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Basic Universal Triple I Restriction Methods for FMP Problem

Yiming Tang[1,2,3] *and Yanxiang Chen*[1]

[1] AnHui Province Key Laboratory of Affective Computing & Advanced Intelligent Machine, School of Computer and Information, Hefei University of Technology, Hefei 230009, China
[2] State Key Laboratory of Virtual Reality Technology and Systems, Beihang University, Beijing 100191, China
[3] Information and Communication Engineering Postdoctoral Research Station, Hefei University of Technology, Hefei 230009, China

Abstract: A new fuzzy reasoning method called basic universal triple I restriction method is put forward, which is a generalization of the basic triple I restriction method. The basic triple I restriction principle is improved, and unified forms of this new method are established for fuzzy modus ponens (FMP) problem. Furthermore, its optimal restriction solutions are obtained for some specific implications. As its particular case, the related conclusions of basic triple I restriction method are achieved and improved.

Keywords: Fuzzy reasoning, unified form, triple I restriction method, universal triple I method.

1. Introduction

Currently fuzzy reasoning plays a vital role in the areas of fuzzy control, artificial intelligence, decision making, affective computing and so on [1–4]. The most basic form of fuzzy reasoning is fuzzy modus ponens (FMP) as following:

FMP: for a given rule $A \to B$ and input A^*,

to compute B^* (output),, (1)

where $A, A^* \in F(U)$, $B, B^* \in F(V)$, and $F(U), F(V)$ denote the set of all fuzzy subsets of U, V, respectively. To solve it, an excellent method is the triple I method proposed by Wang in 1999 [5]. The triple I method is broadly researched by lots of scholars [6–9], and such results illustrate that it has some ideal advantages including strict logic basis and consistency.

Song et al. [10] further established the α-triple I restriction method. Its solution is the largest B^* (or smallest A^*) such that the following formula holds for all $u \in U, v \in V$:

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \le \alpha, \quad (2)$$

where $\alpha \in (0, 1]$.

Later, Peng [11] proposed the basic triple I restriction method. Its solution is the largest B^* (or smallest A^*) making the left-hand of (2), i.e.

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \quad (3)$$

takes its minimum for all $u \in U, v \in V$. Wang et al. [12] analyzed such two triple I restriction methods, and gave related expressions for family of implication $L - \lambda - G$. Then Liu, Wang [13] obtained the unified forms of α-triple I restriction method.

Moreover, from the viewpoint of fuzzy system, we generalized the triple I method to the universal triple I method [14,15] (which has more comprehensive advantages). Then we further put forward the α-universal triple I restriction method [16] derived from

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \le \alpha, \quad (4)$$

in which $\alpha \in (0, 1]$. It is found that the α-universal triple I restriction method is superior to the α-triple I restriction method.

Similar to (4), we also need to research the restriction method from the left-hand of (4), i.e.,

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)), \quad (5)$$

which is called basic universal triple I restriction method. The aim of this paper is to systematically investigate the basic universal triple I restriction methods for FMP.

The rest of the paper is organized as follows. Section 2 is the preliminaries. In Section 3, the unified forms of basic universal triple I restriction method are presented for FMP problem. In Section 4, the related results of basic triple I restriction method are achieved and improved. Section 5 summarizes this paper.

* Corresponding author: e-mail: tym608@163.com

2. Preliminaries

Definition 1. *If a mapping $I : [0,1]^2 \to [0,1]$ satisfying*

$$I(0,0) = I(0,1) = I(1,1) = 1,\ I(1,0) = 0,$$

then it is said to be an implication on $[0,1]$. $I(a,b)$ is also written as $a \to b$ for any $a, b \in [0,1]$.

Definition 2. *An implication I is defined as a residual implication if the following conditions hold:*

(C1) $I(a,b)$ is non-decreasing w.r.t. b $(a, b \in [0,1])$.

(C2) $I(a,b)$ is right-continuous w.r.t. b $(a \in [0,1]$, $b \in [0,1)$).

(C3) $\{x \in [0,1] | \, I(a,x) = 1\} \neq \varnothing$ $(a \in [0,1]$).

Especially, if I also satisfies

(C4) $a \leq b$ iff $I(a,b) = 1$ $(a, b \in [0,1]$, and iff denotes "if and only if"),

then I is said to be a strongly residual implication.

Definition 3. *Suppose that T, I are two $[0,1]^2 \to [0,1]$ mappings, then (T, I) is said to be a residual pair or, T and I are residual to each other, if the residual condition as following holds for any $a, b, c \in [0,1]$:*

$$T(a,b) \leq c \text{ iff } b \leq I(a,c).$$

It is easy to find that for a mapping I which has a residual pair, its residual mapping T is unique, and vice versa.

Theorem 1([14]). *If a mapping $I : [0,1]^2 \to [0,1]$ satisfies (C1), (C2) and (C3), and construct $T : [0,1]^2 \to [0,1]$ as the following:*

$$T(a,b) = \inf\{x \in [0,1] | \, b \leq I(a,x)\},\ a, b \in [0,1] \quad (6)$$

then (T, I) is a residual pair, while the following formula holds:

$$I(a,b) = \sup\{x \in [0,1] | \, T(a,x) \leq b\}. \quad (7)$$

By the definition of residual implication, together with Theorem 1, we can get Theorem 2.

Theorem 2. *Suppose that I is a residual implication and that T is obtained from (6), then (T, I) is a residual pair, and (7) holds.*

Definition 4. *Suppose that Z is any nonempty set, and that $F(Z)$ is the set of all fuzzy subsets on Z, define partial order relation $\leq_F$ on $F(Z)$ (by virtue of pointwise order) as:*

$$A(z) \leq_F B(z) \text{ iff } A(z_0) \leq B(z_0)$$

for any $z_0 \in Z$, in which $A, B \in F(Z)$.

Lemma 1([14]). *$< F(Z), \leq_F >$ is a complete lattice.*

3. Basic universal triple I restriction method for FMP

Based on the idea of basic universal triple I restriction method, we can achieve the key principle:

Basic universal triple I restriction principle for FMP: The conclusion $B^*(v)$ (in $< F(V), \leq_F >$) of FMP (1) is the largest fuzzy set making (5) get its minimum.

Such principle obviously improves the previous basic triple I restriction principle for FMP in [11].

Definition 5. *Let $A, A^* \in F(U)$, $B \in F(V)$, if B^* (in $< F(V), \leq_F >$) makes (5) get its minimum for any $u \in U$ and $v \in V$, then B^* is called a FMP-universal triple I restriction solution (FMP-universal solution for short).*

Definition 6. *Suppose that $A, A^* \in F(U)$, $B \in F(V)$, and that nonempty set $\mathbb{E}$ is the set of all FMP-universal solutions, and finally that D^* is the supremum of $\mathbb{E}$, then D^* is defined as a SupP-quasi-universal triple I restriction solution (SupP-quasi-universal solution for short). And, if D^* is the maximum of $\mathbb{E}$, then D^* is also called a MaxP-universal triple I restriction solution (MaxP-universal solution for short).*

Theorem 3. *Suppose that $A, A^* \in F(U)$, $B \in F(V)$ and that $\to_2$ satisfies (C1), then the minimum of (5) is as follows:*

$$M(u,v) = (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0),$$

and there exists a unique SupP-quasi-universal solution B^*. Furthermore, if $\to_2$ is left-continuous w.r.t. the second variable, then B^* is the MaxP-universal solution.

Proof. (i) Since $\to_2$ satisfies (C1), it follows that

$$A^*(u) \to_2 B^*(v) \geq A^*(u) \to_2 0,$$

and thus

$$\begin{aligned} &(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \\ &\geq (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0) \\ &= M(u,v) \ \ (u \in U, v \in V). \end{aligned}$$

This implies that $M(u,v)$ is the minimum of (5). Notice that

$$\begin{aligned} \mathbb{E} = \{&D^* \in F(V) \mid (A(u) \to_1 B(v)) \to_2 (A^*(u) \\ &\to_2 D^*(v)) = M(u,v),\ \ u \in U, v \in V\}. \end{aligned}$$

From the fact that $0 \in \mathbb{E}$, we know that $\mathbb{E}$ is not empty. Therefore there exists the SupP-quasi-universal solution B^* which is unique, and $B^* = \sup \mathbb{E}$.

(ii) When $\to_2$ is left-continuous w.r.t. the second variable, we show that $B^* \in \mathbb{E}$ (where B^* is obtained in (i)). In fact, suppose on the contrary that $B^* \notin \mathbb{E}$, then there exist fuzzy sets $D_1, D_2, \cdots$ in $\mathbb{E}$ such that

$$\lim_{n\to\infty} D_n(v) = B^*(v) \ \ (v \in V).$$

From the fact that $B^* = \sup \mathbb{E}$, we get

$$D_n(v) \leq B^*(v) \ (v \in V)$$

and hence $B^*(v)$ is the left limit of

$$\{D_n(v)|\ n = 1, 2, \cdots\}(v \in V).$$

Since $D_1, D_2, \cdots \in \mathbb{E}$, it follows that $(n = 1, 2, \cdots; u \in U, v \in V)$:

$$M(u, v) = (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 D_n(v)).$$

Because $\to_2$ is left-continuous w.r.t. the second variable and $\to_2$ satisfies (C1), we obtain $(u \in U, v \in V)$:

$$\begin{aligned} &M(u, v) \\ &= \lim_{n\to\infty} \{(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 D_n(v))\} \\ &= (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)). \end{aligned}$$

Thus $B^* \in \mathbb{E}$, which contradicts the previous suppose.

To sum up, we get $B^* \in \mathbb{E}$. Therefore B^* is the maximum of $\mathbb{E}$, and hence B^* is the MaxP-universal solution.

The proof is completed.

*Remark.*In Definition 5, A, A^*, B should be unchangeable and B^* changeable, while B^* should make (5) get its minimum for any $u \in U$ and $v \in V$. Suppose that $\to_2$ satisfies (C1). For (5), once there exists a FMP-universal solution B^*, then every fuzzy set D which is less than B^* ($D \in F(V)$), will be a FMP-universal solution. This means that there are many different FMP-universal solutions, including

$$B^*(v) \equiv 0.$$

This last is a special solution, for which (5) always takes its minimum no matter what major premise $A \to_1 B$ and minor premise A^* are adopted. Therefore, when the optimal FMP-universal solution exists, it should be the largest one; in other words, it should be the supremum of all solutions (i.e. the supremum of $\mathbb{E}$).

Theorem 4.*If $\to_2$ is a strongly residual implication satisfying the following condition:*

(C5) $I(a, b)$ is strictly increasing w.r.t. b if $a > b$ $(a, b \in [0, 1])$,

and T is the mapping residual to $\to_2$, and $A, A^* \in F(U)$, $B \in F(V)$, then the SupP-quasi-universal solution can be expressed as

$$B^*(v) = 0\chi_{E_1} + \chi_{E_1^c}, \ \ v \in V,$$

where

$$E_1 = \{u \in U|\ T(A^*(u), A(u) \to_1 B(v)) > 0\}$$

and χ_E denotes the characteristic function of the set E which is defined as

$$\chi_E(u) = \begin{cases} 1, & u \in E \\ 0, & u \notin E \end{cases},$$

and $E_1^c = U - E_1$.

*Proof.*Since $\to_2$ is a strongly residual implication, it follows that $\to_2$ satisfies (C1) and (C4).

(i) Suppose that

$$T(A^*(u), A(u) \to_1 B(v)) = 0,$$

i.e. $T(A^*(u), A(u) \to_1 B(v)) \leq 0$. It follows from the residual condition that

$$A(u) \to_1 B(v) \leq A^*(u) \to_2 0$$

holds. From the fact that $\to_2$ satisfies (C1) and (C4), we get

$$M(u, v) = (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0) = 1,$$

and

$$A(u) \to_1 B(v) \leq A^*(u) \to_2 0 \leq A^*(u) \to_2 B^*(v),$$

and hence (5) is equal to $1 = M(u, v)$, which is independent of B^*, thus we should take $B^*(v) = 1$.

(ii) Suppose that

$$T(A^*(u), A(u) \to_1 B(v)) > 0.$$

It follows from the residual condition that

$$A(u) \to_1 B(v) > A^*(u) \to_2 0$$

holds (actually, if $A(u) \to_1 B(v) \leq A^*(u) \to_2 0$, then

$$T(A^*(u), A(u) \to_1 B(v)) \leq 0,$$

a contradiction). Thus it is not difficult to get $A^*(u) > 0$ and $M(u, v) < 1$ (noting that $\to_2$ satisfies (C4)).

We shall show that (5) is equal to $M(u, v)$ iff $B^*(v) = 0$. If $B^*(v) = 0$, then obviously (5) is equal to $M(u, v)$.

If (5) is equal to $M(u, v)$, i.e.

$$\begin{aligned} &(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \\ &= M(u, v) = (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0). \end{aligned}$$

Suppose, on the contrary, that $B^*(v) > 0$.

(a) If $A^*(u) \leq B^*(v)$ or

$$A(u) \to_1 B(v) \leq A^*(u) \to_2 B^*(v),$$

then considering $\to_2$ satisfies (C4), it follows that (5) is equal to

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) = 1 > M(u, v),$$

a contradiction.

(b) If $A^*(u) > B^*(v)$ and

$$A(u) \to_1 B(v) > A^*(u) \to_2 B^*(v),$$

then considering $\to_2$ satisfies (C5), we have

$$A^*(u) \to_2 B^*(v) > A^*(u) \to_2 0,$$

and hence

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v))$$
$$> (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0) = M(u,v),$$

a contradiction. Therefore we achieve $B^*(v) = 0$.

To sum up, $B^*(v) = 0\chi_{E_1} + \chi_{E_1^c}$ (where $E_1 = \{u \in U|\ T(A^*(u), A(u) \to_1 B(v)) > 0\}$) is the SupP-quasi-universal solution (by Definition 6).

The proof is completed.

It is noted that in Theorem 4, $B^*(v) = 0\chi_{E_1} + \chi_{E_1^c}$ ($v \in V$) means that

$$B^*(v) = \begin{cases} 0, & u \in E_1 \\ 1, & u \in E_1^c \end{cases}.$$

Theorem 5. *Suppose that $\to_2$ is a strongly residual implication satisfying (C5), and that $A, A^* \in F(U)$, $B \in F(V)$, then the SupP-quasi-universal solution B^* is the MaxP-universal solution.*

Proof. From the proving process of Theorem 4, we can get that $B^*(v) = 1$ is a FMP-universal solution if

$$T(A^*(u), A(u) \to_1 B(v)) = 0,$$

and that $B^*(v) = 0$ is a FMP-universal solution if

$$T(A^*(u), A(u) \to_1 B(v)) > 0.$$

As a result, the SupP-quasi-universal solution $B^*(v) = 0\chi_{E_1} + \chi_{E_1^c}$ is a FMP-universal solution, thus B^* is the maximum in all FMP-universal solutions, i.e., it is the MaxP-universal solution.

The proof is completed.

Example 1. These implications as following are strongly residual implications satisfying (C5) where $I_{ep}, I_{y-0.5}$ are from [14, 17], and I_9 is from [18].

$I_L(a,b) = \begin{cases} 1, & a \le b \\ a' + b, & a > b \end{cases}$ (Lukasiewicz implication);

$I_G(a,b) = \begin{cases} 1, & a \le b \\ b, & a > b \end{cases}$ (Gödel implication);

$I_{Go}(a,b) = \begin{cases} 1, & a = 0 \\ (b/a) \wedge 1, & a \ne 0 \end{cases}$ (Goguen implication);

$I_{ep}(a,b) = \begin{cases} 1, & a \le b \\ (2b - ab)/(a + b - ab), & a > b \end{cases}$;

$I_{y-0.5}(a,b) = \begin{cases} 1, & a \le b \\ 1 - (\sqrt{1-b} - \sqrt{1-a})^2, & a > b \end{cases}$;

$I_9(a,b) = \begin{cases} 1, & a \le b \\ 1 - a + ab, & a > b \end{cases}$

(revised Reichenbach implication).

We get from Theorems 4 and 5 that if $\to_2 \in \{I_L, I_G, I_{Go}, I_{ep}, I_{y-0.5}, I_9\}$ then the SupP-quasi-universal solution (also the MaxP-universal solution) is

$$B^*(v) = 0\chi_{E_1} + \chi_{E_1^c}\ (v \in V).$$

Moreover, we get the following results by computing: (i) If $\to_2 \in \{I_L, I_9\}$, then

$$E_1 = \{u \in U|\ A^*(u) + (A(u) \to_1 B(v)) > 1\}.$$

(ii) If $\to_2 \in \{I_G, I_{Go}, I_{ep}\}$, then

$$E_1 = \{u \in U|\ A^*(u) \wedge (A(u) \to_1 B(v)) > 0\}.$$

(iii) If $\to_2 \in \{I_{y-0.5}\}$, then

$$E_1 = \{u \in U|\sqrt{1 - A^*(u)} + \sqrt{1 - (A(u) \to_1 B(v))} < 1\}.$$

We only prove the case $\to_2 \in \{I_L\}$ as an example. It is easy to get

$$T_L(a,b) = \begin{cases} a + b - 1, & a + b > 1 \\ 0, & a + b \le 1 \end{cases}$$

is the mapping residual to I_L, so

$$E_1 = \{u \in U|\ T(A^*(u), A(u) \to_1 B(v)) > 0\}$$
$$= \{u \in U|\ A^*(u) + (A(u) \to_1 B(v)) > 1\}.$$

Theorem 6. *Suppose that $\to_2$ is a residual implication satisfying the following conditions:*

(C6) $I(0,b) = 1$, $I(a,1) = 1$ ($a, b \in [0,1]$),

(C7) $I(a,b)$ is strictly increasing w.r.t. b if $a > 0$ ($a, b \in [0,1]$),

and that $A, A^* \in F(U)$, $B \in F(V)$, then the SupP-quasi-universal solution can be computed as

$$B^*(v) = 0\chi_{E_2} + \chi_{E_2^c}\ (v \in V)$$

where

$$E_2 = \{u \in U|\ A^*(u) \wedge (A(u) \to_1 B(v)) > 0\}.$$

Proof. Since $\to_2$ is a residual implication, it follows that $\to_2$ satisfies (C1).

(i) Suppose that $A^*(u) = 0$ or $A(u) \to_1 B(v) = 0$. From the fact that $\to_2$ satisfies (C6), we get that

$$M(u,v) = (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0) = 1$$

and hence (5) is equal to $1 = M(u,v)$, which is independent of B^*, thus we should take $B^*(v) = 1$.

(ii) Suppose that $A^*(u) > 0$ and $A(u) \to_1 B(v) > 0$. We shall show that (5) is equal to $M(u,v)$ iff $B^*(v) = 0$.

(a) If $B^*(v) = 0$, then it is obvious that (5) is equal to $M(u,v)$.

(b) If (5) is equal to $M(u,v)$, i.e.

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) = M(u,v)$$
$$= (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0).$$

Suppose, on the contrary, that $B^*(v) > 0$. Since $\to_2$ satisfies (C7), we obtain (noting that $A^*(u) > 0$, $A(u) \to_1 B(v) > 0$):

$$A^*(u) \to_2 B^*(v) > A^*(u) \to_2 0,$$

and hence

$$\begin{aligned}&(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v))\\&> (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0)\\&= M(u,v),\end{aligned}$$

a contradiction. As a result, we achieve $B^*(v) = 0$.

it follows from Definition 6 that the conclusion is correct.

Summarizing above, $B^*(v) = 0\chi_{E_2} + \chi_{E_2^c}$ (where $E_2 = \{u \in U |\ A^*(u) \wedge (A(u) \to_1 B(v)) > 0\}$) is the SupP-quasi-universal solution (by Definition 6).

The proof is completed.

Similar to Theorem 5, we can prove Theorem 7.

Theorem 7. *Suppose that $\to_2$ is a residual implication satisfying (C6) and (C7), and that $A, A^* \in F(U)$, $B \in F(V)$, then the SupP-quasi-universal solution B^* is the MaxP-universal solution.*

Example 2. The following two implications are all residual implications satisfying (C6) and (C7):

$I_Y(a,b) = b^a$ $(I_Y(0,0) = 1)$ (Yager implication);

$I_R(a,b) = 1 - a + ab$ (Reichenbach implication).

Thus by Theorems 6 and 7, we can get that if $\to_2 \in \{I_R, I_Y\}$, then the SupP-quasi-universal solution (which is also the MaxP-universal solution) is

$$B^*(v) = 0\chi_{E_2} + \chi_{E_2^c}\ (v \in V).$$

Theorem 8. *If $\to_2$ is a strongly residual implication satisfying the following condition:*

(C8) $I(a,b) = f(a)$ if $a > b$, in which $f(a)$ is a function which is independent of b $(a, b \in [0,1])$,

and T is the mapping residual to $\to_2$, and $A, A^* \in F(U)$, $B \in F(V)$, then the SupP-quasi-universal solution can be expressed as

$$B^*(v) = \inf_{u\in E_1}\{A^*(u)\}\chi_{E_1} + \chi_{E_1^c}\ (v \in V).$$

Proof. Since $\to_2$ is a strongly residual implication, it follows that $\to_2$ satisfies (C1) and (C4).

(i) Suppose that

$$T(A^*(u), A(u) \to_1 B(v)) = 0,$$

then it is similar to Theorem 4(i) that we get that (5) is equal to $1 = M(u,v)$, which is independent of B^*, thus we should take $B^*(v) = 1$.

(ii) Suppose that

$$T(A^*(u), A(u) \to_1 B(v)) > 0.$$

By the residual condition we have

$$A(u) \to_1 B(v) > A^*(u) \to_2 0$$

holds (actually, if $A(u) \to_1 B(v) \le A^*(u) \to_2 0$, then

$$T(A^*(u), A(u) \to_1 B(v)) \le 0,$$

a contradiction). Thus it is not difficult to get $A^*(u) > 0$, and

$$\begin{aligned}M(u,v) &= (A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 0)\\&= f(A(u) \to_1 B(v))\\&< 1,\end{aligned}$$

where note that $\to_2$ satisfies (C4).

We shall show that (5) is equal to $M(u,v)$ iff $B^*(v) < A^*(u)$.

(a) If $B^*(v) < A^*(u)$, then

$$A^*(u) \to_2 B^*(v) = f(A^*(u)) = A^*(u) \to_2 0,$$

thus (5) is equal to $M(u,v)$.

(b) If (5) is equal to $M(u,v)$, that is,

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) = M(u,v).$$

Suppose, on the contrary, that $A^*(u) \le B^*(v)$. Since $\to_2$ satisfies (C4) we have that $A^*(u) \to_2 B^*(v) = 1$, and hence (5) is equal to $1 > M(u,v)$, a contradiction. Thus we get $B^*(v) < A^*(u)$.

To sum up,

$$B^*(v) = \inf_{u\in E_1}\{A^*(u)\}\chi_{E_1} + \chi_{E_1^c}$$

is the SupP-quasi-universal solution (by Definition 6).

The proof is completed.

Theorem 9. *Suppose that $\to_2$ is a strongly residual implication satisfying (C8), and that $A, A^* \in F(U)$, $B \in F(V)$, and finally that*

$$A^*(u) > \inf_{u\in E_1}\{A^*(u)\}$$

holds for any $u \in E_1$, then the SupP-quasi-universal solution B^* is the MaxP-universal solution.

Proof. From the proving process of Theorem 8, if

$$T(A^*(u), A(u) \to_1 B(v)) = 0,$$

we can get that $B^*(v) = 1$ is a FMP-universal solution; if

$$T(A^*(u), A(u) \to_1 B(v)) > 0,$$

i.e., $u \in E_1$, it follows from given conditions that

$$B^*(v) = \inf_{u\in E_1}\{A^*(u)\} < A^*(u)$$

holds, thus B^* is also a FMP-universal solution.

As a result, the SupP-quasi-universal solution

$$B^*(v) = \inf_{u\in E_1}\{A^*(u)\}\chi_{E_1} + \chi_{E_1^c}$$

is a FMP-universal solution, thus B^* is the MaxP-universal solution.

The proof is completed.

*Example 3.*The following two implications are all the strongly residual implications satisfying (C8):

$$I_{GR}(a,b)=\begin{cases}1, & a\le b\\ 0, & a>b\end{cases} \text{ (Gaines-Rescher implication)};$$

$$I_{10}(a,b)=\begin{cases}1, & a\le b\\ a', & a>b\end{cases} \text{ (from [19])}.$$

Thus it follows from Theorems 8 and 9 that if $\rightarrow_2 \in \{I_{GR}, I_{10}\}$, then the SupP-quasi-universal solution is

$$B^*(v)=\inf_{u\in E_1}\{A^*(u)\}\chi_{E_1}+\chi_{E_1^c}\ (v\in V).$$

Moreover, if

$$A^*(u)>\inf_{u\in E_1}\{A^*(u)\}$$

holds for any $u\in E_1$, then $B^*(v)$ is the MaxP-universal solution. By computing, we get that

$$E_1=\{u\in U|\ A^*(u)\wedge(A(u)\rightarrow_1 B(v))>0\}$$

if $\rightarrow_2$ takes I_{GR}, and that

$$E_1=\{u\in U|\ A(u)\rightarrow_1 B(v)>(A^*(u))'\}$$

if $\rightarrow_2$ takes I_{10}.

4. Basic triple I restriction method for FMP

When $\rightarrow_1=\rightarrow_2$, the basic universal triple I restriction method degenerates into the basic triple I restriction method. Denote $\rightarrow\triangleq\rightarrow_1=\rightarrow_2$. Inspecting the results mentioned above, we can similarly get the related definitions (including FMP-solution, SupP-quasi-solution and MaxP-solution) and following conclusions of the basic triple I restriction method for FMP.

Definition 7.*Let $A,A^*\in F(U)$, $B\in F(V)$, if B^* (in $<F(V),\le_F>$) makes (3) get its minimum for any $u\in U$ and $v\in V$, then B^* is called a FMP-solution.*

Definition 8.*Suppose that $A,A^*\in F(U)$, $B\in F(V)$, and that nonempty set $\mathbb{E}_1$ is the set of all FMP-solutions, and finally that D^* is the supremum of $\mathbb{E}_1$, then D^* is called a SupP-quasi-triple I restriction solution (SupP-quasi-solution for short). And, if D^* is the maximum of $\mathbb{E}_1$, then D^* is also called a MaxP-triple I restriction solution (MaxP-solution for short).*

Proposition 1.*Suppose that $A,A^*\in F(U)$, $B\in F(V)$ and that $\rightarrow$ satisfies (C1), then the minimum of (3) is*

$$M_1(u,v)=(A(u)\rightarrow B(v))\rightarrow(A^*(u)\rightarrow 0),$$

and there exists a unique SupP-quasi-solution B^*. Furthermore, if $\rightarrow$ is left-continuous w.r.t. the second variable, then B^* is the MaxP-solution.

*Remark.*In [11], the FMP-triple I restriction method was investigated. Theorem 1.1.1 in [11] coincides with Proposition 1 in this paper (noting that the MaxP-solution is called the solution of triple I restriction method for the FMP problem in [11]). Further, we point out by Proposition 1 that if $\rightarrow$ satisfies (C1), then there exists a unique SupP-quasi-solution. Thus Proposition 1 includes the conclusions of Theorem 1.1.1 in [11].

Proposition 2.*Suppose that $\rightarrow$ is a strongly residual implication satisfying (C5), and that T is the mapping residual to $\rightarrow$, then the SupP-quasi-solution is*

$$B^*(v)=0\chi_{E_3}+\chi_{E_3^c}\ (v\in V)$$

where

$$E_3=\{u\in U|\ T(A^*(u),A(u)\rightarrow B(v))>0\}.$$

Moreover, B^* is also the MaxP-solution.

Proposition 3.*Suppose that $\rightarrow$ is a residual implication satisfying (C6) and (C7), then the SupP-quasi-solution is*

$$B^*(v)=0\chi_{E_4}+\chi_{E_4^c}\ (v\in V)$$

where

$$E_4=\{u\in U|\ A^*(u)\wedge(A(u)\rightarrow B(v))>0\}.$$

Moreover, B^* is also the MaxP-solution.

Proposition 4.*Suppose that $\rightarrow$ is a strongly residual implication satisfying (C8), and that T is the mapping residual to $\rightarrow$, then the SupP-quasi-solution is*

$$B^*(v)=\inf_{u\in E_3}\{A^*(u)\}\chi_{E_3}+\chi_{E_3^c}\ (v\in V).$$

Moreover, if

$$A^*(u)>\inf_{u\in E_3}\{A^*(u)\}$$

holds for any $u\in E_3$, then B^* is also the MaxP-solution.

*Example 4.*Let $\rightarrow\in\{I_L,I_G,I_{Go},I_{ep},I_{y-0.5},I_9,I_R,I_Y, I_{GR},I_{10}\}$ in (3), the results of FMP-solutions are as follows:

(i) Suppose $\rightarrow\in\{I_L,I_G,I_{Go},I_{ep},I_{y-0.5},I_9\}$, then the SupP-quasi-solution (which is also the MaxP-solution) is

$$B^*(v)=0\chi_{E_3}+\chi_{E_3^c},\ v\in V.$$

Moreover, if $\rightarrow\in\{I_L,I_9\}$, then

$$E_3=\{u\in U|\ A^*(u)+(A(u)\rightarrow B(v))>1\};$$

if $\rightarrow\in\{I_G,I_{Go},I_{ep}\}$, then

$$E_3=\{u\in U|\ A^*(u)\wedge(A(u)\rightarrow B(v))>0\};$$

if $\rightarrow\in\{I_{y-0.5}\}$, then

$$E_3=\{u\in U|\sqrt{1-A^*(u)}+\sqrt{1-(A(u)\rightarrow B(v))}<1\}.$$

(ii) Suppose $\rightarrow \in \{I_R, I_Y\}$, then the SupP-quasi-solution (which is also the MaxP-solution) is

$$B^*(v) = 0\chi_{E_4} + \chi_{E_4^c},\ v \in V.$$

(iii) Suppose $\rightarrow \in \{I_{GR}, I_{10}\}$, then the SupP-quasi-solution is

$$B^*(v) = \inf_{u \in E_3} \{A^*(u)\}\chi_{E_3} + \chi_{E_3^c},\ v \in V.$$

If $\rightarrow \in \{I_{GR}\}$, then

$$E_3 = \{u \in U |\ A^*(u) \wedge (A(u) \rightarrow B(v)) > 0\};$$

if $\rightarrow \in \{I_{10}\}$, then

$$E_3 = \{u \in U |\ A(u) \rightarrow B(v) > (A^*(u))'\}.$$

Further, if

$$A^*(u) > \inf_{u \in E_3} \{A^*(u)\}$$

holds for any $u \in E_3$, then B^* is the MaxP-solution.

*Remark.*By Theorems 1.1.4 and 1.1.5 in [11], Peng provided the MaxP-solutions based on I_L, I_{Go}, I_R, I_Y. It is not difficult to find that these MaxP-solutions are the same as the related conclusions of Example 4(i)(ii) in this paper. What is more, notice that Example 4(i)(ii) can be deduced by Propositions 2 and 3 in this paper, thus Theorems 1.1.4 and 1.1.5 in [11] are special cases of Propositions 2 and 3.

5. Conclusion

The basic universal triple I restriction method is proposed and researched, which includes:

(i) New basic universal triple I restriction principle for FMP is put forward, which improves the basic triple I restriction principle for FMP.

(ii) The existence condition of FMP-universal solutions, and the condition (that the SupP-quasi-universal solution is the MaxP-universal solution) are achieved, and then the unified forms of basic universal triple I restriction method for FMP are established.

Moreover, the related SupP-quasi-universal solutions (or MaxP-universal solutions) are achieved for 5 familiar implications that $\rightarrow_2$ takes respectively.

(iii) As a particular case of basic universal triple I restriction method, the related conclusions of basic triple I restriction method are achieved and improved.

How can we apply the basic universal triple I restriction method to fuzzy control, artificial intelligence [20–25] and so forth? It will be our next work.

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Yiming Tang received his M.S. degree and Ph.D. degree from School of Computer and Information, Hefei University of Technology in 2008 and 2011 respectively. He is a lecturer now in Hefei University of Technology, and his main research interests include fuzzy theory, fuzzy system, computer-aided design, affective computing and natural language processing. He is currently a member of the China Computer Federation, and ACM. Moreover, he serves as a Reviewer of several international conferences and journals.

Yanxiang Chen received her Ph.D. degree from University of Science and Technology of China in 2004. From 2006 to 2008 as a Visiting Scholar she did research in Department of Electrical Computer Engineering in University of Illinois at Urbana and Champaign, America. She is now an Associate Professor of School of Computer and Information in Hefei University of Technology, Tutor of Masters. She has been engaged in teaching and researches in fuzzy theory, speech communication and signal processing with professional interests in speech recognition, speaker recognition, speech retrieval and emotional speech recognition.