Re_1^2-1}+\Re_2^{2v}\dfrac{\Re_2^{2s+2}-1}{\Re_2^2-1}\\ \qquad -2(\Re_1\Re_2)^v\dfrac{(\Re_1\Re_2)^{s+1}-1}{\Re_1\Re_2-1}\end{array}\right]$$

$$=\frac{1}{5}\left[\begin{array}{l}\dfrac{\Re_1^{2v+2s+2}-\Re_1^{2v}}{\Re_1}+\dfrac{\Re_2^{2v+2s+2}-\Re_2^{2v}}{\Re_2}\\ \qquad +\left\{(-1)^{v+s+1}-(-1)^v\right\}\end{array}\right]$$

$$=\frac{-1}{10}\left[\begin{array}{l}\left(\Re_1^{2v+2s+2}+\Re_2^{2v+2s+2}\right)\\ -\left(\Re_1^{2v}+\Re_2^{2v}\right)-\sqrt{5}\left(\Re_1^{2v+2s+2}-\Re_2^{2v+2s+2}\right)\\ \qquad +\sqrt{5}\left(\Re_1^{2v}-\Re_2^{2v}\right)(-2)\left\{(-1)^{v+s+1}-(-1)^v\right\}\end{array}\right] \quad (2.9)$$

$$\sum_{k=o}^{s}F_{v+k}^2=\frac{1}{2}\left[\begin{array}{l}\left(F_{2v+2s+2}-F_{2v}\right)-\dfrac{1}{5}\left(L_{2v+2s+2}-L_{2v}\right)\\ \qquad +\dfrac{2}{5}\left\{(-1)^{v+s+1}-(-1)^v\right\}\end{array}\right]$$

This completes the proof.

Proposition 4: For $(s,v)\in N_{**}$ the following equality holds:

$$\sum_{k=o}^{s}(-1)^k F_{v+k}=\frac{1}{2}\left[\begin{array}{l}3F_v-L_v\\ \quad +(-1)^s\left\{3F_{v+s+1}-L_{v+s+1}\right\}\end{array}\right] \quad (2.10)$$

Proof: By Binet's formula, we have

$$\sum_{k=0}^{s}(-1)^k F_{v+k}=\sum_{k=0}^{s}(-1)^k\frac{\Re_1^{v+k}-\Re_2^{v+k}}{\Re_1-\Re_2} \quad (2.11)$$

$$=\frac{1}{\sqrt{5}}\sum_{k=0}^{s}(-1)^k\left(\Re_1^{v+k}-\Re_2^{v+k}\right)$$

$$=\frac{1}{\sqrt{5}}\left[\Re_1^v\frac{(-\Re_1)^{s+1}-1}{-\Re_1-1}-\Re_2^v\frac{(-\Re_2)^{s+1}-1}{-\Re_2-1}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\frac{\Re_1^v-(-1)^{s+1}\Re_1^{v+s+1}}{\Re_1+1}-\frac{\Re_2^v-(-1)^{s+1}\Re_2^{v+s+1}}{\Re_2+1}\right]$$

$$=\frac{1}{2\sqrt{5}}\left[\begin{array}{l}3\left(\Re_1^v-\Re_2^v\right)\\ +3(-1)^s\left(\Re_1^{v+s+1}-\Re_2^{v+s+1}\right)-\sqrt{5}\left(\Re_1^v+\Re_2^v\right)\\ \qquad -\sqrt{5}(-1)^s\left(\Re_1^{v+s+1}+\Re_2^{v+s+1}\right)\end{array}\right] \quad (2.12)$$

$$\sum_{k=o}^{s}(-1)^k F_{v+k}=\frac{1}{2}\left[\begin{array}{l}3F_v-L_v\\ \quad +(-1)^s\left\{3F_{v+s+1}-L_{v+s+1}\right\}\end{array}\right]$$

This completes the proof.

Proposition 5: For $(s,v)\in N_{**}$ the following equality holds:

$$\sum_{k=o}^{s}(-1)^k F_{2v+2k}=\frac{1}{10}\left[\begin{array}{l}5F_{2v}-5F_{2v+2s+2}\\ \qquad -L_{2v}+L_{2v+2s+2}\end{array}\right] \quad (2.13)$$

Proof: By Binet's formula, we have

$$\sum_{k=o}^{s}(-1)^k F_{2v+2k}=\sum_{k=0}^{s}(-1)^k\frac{\Re_1^{2v+2k}-\Re_2^{2v+2k}}{\Re_1-\Re_2} \quad (2.14)$$

$$=\frac{1}{\sqrt{5}}\left[\frac{\Re_1^{2v}-(-1)^{2s+2}\Re_1^{2v+2s+2}}{\Re_1^2+1}-\frac{\Re_2^{2v}-(-1)^{2s+2}\Re_2^{2v+2s+2}}{\Re_2^2+1}\right]$$

$$=\frac{2}{\sqrt{5}}\left[\frac{\Re_1^{2v}-\Re_1^{2v+2s+2}}{5+\sqrt{5}}-\frac{\Re_2^{2v}-\Re_2^{2v+2s+2}}{5-\sqrt{5}}\right]$$

$$=\frac{1}{10\sqrt{5}}\left[\begin{array}{l}5\left(\Re_1^{2v}-\Re_2^{2v}\right)-\sqrt{5}\left(\Re_1^{2v}+\Re_2^{2v}\right)\\ \qquad -5\left(\Re_1^{2v+2s+2}-\Re_2^{2v+2s+2}\right)\\ \qquad +\sqrt{5}\left(\Re_1^{2v+2s+2}+\Re_2^{2v+2s+2}\right)\end{array}\right] \quad (2.15)$$

$$\sum_{k=o}^{s}(-1)^k F_{2v+2k}=\frac{1}{10}\left[\begin{array}{l}5F_{2v}-5F_{2v+2s+2}\\ \qquad -L_{2v}+L_{2v+2s+2}\end{array}\right]$$

This completes the proof.

Proposition 6: For $(s,v)\in N_{**}$ the following equality holds:

$$\sum_{k=o}^{s}F_{v+k}F_{v+k+1}=\frac{1}{2}\left[\begin{array}{l}F_{2v+2s+3}-F_{2v+1}\\ \quad -\dfrac{1}{5}\left(L_{2v+2s+3}-L_{2v+1}\right)\\ \qquad +\dfrac{1}{5}\left\{(-1)^{v+s+1}-(-1)^v\right\}\end{array}\right] \quad (2.16)$$

Proof: By Binet's formula, we have

$$\sum_{k=o}^{s}F_{v+k}F_{v+k+1}=\sum_{k=0}^{s}\left(\frac{\Re_1^{v+k}-\Re_2^{v+k}}{\Re_1-\Re_2}\right)\left(\frac{\Re_1^{v+k+1}-\Re_2^{v+k+1}}{\Re_1-\Re_2}\right) \quad (2.17)$$

$$=\frac{1}{5}\sum_{k=o}^{s}\Re_1^{2v+2k+1}+\frac{1}{5}\sum_{k=o}^{s}\Re_2^{2v+2k+1}$$
$$-\left(\Re_1+\Re_2\right)\sum_{k=0}^{s}\left(\Re_1\Re_2\right)^{v+k}$$

$$=\frac{1}{5}\left[\begin{array}{c}\Re_1^{2v+1}\frac{\Re_1^{2s+2}-1}{\Re_1^2-1}+\Re_2^{2v+1}\frac{\Re_2^{2s+2}-1}{\Re_2^2-1}\\ -\left(\Re_1+\Re_2\right)\left(\Re_1\Re_2\right)^{v}\frac{\left(\Re_1\Re_2\right)^{s+1}-1}{\Re_1\Re_2-1}\end{array}\right]$$

Because $\left(\Re_1^2-1\right)=\Re_1$ and $\left(\Re_2^2-1\right)=\Re_2$, we get

$$=\frac{1}{5}\left[\begin{array}{c}\frac{\Re_1^{2v+2s+3}-\Re_1^{2v+1}}{\Re_1}\\ +\frac{\Re_2^{2v+2s+3}-\Re_2^{2v+1}}{\Re_2}\\ -\frac{\left(\Re_1\Re_2\right)^{v+s+1}-\left(\Re_1\Re_2\right)^{v}}{\Re_1\Re_2-1}\end{array}\right]$$

$$=\frac{-1}{10}\left[\begin{array}{c}\left(\Re_1^{2v+2s+3}+\Re_2^{2v+2s+3}\right)\\ -\left(\Re_1^{2v+1}+\Re_2^{2v+1}\right)-\sqrt{5}\left(\Re_1^{2v+2s+3}-\Re_2^{2v+2s+3}\right)\\ +\sqrt{5}\left(\Re_1^{2v+1}-\Re_2^{2v+1}\right)-\left\{\left(-1\right)^{v+s+1}-\left(-1\right)^{v}\right\}\end{array}\right] \quad (2.18)$$

This completes the proof.

3. Conclusion

In this paper, we present many properties. We define the sum of $s+1$ consecutive members of Fibonacci sequence and the same thing for even and for odd and their product of adjacent Fibonacci numbers.

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