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Core Expansion and Spiral Breakup in Oscillatory Recovering Media

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Abstract: We studied a new type of meandering of spiral waves in oscillatory recovering media using the modified Barkley's model. The core region expands, and the spiral tip follows itself an outward spiral-like trajectory. The spiral waves then break up near the center of rotation, leading to spatiotemporal irregularity. This mechanism, as pointed out by Garfinkel, is still unknown. We investigate the effect of the local dynamics and fixed points and study the transition to meandering and core expansion as the control parameter is varied. The resulting non-monotonic dispersion curve has implications about similar behavior observed in the restitution curve and attributed to double-wave reentry in cardiac tissue as indicated by other authors (Chaos 29 (2019), 073108).

Keywords: Spiral breakup, core expansion, oscillatory recovery, dispersion curve.

1. Introduction

Spiral waves have been frequently observed in many excitable and oscillatory media. Examples include chemical reactive solutions [1,2], in bacterial colonies [3], and in cardiac tissue [4,5]. Understanding the mechanisms of these spiral waves remains one of the challenging problems in nonlinear dynamics. The spiral waves dynamics are determined by the values of the control parameters. As the parameters values are changed, transitions are observed from regular patterns to spiral breakup, leading to spatiotemporal chaos or irregularity [6,7,8,9,10]. This behavior is responsible for the onset of cardiac fibrillation [4,11,12]. Using Optical mapping in the epicardial surface of an explanted human heart, spiral wave reentry has been observed as the route from tachycardia to fibrillation [12].

Spiral breakup has also been encountered in experiments with the Belouzov-Zhabotinsky (BZ) reaction [13,14]. One common scenario for the breakup in oscillatory and excitable media is the Eckhaus instability [7,8,9]. It occurs when a long wavelength modulation appears, so that the minimum of the local period violates the dispersion relation. Spiral breakup can also occur without any modulation mode. It occurs in an oscillatory reaction diffusion system when the rotation center of the stable spiral is located at the unstable focus [15]. A similar phenomenon occurs in excitable media when the recovery is oscillatory and not monotonic, and it can be attributed to core expansion [6]. We investigate it in this paper using the modified Barkley's model [16].

2. The model and numerical methods

The reaction diffusion model here is given by:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{\varepsilon}u(1-u)[u-((b+v)/a)]+\nabla^2 u, \\ \frac{\partial v}{\partial t} &= u^3 - v \end{aligned} \tag{1}$$

where u and v are the excitatory and recovery (inhibitory) variables,

In the local dynamics, there are three fixed points: (0, 0), (1, 1) and the point (u_o,v_o) given by the solution of the equation: $u^3 + b = au$. For the parameter values $a = 0.75$, $b = 0.06$, only (0, 0) is stable. The other two fixed points, (1, 1) and $(u_o, v_o) = (0.080701, 0.000526)$ for $b = 0.06$, $a = 0.75$ and $\varepsilon^{-1} = 13.5$, are saddle points. The eigenvalues of (u_o, v_o) are given by the solutions of the

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following equation:

$$\lambda^2 + [1 + u_o(u_o - 1)/\varepsilon]\lambda + u_o(u_o - 1)/\varepsilon \\ -3u_o^3(u_o - 1)/a\varepsilon = 0 \qquad (2)$$

The local dynamics are excitable. Depending on the value of the parameter ε, the recovery to the resting state is monotonic for low values of ε or damped oscillatory as ε is increased. The inverse of the parameter ε specifies the recovery time and characterizes the abruptness of excitation. In the standard Barkley's model, the local dynamics term in Eq. (1) is $(u - v)$. This implies that the recovery rate here is slower. The use of this nonlinear form $(u^3 - v)$ allows here maintaining the wave propagation in one spatial dimension for high values of ε^{-1}, due to the delay in the production of the recovery variable v. For some values of the system control parameters, the spiral wave undergoes simple rigid rotation around a circular core. As the control parameter ε is varied, the spiral end follows other meandering trajectories, as shown in Fig. 1.

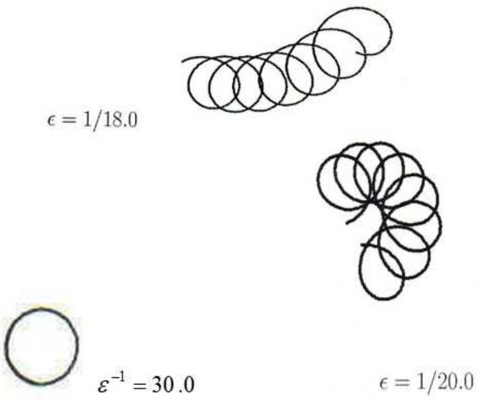

Fig. 1: Tip meandering patterns for different values of ε.

3. Discussion

A spiral wave was created by the coarse gradient cross-field method: $u = 0$ in the left half of the medium and $u = 1$ in the right half; $v = a/2$ in the upper half and $v = 0$ in the lower half. In Fig. 2, snapshots show the dynamics of the system at various times. The first snapshot corresponds to t = 75 t.u, the third is at t = 86 t.u, the 6th is at 103 t.u, the 7th is at 107 t.u. As we increased the value of the parameter ε, the spiral wave deviates from stable simple rotation, the tip meanders and with further increase in the parameter ε, interesting new meandering types occur as shown in Fig. 3, as the spiral inner core region expands in time. The tip is defined here as the intersection of the two isolines $u = 0.5$ and $v = 0.5u - b$. A new phenomenon is observed in Fig.3, as the core grows, the tip itself moves out along a "spiraling" trajectory. Spiral breakup occurs when the spiral period reaches the threshold value of the period that corresponds to plane wave propagation for this value of ε [17]. Then the spiral rotates more rapidly than plane waves can propagate. Notice that the recovery rate is slower for the cubic local dynamics term $(u^3 - v)$ compared to the linear kinetics $(u - v)$. Since the core is a source of waves, the tip moves outward and forward, the spiral period changes due to Doppler shift. At some locations within the excitable medium, for critical values of ε, the period will reach the minimum period compatible with plane wave propagation. At this instability, the spiral breaks down into wave breaks, which would generate new spiral waves leading to spatiotemporal chaos. Unstable meandering due to lowered excitability has been shown in [18] to lead to spiral break up (BU) and similarly attributed to Doppler effect. But the tip trajectory was not like a spiral path. Such Doppler instability at the spiral center has also been initiated by [19] and lead to turbulence that was controlled by injecting feedback signals which resulted in stabilizing the spiral wave as a means to prevent spiral breakup. However, no core expansion was reported and the reason behind the instability was not described. The Doppler-induced instability leading to spiral break up, is seen here to be due to the core expansion, which was attributed to non-local effects and curvature effects in the case the medium recovers in an non-monotonic manner. Such spiral break up (BU) near the core region has been attributed to Eckhaus instability in [7], but later this claim was disproved by [10] and the BU near the core region was attributed to the nonlinear Doppler effect resulting from the meandering instability. The value in [10] at the onset the instability for the parameter was $\varepsilon = 0.077$, which is very close to the critical value that we obtained here $\varepsilon^{-1} = 13.5$.

4. Spiral waves core expansion

A similar core expansion was investigated by Yang and Garfinkel [15] in oscillatory media. The value of the parameter a was changed and for lower values of a, in the stable spiral regime, the spiral wave breaks up near the tip region. The spiral core expands and the tip spirals out along the previous spiral arm. They attributed this behavior to the presence of the unstable focus at the location of the spiral center. For low values of a, the rotation center cannot maintain periodic waves anymore as the faster small amplitude oscillations dominate, which leads to the destruction of the stable spiral wave. In our case, the excitable medium recovers in an oscillatory way, and that lead to core expansion, without the presence of an unstable focus. According to [15], in the oscillatory

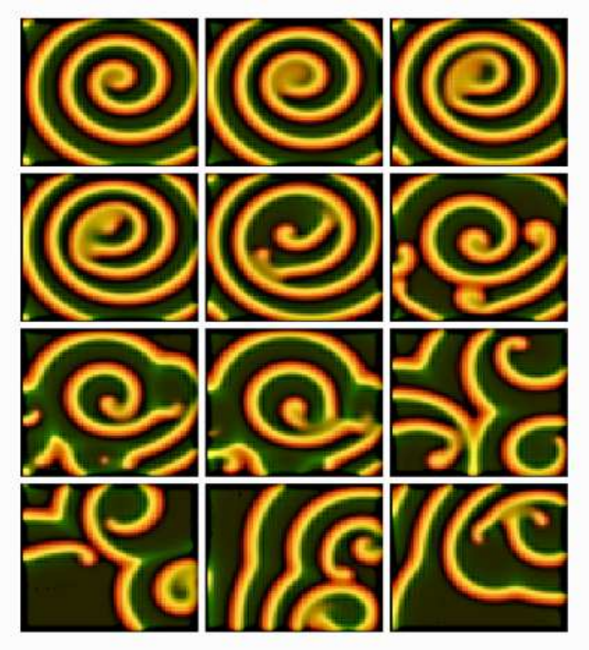

Fig. 2: Snapshots where the time intervals between snapshots are not equal, showing the scenario leading to spatiotemporal chaos with grid size 95×95, $\varepsilon^{-1} = 13.5$, $a = 0.75$, $b = 0.06$, $dt = 0.052$, $dx = 0.51$, $L = 48$.

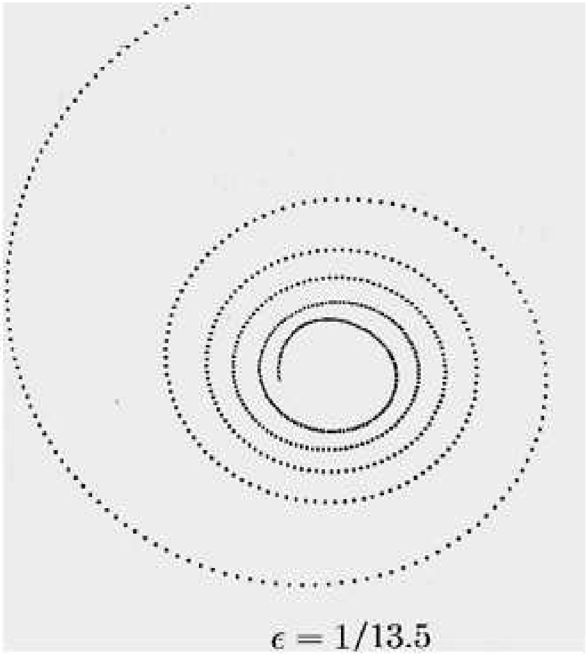

Fig. 3: The outward "spiral" motion of the tip for $\varepsilon^{-1} = 13.5$.

medium, the spiral break up occurred directly from the stable spiral regime. In our case, with the change of the parameter ε, meandering patterns occurred first (Fig. 1), and with further increase of ε, core expansion is observed. The value of $\varepsilon = 0.02$ in [15], and the control parameter was a. Here, the value of $a = 0.75$, and the value of ε is increased to 1/13.5. The use of such large values of ε is possible since we are using the cubic term ($u^3 - v$) instead of the linear term ($u - v$) like in [15]. The recovery rate is slowed down by using the cubic dependence on u, i.e., the modified Barkley's model. In a monotonically recovering medium, perturbations near the spiral core are usually weakened by repulsive wave-front interactions. In the case of oscillatory recovery, when we increase the values ε, the spiral core expands, and the tip follows an outward motion along a spiraling trajectory. This expansion was predicted by the theory of non-local effects [20,21] and it is attributed to wave-front interactions and curvature effects. The interaction of wave-fronts propagating in an excitable medium is determined by the manner of recovery to the rest state. If the excitable medium recovers in an oscillatory manner, the wave-fronts would lock up at fixed distances from the preceding ones as shown below.

The solution of the reaction-diffusion system is written as the resultant of two solitary waves with an additional small perturbation term which is negligible in the limit of infinite distance between the two waves:

$$u(z) = u(z - z_1) + u(z - z_2) + R \tag{3}$$

where, $z_1 = x_1 - ct$ and $z_2 = x_2 - ct$, x_1 and x_2 are the waves positions, and c is the propagation speed of the solitary waves. For large $|z_1|$ and $|z_2|$, the tail of the wave determines the medium's recovery type to the resting state: When the recovery is damped oscillatory, the wave's tail is given by $u(z) \propto e^{\eta z}\cos(\nu z + \psi)$; when the recovery is non-oscillatory, $u(z) \propto e^{\eta z}$. In both cases, the leading edge of the wave is assumed to be of the form $u(z) \propto e^{-\mu z}$. Equations for x_1 and x_2 were obtained based on the solvability conditions that remove singularities from [22]:

$$\frac{\delta x_1}{\delta t} = c + a_R e^{-\mu(x_1 - x_2)} \tag{4}$$

$$\frac{\delta x_2}{\delta t} = c + a_L e^{-\eta(x_1 - x_2)} cos(\nu(x_1 - x_2) + \Psi) \tag{5}$$

The second term on the right-hand side of Eq. (4) corresponds to the effect exerted by the second wave on the propagation of the first wave. In excitable media, it is usually neglected. The second term on the right-hand side of Eq. (5) corresponds to the effect of the refractory wake of the first wave on the propagation of the second wave. Using Eq. (4) and Eq. (5), the distance between the two waves $\lambda = x_1 - x_2$ is governed by the equation:

$$\frac{d\lambda}{dt} = a_L e^{-\eta\lambda} cos(\nu\lambda + \Psi) \tag{6}$$

If $\nu \neq 0$, the excitable medium recovers in an oscillatory way. Then, it follows from Eq. (6) that an infinite number of steady state solutions exist. This means that the spacing between the wave-fronts takes one of many possible values. This agrees with the results in [23] where there was alternation of the time intervals between the two waves in double-wave reentry in an anatomic circuit consisting of a mono-layer of chick embryo cardiac cells grown in an annular geometry. The corresponding velocity restitution curve was non-monotonic. Such non-monotonic restitution relationships result in cases in which various spacing values between circulating waves are possible.

Different solutions were obtained for the parameter values $a = 0.75$ and $b = 0.06$, $\varepsilon^{-1} = 13.5$ and 14.5. The numerical solution in Fig.4 is seen to approach the resting state in an oscillatory manner. For lower values of the parameter ε, it reaches the resting state (0, 0, 0) in a non-oscillatory manner for $\varepsilon^{-1} = 50.0$. For this value of ε, the spiral undergoes circular rigid rotation. If $\varepsilon^{-1} = 20.0$, the recovery is also monotonic, but there is meandering rather than rigid rotation. The spiral end follows an epicycle-like trajectory (shown in Fig. 1). Unlike the results for $\varepsilon^{-1} = 13.5$, the spiraling motion of the tip due to core expansion was not observed for $\varepsilon^{-1} = 14.5$, even though the recovery is of the oscillatory type. The tip does not follow an outward “spiraling” trajectory. The meandering type is like an epicycle trajectory. More specifically, our results showed “spiraling” tip trajectory for $13.0 < \varepsilon^{-1} < 13.8$. For $\varepsilon^{-1} > 13.9$, it meanders but not along a spiraling orbit. We conclude then that oscillatory recovery does not always imply core expansion. Final conclusions about the spiral breakup (BU) mechanism are not in general easy to obtain from numerical simulations alone. It is worth noting, however, that while it is true the oscillatory recovery causes core expansion, the reverse conclusion − core expansion ”proves” oscillatory recovery − in not valid.

In Fig. 5, the dispersion curve, wave speed as a function of period or the time interval since the end of the preceding excitation at the same point, is shown for $\varepsilon^{-1} =$ 13.5 and 50.0, computed by repetitively stimulating at one of the ends of an open line. It exhibits damped oscillatory character for $\varepsilon^{-1} = 13.5$. This is in total agreement with the non-monotonic velocity restitution curve in [23]. However, in [23], it corresponded to complex eigenvalues. In Fig. 5, the first super-normal period during which the excitability is higher than that of the rest state is very pronounced. However, for $\varepsilon^{-1} = 50.0$, as expected, the monotonically recovering system corresponds to a monotonic increase in the propagation velocity until the limit set by the solitary wave velocity is reached. Such oscillatory restitution curve was obtained in [23]. Two re-entrant excitation waves circulated as a doublet in a closed loop in chick embryo cardiac tissue. Such behavior is characteristic of cardiac arrhythmias. It commonly occurs during tachycardia during which the heart beats abnormally too rapidly. The velocity restitution curve is not of the typical type, where the propagation velocity increases with as the time interval increases. It is of the oscillatory type like the one we obtained here in the case of oscillatory recovering excitable.

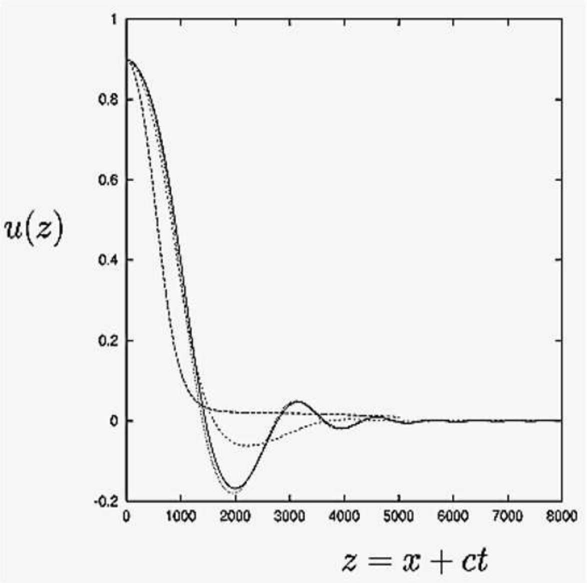

Fig. 4: Solitary traveling solution $u(z) = u(x + ct)$ of Eq. (1) illustrating damped oscillatory recovery for $\varepsilon^{-1} = 13.5$ and 14.5, and monotonic recovery for $\varepsilon^{-1} = 20.0$ and 50.0.

In Fig. 6, we show the irregularity in the 1D dynamics for $\varepsilon^{-1} = 13.5$ and the regular behavior for $\varepsilon^{-1} = 50.0$. The corresponding time variation show that the onset of spiral chaos for $\varepsilon^{-1} = 13.5$ which corresponds to damped oscillatory recovery, while there is no irregularity for $\varepsilon^{-1} = 50.0$ which corresponds to monotonic recovery.

5. Conclusions

We present a new scenario for the breakup (BU) using a particular meandering type along an outward spiral-like trajectory, leading to spiral chaos. This type of destabilization of simple rigid rotation as the parameter ε is decreased is attributed to curvature effects and wavefronts interactions in the case of oscillatory damped recovery to the resting state in excitable media. It occurs in this excitable reaction diffusion system due to the non-monotonic recovery, and without any unstable focus as it is the case for an oscillatory medium. The spiral wave is found to break up into smaller spirals near the center of rotation. This is a Doppler-induced instability and not an Eckhaus instability as it was described by [7]. It is in agreement with the analysis in [10].

We also presented here some understanding about the recovery type that corresponds to double-wave reentry in the context of physiological properties of cardiac tissue.

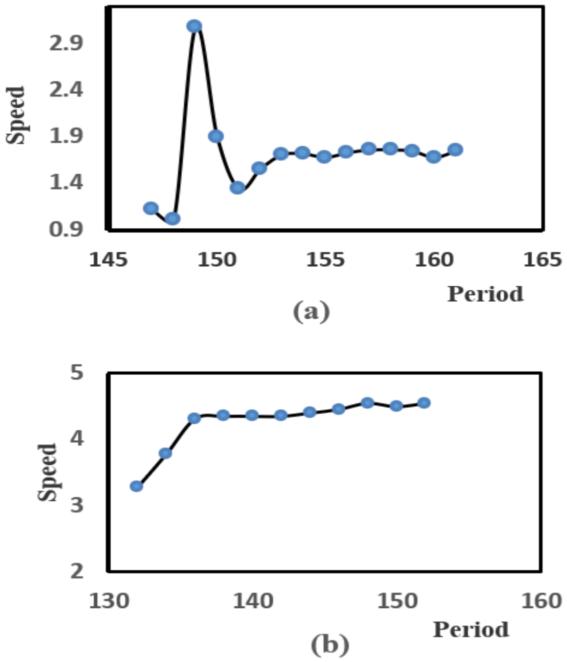

Fig. 5: Dispersion curve: Speed as a function of the period for $\varepsilon^{-1} = 13.5$, and monotonic recovery for $\varepsilon^{-1} = 50.0$.

The non-monotonic dispersion curve that was obtained here is typically observed for the case of double-wave reentry in cardiac tissue. The Doppler-induced instability leading to spiral chaos, is initially started by the core expansion, due to non-local effects, and is found to be characteristic of non-monotonically recovering excitable media. This transition from regular patterns to spatiotemporal chaos in such excitable media remains however a challenging problem and will be the subject of further investigation.

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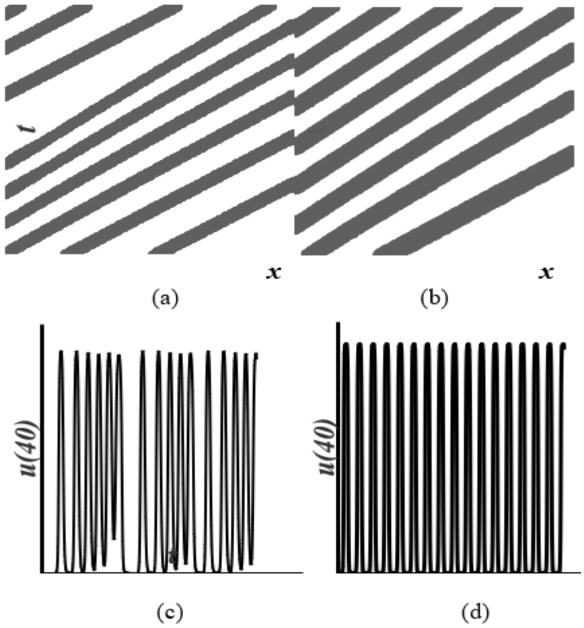

Fig. 6: The 1D dynamics for *(a)* $\varepsilon^{-1} = 13.5$, and *(b)* $\varepsilon^{-1} = 50.0$. The corresponding time dependence is shown at site #40 for *(c)* $\varepsilon^{-1} = 13.5$, *(d)* $\varepsilon^{-1} = 50.0$.

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