Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/120102

Modified shifted Chebyshev residual spectral scheme for even-order BVPs

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Received: 25 Jul. 2022, Revised: 28 Nov. 2022, Accepted: 25 Dec. 2022
Published online: 1 Jan. 2023

Abstract: This paper aims to determine approximate numerical solutions for various linear and nonlinear even-order boundary value problems. The proposed method relies on a direct numerical technique utilizing modified shifted first-kind Chebyshev polynomials. By expressing an approximate solution as a finite sum of polynomials with unknown coefficients, we aim to derive an approximation that satisfies the given boundary conditions. Various linear and nonlinear even-order boundary-value problems have been solved. The obtained results were compared with the other methods. The comparisons showed the efficiency and accuracy of the introduced scheme.

Keywords: Shifted Chebyshev polynomials; Residual spectral method; Boundary value problems

1 Introductions

Boundary value problems (BVPs) have gained significant importance as they are used in various scientific applications domains [1–6]. However, most of these applications need exact analytical solutions. Hence, approximate numerical methods will be required. To this end, several numerical methods exist, each with advantages and disadvantages. Some of the authors choose to use the finite difference method [7]. The finite element method was applied in [8].

Spectral methods have gained popularity due to their ability to obtain semi-analytic solutions. The spectral method has been demonstrated as an effective and precise technique for solving various differential equations, particularly those with smooth solutions. Nonetheless, its implementation can prove more intricate than alternative numerical methods, requiring more excellent computational resources when tackling more considerable problems.

The spectral methods consist of three main types: Galerkin, Tau, and collocation. This technique assumes an approximate solution $V(t)$ as a set of unknown constants multiplied by orthogonal polynomials well-suited to the problem.

$$V(t) \approx V_r(t) = \sum_{j=0}^{r} c_j p_j(t). \tag{1}$$

Once spectral methods have been applied, the differential equation is transformed into a system of algebraic equations with unspecified constants. At this point, one can use various numerical methods, such as Newton's method or Gaussian elimination, to determine these constants. When the constants have been determined, they can be used to obtain an approximate solution to the equation.

The Chebyshev polynomials are chosen to be the base function in [9, 10]. The authors in [11] used a technique that depends on Legendre polynomials. Also, the ultra-spherical is applied in [12]. While in [13], the authors select the Monic Chebyshev. However, some authors choose the derivative of the polynomials as its base function [15].

Here we choose the shifted Chebyshev polynomials and modify them. Then, we used them in the Galerkin method. This paper is molded into four sections. All necessary definitions and relations are discussed in

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section two. The proposed technique will be presented in section three to find the approximate solutions. Finally, through section four, some of the BVPs were solved to prove the efficiency of our method.

2 Preliminaries

Some essential relations of shifted Chebyshev polynomials (SCHPs) in the interval $[0,1]$ will be presented. The recurrence relation expressed as [16, 17]:

$$T^*_{i+2}(t) = 2(2t-1)T^*_{i+1}(t) - T^*_i(t), \quad k = 0,1,2,\cdots, \tag{2}$$

with initials $T^*_0(t) = 1$ and $T^*_1(t) = 2t-1$.
Their inequalities:

$$T^*_i(0) = (-1)^i, \quad T^*_i(1) = 1, \tag{3}$$

$$T^{*'}_i(0) = 2(-1)^{i-1}i^2, \quad T^{*'}_i(1) = 2i^2, \tag{4}$$

$$|T^*_i(t)| \leq 1, \quad |T^*_i(t)| \leq 2i^2. \tag{5}$$

The SCHPS is defined in power series as:

$$T^*_i(t) = i\sum_{j=0}^{i} \frac{(-1)^{i-j}2^{2j}(i+j-1)!}{(i-j)!(2j)!}t^i. \tag{6}$$

The following section will discuss the method for obtaining approximate solutions for various BVPs. Also, the problem formulation will be presented.

3 The Proposed Method and Problem Formulation

Assume the BVP with even-order m

$$V^{(m)}(t) = f\left(t, V(t), V'(t), ..., V^{(m-1)}\right), \quad t \in [0,1], \tag{7}$$

with the homogeneous initial and boundary conditions:

$$\begin{aligned} V(0) = V'(0) = V''(0) = \cdots = V^{(\frac{m}{2}-1)}(0) = 0, \\ V(1) = V'(1) = V''(1) = \cdots = V^{(\frac{m}{2}-1)}(1) = 0. \end{aligned} \tag{8}$$

As we mentioned in Equation (1), we choose the orthogonal polynomials $U(t)$ which defined as:

$$U_{n,j}(t) = t^n(1-t)^nT^*_j(t); \quad n, j = 0,1,\cdots. \tag{9}$$

Suppose the approximate solution of the BVP (7) and (8) as:

$$\begin{aligned} V(t) \approx V_r(t) &= \sum_{i=0}^{r} c_iU_{n,i}(t), \\ V'(t) \approx V'_r(t) &= \sum_{i=0}^{r} c_iU'_{n,i}(t), \\ &\vdots \\ V^{(m)}(t) \approx V^{(m)}_r(t) &= \sum_{i=0}^{r} c_iU^{(m)}_{n,i}(t), \end{aligned} \tag{10}$$

where c_i are constants.

By substituting from Equation (10) into Equations (7) and (8) to get the residual:

$$\begin{aligned} R_r(t) = \sum_{i=0}^{r} c_iU^{(m)}_{n,i}(t) - \\ f\left(t, \sum_{i=0}^{r} c_iU_{n,i}(t), \sum_{i=0}^{r} c_iU'_{n,i}(t), ..., \sum_{i=0}^{r} c_iU^{(m-1)}_{n,i}(t)\right). \end{aligned} \tag{11}$$

Apply the Galerkin method to obtain the following algebraic system:

$$R(t_j) = 0; \quad j = 0,1,2,\cdots r. \tag{12}$$

The system of equations with unknown coefficients c_i is generated from the previous equation. Then, by using any numerical technique, we can find the coefficients c_i. Finally, the approximate solution is formed. We can present the technique of the solution in Algorithm 1:

Algorithm 1 Steps for solving BVPs via $U_{n,i}(t)$

Step 1: Enter $r \in \mathbb{N}$;
Step 2: Shift the range to [0,1];
Step 3: Choose the point $\{t_i\}_{i=0}^{r}$;
Step 5: Expressed the BVP using equation (10);
Step 6: Use the system from step 5 to find c_i;
Step 7: Use the c_i from the previous step to get the approximate solution.

4 Numerical Examples

Within this section, we shall solve some BVPs aimed at explicating the efficacy and precision of the proposed methodology. Additionally, convert the problems, which are non-homogeneous initial and boundary conditions.

*Example 1.*Consider the fourth-order boundary value problem, which describes the model of bending of a beam pinned from both sides:

$$\begin{aligned} U^{(4)}(t) = (t^4 + 14t^3 + 49t^2 + 32t - 12)e^t, \quad t \in [0,1], \\ U(0) = U'(0) = U(1) = U'(1) = 0. \end{aligned} \tag{13}$$

While its exact solution is:

$$U(t) = t^2(1-t)^2e^t \tag{14}$$

Table (1) shows the MAE compared with another method.

Table 1: The MAE for Example 1.

N	Proposed method	[18]
6	$1.1 * 10^{-8}$	-
8	$2.4 * 10^{-11}$	$9.3 * 10^{-8}$
16	$4.2 * 10^{-16}$	$4.7 * 10^{-13}$

*Example 2.*Consider the fourth-order equation:

$$32U^{(4)}(t) - 8U^{(2)}(t) - 2U(t) = (t-5)e^{\frac{t+1}{2}}, \quad t \in [-1,1], U(-1) = 1, U(1) = 0, \quad U'(-1) = 0, U'(1) = \frac{-e}{2}. \tag{15}$$

While its exact solution is:

$$U(t) = \frac{1-t}{2} e^{\frac{1+t}{2}} \tag{16}$$

We shifted to the range to be $[0,1]$ and converted the conditions to be homogeneous at $n = 2$. Table (2) and Table (3) show the efficiency and accuracy of the proposed method. Also, Fig. (1) shows the stability of the error through the interval $[0,1]$.

Table 2: The MAE for Example 2.

N	Proposed method	[19]
8	$1.70*10^{-14}$	$4.69*10^{-3}$
12	$1.31*10^{-16}$	$9.47*10^{-7}$

Table 3: The absolute error (AE) for Example 2.

t	The proposed method $N=10$	MTA [19] $N=20$	SSM [20]
-1	0	0	0
-0.6	$1.43*10^{-16}$	$2.17*10^{-14}$	$6.53*10^{-4}$
-0.2	$6.77*10^{-17}$	$3.46*10^{-14}$	$1.75*10^{-3}$
0.2	$2.08*10^{-17}$	$5.28*10^{-14}$	$1.78*10^{-3}$
0.6	$7.11*10^{-17}$	$1.23*10^{-14}$	$6.85*10^{-4}$
1	0	0	0

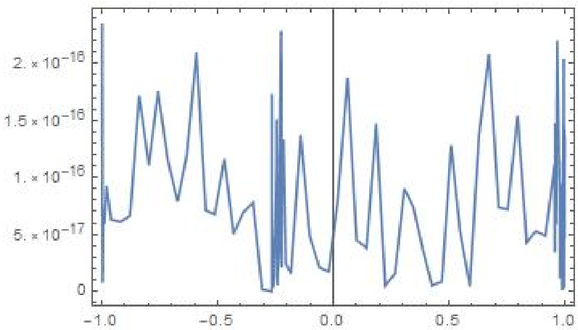

Fig. 1: The absolute error by at $N = 10$ for Example 2.

*Example 3.*Consider the following nonlinear differential equation:

$$U^{(4)}(t) + (U''(xt)^2 = \sin t + \sin^2 t \quad t \in [0,1], \quad U(0) = 0, U'(0) = 1, U(1) = \sin(1), U'(1) = \cos(1). \tag{17}$$

While its exact solution is:

$$U(x) = \sin t. \tag{18}$$

Table (4) shows the correctness of the presented method and polynomials. In addition, the stability showed Fig. (2).

Table 4: The AE of different methods for Example 3 at $N = 5$.

t	Proposed method	[21]
0	0	$1.65*10^{-25}$
0.1	$8.00*10^{-12}$	$1.20*10^{-10}$
0.2	$1.43*10^{-11}$	$4.01*10^{-10}$
0.3	$1.91*10^{-11}$	$7.30*10^{-10}$
0.4	$2.24*10^{-11}$	$1.00*10^{-9}$
0.5	$2.38*10^{-11}$	$1.12*10^{-9}$
0.6	$2.36*10^{-11}$	$1.03*10^{-9}$
0.7	$2.11*10^{-11}$	$7.64*10^{-10}$
0.8	$1.66*10^{-11}$	$4.17*10^{-10}$
0.9	$9.70*10^{-12}$	$1.21*10^{-10}$
1	$2.22*10^{-16}$	$1.46*10^{-23}$

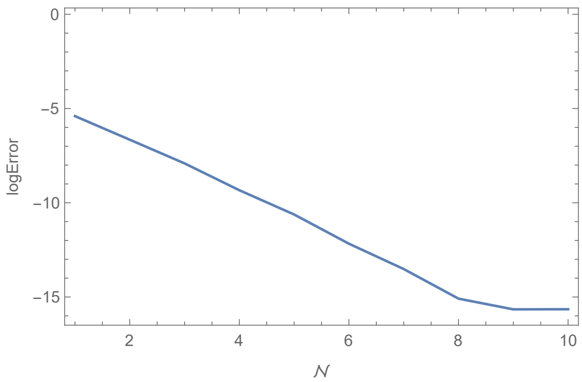

Fig. 2: The Log-error for Example 3.

5 Concluding Remarks

This paper has successfully demonstrated the feasibility of utilizing a direct numerical technique based on modified shifted Chebyshev polynomials to determine approximate numerical solutions of various linear and nonlinear even boundary-value problems. The approach involved expressing an approximate solution as a finite sum of polynomials and unknown coefficients. The proposed method effectively and accurately solved both linear and nonlinear even boundary value problems.

Availability of data and material

The authors did not use any scientific data during this research.

Conflict of interest

The authors declare that they have no conflict of interest.

Funding

The authors received no financial support for the research.

Authors' contributions

All authors contributed equally work.

Acknowledgement

The authors would like to thank the anonymous reviewers for carefully reading the article and also for their constructive and valuable comments, which have improved the paper in its present form. The authors are also sincerely grateful to the Helwan School of Numerical Analysis in Egypt (HSNAE) members for their valuable effort and support.

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