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Review of Longitudinal Waves Propagation in Layered Media: Theory, Modeling Techniques, and Applications

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Abstract: The propagation of longitudinal waves in layered media plays a fundamental role in geophysics, civil engineering, materials science, and non-destructive testing. The longitudinal wave is a type of wave in which all particles within the medium (such as wire, air, water, glass, rock) oscillate parallel to the direction of propagation. Earlier research on the propagation of longitudinal waves within stratified media was founded on their elastic characteristics. However, recent studies on longitudinal waves indicate that the confinement of longitudinal waves, as well as their propagation behavior, can be engineered through the design of surface environments, shapes, structures, materials, etc., enabling a unique and remarkable system for the propagation of longitudinal waves in layered media. In addition to the fundamental elastic properties of longitudinal waves, as well as the elastic properties of layered media, the study of longitudinal waves may pave the way for highly integrated platforms for material models, interface conditions, and attenuation characteristics of longitudinal waves in both basic sciences and practical applications. In this study, we present a review of the propagation of longitudinal waves in elastic layered media, in a sequence of theoretical framework, analytical and numerical solutions, with the main applications of longitudinal wave propagation in engineering and scientific fields, focusing on the dispersion of elastic longitudinal waves. Furthermore, this study examines prevailing research trends, future challenges, and prospective directions for the propagation of longitudinal waves in stratified elastic media.

Keywords: Longitudinal waves; Layered media; Equation of Motion; Transfer matrix method; Finite element method; Dispersion waves.

1 Introduction

Longitudinal wave propagation in layered media has been the subject of sustained scientific investigation for more than a century due to its fundamental role in elastic wave theory and its critical importance in engineering and geophysical applications. Longitudinal waves also referred to as compressional or dilatational waves are characterized by particle motion parallel to the direction of wave travel and represent the primary mechanism for stress and energy transmission in elastic solids [1–3]. In layered media, where material properties vary discretely or continuously across spatial dimensions, the propagation characteristics of longitudinal waves become markedly more complex than in homogeneous continua.

Layered elastic media arise naturally in geological formations, biological tissues, and engineered materials. Stratified Earth layers, sedimentary basins, and crystalline rock formations exhibit strong elastic contrasts that significantly influence the propagation of compressional seismic waves [4, 5]. In engineering practice, layered configurations are ubiquitous in laminated composites, thin coatings, bonded joints, sandwich structures, microelectronic devices, and functionally graded materials [6–8]. The increasing reliance on such materials has intensified interest in understanding how longitudinal waves interact with interfaces, inclusions, and periodic heterogeneities.

From a theoretical perspective, longitudinal wave propagation originates from the classical equations of linear elasticity, derived from conservation of momentum and constitutive stress–strain relations [9, 10]. In one-dimensional homogeneous rods, these equations yield a simple wave equation with constant wave speed

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and nondispersive behavior [11]. However, the introduction of layering leads to spatially varying elastic moduli and densities, rendering the governing equations piecewise-defined and necessitating the enforcement of continuity conditions at interfaces [12, 13]. As a consequence, wave speed, impedance, and attenuation become strongly dependent on frequency, layer thickness, and material contrast.

One of the most important physical phenomena associated with propagation of longitudinal waves in layered media is the reflection and transmission waves at the interfaces of the media. When a longitudinal wave encounters a boundary between two layers with different acoustic impedances, part of the incident energy is reflected while the remainder is transmitted into the adjacent layer [14]. In multilayered systems, repeated reflections lead to multiple scattering, constructive and destructive interference, and resonance effects that profoundly alter the overall wave field [15, 16]. These mechanisms are responsible for the emergence of dispersion and attenuation even in lossless elastic materials.

Periodic layered media represent a particularly important class of structures, as they give rise to band-gap behavior in which wave propagation is prohibited over specific frequency ranges [17–19]. This phenomenon, analogous to electronic band gaps in solid-state physics, has been extensively studied in the context of photonic crystals and elastic metamaterials [20–22]. Longitudinal waves in periodic layered rods and bars exhibit Bragg scattering and local resonance effects, enabling unprecedented control over wave transmission, filtering, and vibration suppression [23–25].

To analyze longitudinal wave propagation in layered media, a wide range of analytical and semi-analytical methods has been developed. Early work employed classical solutions based on plane wave expansions and impedance matching techniques [26]. Subsequently, matrix-based approaches such as the transfer matrix method, Thomson–Haskell method, and propagator matrix method became standard tools for modeling wave propagation in stratified media [27, 28]. These methods allow for systematic treatment of multilayer systems and facilitate the derivation of dispersion relations, reflection coefficients, and transmission spectra.

Despite their effectiveness, traditional transfer matrix formulations can suffer from numerical instability at high frequencies or for systems with many layers [29]. To address these limitations, alternative formulations such as the global matrix method, stiffness matrix method, and recursive algorithms have been proposed and widely adopted [30–32]. These approaches have proven particularly useful in modeling longitudinal waves in thick multilayered composites, coated rods, and anisotropic laminates.

Dispersion analysis constitutes a central theme in the study of longitudinal wave propagation in layered media. Dispersion curves reveal how phase and group velocities vary with frequency and provide critical insight into wave guiding behavior, cutoff frequencies, and energy transport mechanisms [33–35]. In layered rods and plates, dispersion characteristics are strongly influenced by geometric confinement, interfacial stiffness, and periodicity [36, 37]. Such insights are essential for the design of wave-based sensing systems and vibration mitigation devices.

With the advent of modern computational resources, numerical methods have become indispensable for investigating longitudinal wave propagation in layered structures. Finite element methods (FEM) enable time-domain and frequency-domain simulations of wave motion in complex geometries, accounting for material anisotropy, damping, and imperfect interfaces [38, 39]. Spectral element methods and wave finite element techniques further enhance computational efficiency and accuracy, particularly for high-frequency wave problems [40, 41].

Experimental research has been instrumental in confirming the validity of theoretical and computational models. Ultrasonic testing techniques employing longitudinal guided waves have been extensively used to characterize layered materials, detect defects, and evaluate interfacial integrity [42–44]. Advances in laser ultrasonic and noncontact measurement methods have enabled high-resolution observation of wave propagation phenomena in multilayered systems [45, 46].

The practical significance of longitudinal wave propagation in layered media extends across multiple disciplines. In nondestructive evaluation, longitudinal waves are used to detect cracks, delamination, and corrosion in layered structural components [47, 48]. In geophysics, compressional wave analysis forms the backbone of seismic imaging and subsurface characterization [49, 50]. In biomedical engineering, the interaction of longitudinal ultrasound waves with layered tissues underpins diagnostic imaging and electrography techniques [51, 52].

A thorough study that incorporates classical theory, contemporary modeling methods, and experimental developments pertaining to longitudinal wave propagation in layered media is obviously needed given the large and quickly growing body of research. This review aims to provide such a synthesis, emphasizing dispersion mechanisms, interface effects, analytical and numerical methodologies, and emerging applications, while also identifying open challenges and future research directions.

The rest of this article is structured as follows. Section 2 introduces real live examples of longitudinal waves. Section 3 introduces the governing equations and theoretical background of longitudinal wave propagation in layered media, and emphasizes the continuum mechanics framework, equation of motion, constitutive relations, non-dispersive longitudinal wave equation in layered media, interface and boundary conditions, reflection and transmission coefficients, phase and group velocities of longitudinal wave, and dispersion, attenuation, and scattering of a longitudinal wave. Section 4 introduces the analytical modeling approaches, which play a crucial role in understanding longitudinal wave propagation in layered media, presenting the transfer matrix method, which the one of the most widely used techniques, in which each layer is represented by a matrix relating wave field variables at its boundaries. By multiplying the matrices of individual layers, the global response of the system is obtained. Section 5 introduces the numerical modeling techniques, which are indispensable for modeling wave propagation in layered media with complex geometries, material anisotropy, and damping, and emphasizes the finite element method (FEM), which is widely used due to its flexibility and ability to handle arbitrary geometries and boundary conditions. Both time-domain and frequency-domain FEM formulations are employed in wave analysis. Section 6 introduces some important applications of the longitudinal wave in layered media in science and engineering. Section 7 introduces the modern developments and research trends, presenting the recent advances in computing and machine learning and data-driven modeling. Finally, Section 8 introduces the current challenges and future directions, presenting the accurate modeling of nonlinear behavior, strong heterogeneity, and uncertainty in material properties. In addition, this section emphasize the future research, which is expected to focus on multiscale modeling, real-time wave monitoring, and the design of engineered layered materials with tailored wave properties.

2 Examples of Longitudinal Waves

Waves that oscillate parallel to their path of travel and whose displacement is in the same (or opposite) direction of the wave's propagation are known as longitudinal waves. Because they create compression and rarefaction when passing through a material, mechanical longitudinal waves are also known as compressional or compression waves, and pressure waves because they cause pressure increases and reductions. An effective visualization is a wave that runs the length of a stretched Slinky toy, where the distance between coils varies. Examples from the real world include seismic P waves (produced by earthquakes and explosions) and sound waves (vibrations in pressure, a particle of displacement, and particle velocity conveyed in an elastic medium). Many natural and man-made processes exhibit longitudinal waves, demonstrating their prevalence and importance. Longitudinal waves, in which particles oscillate parallel to the direction of wave propagation, are exemplified by sound waves passing through solids, water, or air. Longitudinal properties are also present in seismic waves produced by earthquakes. For example, primary (P-waves) move into the Earth's interior as compressional waves, giving vital information about subsurface structures. Additionally, non-invasive observation of interior organs and anomalies is made possible by the longitudinal propagation of ultrasonic waves through tissues in medical imaging. These illustrations show how longitudinal waves have a wide range of uses and ramifications in many industries.

Real Life Examples of Longitudinal Waves (Fig. 1):

– Speaking on the mic: One important illustration of a longitudinal wave is a sound wave. Speaking in front of a microphone causes a speaker to hit the air thousands of times per second at various frequencies. To create sound, the sound particles enter the microphone along with the air particles.
– Clapping: Do you know what causes our hands to make that well-known sound when we clap when singing a birthday song or on any other occasion? When we applaud, the air particles between our palms compress and shift for a brief moment, creating the familiar sound of a clap.
– Vibrating Drumheads: We have all heard the sound of a drum, and most of us have experimented with different beats. The drumhead vibrates and produces sound waves when we strike the drum with the mallet. Because the drumhead moves both inward and outward, air particles move (vibrate) in the same direction, producing soundwaves.
– Tsunami Waves: People who live near the ocean are terrified of tsunamis because they can harm coastal areas. Most people believe that because sea waves rise and fall, they are transverse waves. Nonetheless, tsunamis and other sea waves are examples of both longitudinal and transverse waves. The waves become thinner and smaller as they approach the coast or smaller areas, and the water, particles travel parallel to the wave, making it a longitudinal wave.
– Earthquake (Seismic-P wave): It is believed that animals were able to detect earthquake waves long before humans. They can detect seismic P waves, which are limited to the earth's interior. Although we are largely unaware of them, even humans may feel the little bump and rattle of these waves. The fastest waves are P waves, which need a medium to go through (solid and liquid). These waves result in the longitudinal movement of the earth's interior (tectonic plates), which causes the surface waves (seismic S-wave) that are felt by humans.
– Vibration in Window Panels after a Thunder: We may have noticed the vibration in your home's window panels whenever there is a lot of rain and a thunderstorm; this is caused by sound waves. Our window panels vibrate because of lightning's increase in air pressure and temperature, which produces a shock wave of sound that sounds like a loud boom.
– Music Woofers: When you try to cover a woofer's mouth, have you ever felt air pressure on your palm or noticed the woofer cone moving in and out? The

reason for this is that woofers exploit the longitudinal wave phenomena. They create sound by moving air particles in or out.

3 Governing Equations and Theoretical Background

The governing equations of longitudinal wave propagation in layered media arise from linear elastodynamics combined with appropriate interface conditions. The layered nature introduces impedance mismatches and dispersion effects that fundamentally alter wave behavior compared to homogeneous media. The governing equations of longitudinal wave propagation in layered media (Fig. 2) arise from linear elastodynamics combined with appropriate interface conditions. The layered nature introduces impedance mismatches and dispersion effects that fundamentally alter wave behavior compared to homogeneous media. This theoretical foundation underpins analytical, numerical, and experimental studies across physics and engineering disciplines. Governing equations are the core mathematical relations that describe how a physical or engineered system evolves in space and time, usually in the form of differential equations based on fundamental conservation laws.

3.1 Continuum Mechanics Framework

Longitudinal wave propagation in layered media is commonly analyzed within the framework of linear elastodynamics, assuming small deformations, linear stress–strain relations, and homogeneous material properties within each layer. The medium is modeled as a piecewise homogeneous continuum, where mechanical properties may change abruptly at layer interfaces. Let the displacement field be denoted by

$$u(x,t) = (u_x, u_y, u_z) \tag{1}$$

For longitudinal, it is well known the particle motion is parallel to the direction of propagation. In one-dimensional layered media aligned along the x-axis, then, the displacement reduces to

$$u(x,t) = u_x \tag{2}$$

3.2 Equation of Motion

The governing equation is derived from Newton's second law applied to an infinitesimal volume element:

$$\vec{\nabla} \cdot \sigma_{ij} + \vec{f}(x,y,z) = \rho \frac{\partial^2 u}{\partial t^2}, i,j = 1,2,3 \tag{3}$$

where σ_{ij} is the Cauchy stress tensor, $\vec{f}$ is the body force vector (often neglected), ρ is the mass density.
When body forces are absent, the governing equation for one-dimensional longitudinal motion becomes,

$$\frac{\partial \sigma_{xx}}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{4}$$

3.3 Constitutive Relations

Under linear elastic and isotropic assumptions, stress and strain are related by Hooke's law. The longitudinal strain is:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{5}$$

The corresponding stress is:

$$\sigma_{xx} = E\varepsilon_{xx} = E\frac{\partial u}{\partial x} \tag{6}$$

where E is the Young's modulus.
Substituting from equation (6) into the equation of motion (4) yields the one-dimensional wave equation

$$\frac{\partial y}{\partial x}\left(E\frac{\partial u}{\partial x}\right) = \rho \frac{\partial^2 u}{\partial t^2} \tag{7}$$

3.4 Non-dispersive longitudinal wave equation in layered media

Within each homogeneous layer i, material properties are constant:

$$E_i = \text{ constant}, \quad \rho_i = \text{ constant}. \tag{8}$$

Thus, the equation of motion simplifies to:

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{1}{C_{Li}^2}\frac{\partial^2 u_i}{\partial t^2} \tag{9}$$

where C_{Li} is the longitudinal wave speed:

$$C_{Li} = \sqrt{\frac{E_i}{\rho_i}} \tag{10}$$

3.5 The general harmonic solution is:

$$u_i(x,t) = \left(A_i e^{ik_i x} + B_i e^{-ik_i x}\right) e^{-i\omega t} \tag{11}$$

With $k_i = \frac{\omega}{C_{Li}}$.

3.6 Interface and Boundary Conditions:

At each interface between two adjacent layer i and $i+1$, the following continuity condition must be satisfied:

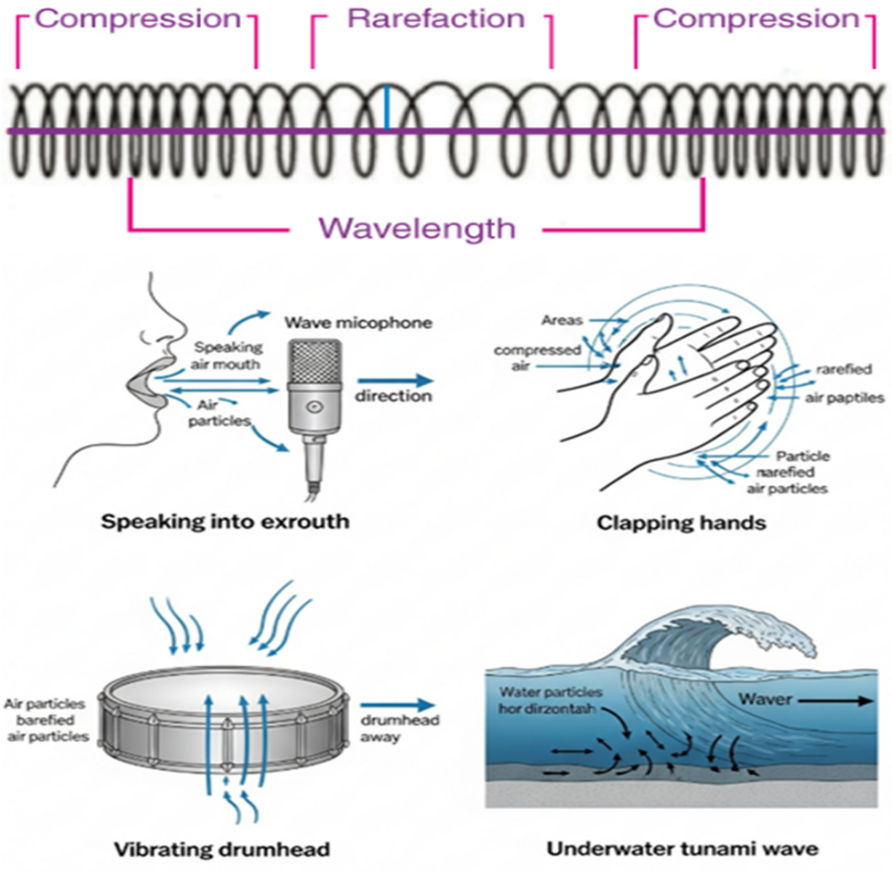

Fig. 1: Examples of longitudinal waves

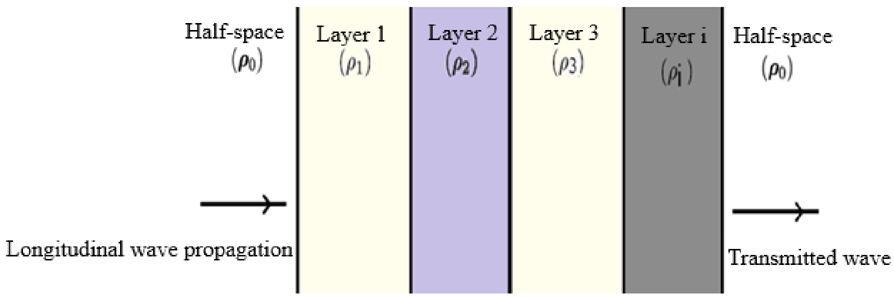

Fig. 2: Geometry of the longitudinal wave propagation in a given layered media

3.6.1 Continuity of displacement:

$$u_i(x,t) = u_{i+1}(x_i,t) \tag{12}$$

3.6.2 Continuity of normal stress:

$$\sigma_{xx}^{(i)}(x_i,t) = \sigma_{xx}^{(i+1)}(x_i,t) \tag{13}$$

These conditions ensure kinematic compatibility and dynamic equilibrium across interfaces.

3.7 Reflection and Transmission Coefficients

For a normally incident longitudinal wave at an interface, the reflection ($\boldsymbol{R}$) and transmission ($\boldsymbol{T}$) coefficients are given by:

$$R = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad T = \frac{2Z_2}{Z_2 + Z_1} \tag{14}$$

These expressions form the basis for analyzing wave propagation in multilayer stacks, phononic crystals, and stratified geological formations.

3.7.1 Incident longitudinal waves:

In the case of incident of longitudinal waves, the Snell's Law for orthotropic medium can be written in the following relations:

$$A_1\delta_1 + A_2\delta_2 + A_3\delta_3 = 0 \tag{15}$$

$$A_1\delta_4 + A_2\delta_5 + A_3\delta_6 = 0 \tag{16}$$

where $A_i, (n = 1,2,3)$ is the longitudinal amplitudes and $\delta_j, (j = 1,2,\ldots,6)$ are

$$\delta_1 = k_1 L(\cos(e_1)) - (\sin(e_1)), \tag{17}$$

$$\delta_2 = k_2 L(\cos(e_2)) - (\sin(e_2)), \tag{18}$$

$$\delta_3 = k_3 L(\cos(e_3)) - (\sin(e_3)), \tag{19}$$

$$\delta_4 = k_4 L(\cos(e_4)) - (\sin(e_4)), \tag{20}$$

$$\delta_5 = k_5 L(\cos(e_5)) - (\sin(e_5)), \tag{21}$$

$$\delta_6 = k_6 L(\cos(e_6)) - (\sin(e_6)), \tag{22}$$

$k_j, (j = 1,2,\ldots,6)$ is the corresponding wave number, $e_j, (j = 1,2,\ldots,6)$ is the incident angles, $L = (\mu_1 + i\mu_2)$ is the complex rigidity coefficient.
Solving equations (15) and (16), we obtained the amplitude ratios in the following form:

$$\frac{A_2}{A_1} = \frac{(\delta_3\delta_4 - \delta_1\delta_6)}{(\delta_2\delta_6 - \delta_3\delta_5)}, \quad \frac{A_3}{A_1} = \frac{(\delta_2\delta_4 - \delta_1\delta_5)}{(\delta_2\delta_6 - \delta_3\delta_5)} \tag{23}$$

3.8 Phase and Group Velocity of longitudinal wave

Longitudinal waves play a fundamental role in the propagation of mechanical disturbances in solids, fluids, and layered media, with important applications in ultrasonic, seismology, nondestructive evaluation, and biomedical imaging. In such waves, particle motion occurs parallel to the direction of wave propagation, and their behavior is governed by the elastic and inertial properties of the medium. Understanding wave kinematics requires distinguishing between phase velocity and group velocity, which describe different aspects of wave propagation and energy transport.

3.8.1 Phase Velocity of longitudinal wave

The speed at which a specific wave phase—such as a crest or zero crossing—propagates over space is represented by the phase velocity. The velocity of a wave's phase is known as its phase velocity. It is limited to a single frequency and a monochromatic signal. The phasor technique makes it simple to find the one-dimensional representation of a sinusoidal wave signal in the time domain, such as the voltage signal on a transmission line:

$$V(x,t) = V_0\cos(\omega t - kx + \alpha) = V_0\cos\left[k\left(\frac{\omega}{k}t - x\right) + \alpha\right]. \tag{24}$$

For a harmonic longitudinal wave, it is defined as the ratio of angular frequency to wavenumber.

$$v_p = \frac{\omega}{k} \tag{25}$$

Phase velocity is particularly relevant in monochromatic wave analysis and governs interference, resonance, and dispersion characteristics in structured or layered media. In homogeneous, non-dispersive media, the phase velocity of longitudinal waves is determined solely by material properties such as density and elastic moduli. However, in dispersive media—including layered, periodic, or frequency-dependent materials—the phase velocity varies with frequency.

3.8.2 Group Velocity of longitudinal wave

In contrast, the group velocity describes the propagation speed of a wave packet or envelope formed by the superposition of multiple frequency components. It is commonly associated with the transport of wave energy and information. For longitudinal waves in dispersive media, group velocity differs from phase velocity and may exhibit complex behavior, including frequency dependence, anomalous dispersion, or even negative values under certain conditions. These effects are especially pronounced in layered media, where multiple reflections, mode coupling, and dispersion significantly alter wave propagation characteristics.

- The group velocity represents the speed of the wave packet / energy transport.
- The group velocity directly related to signal arrival time, energy flow, and pulse propagation in longitudinal waves.

In non-dispersive longitudinal waves (e.g., ideal elastic rods): $v_g = v_p$.
In dispersive media (layered materials, phononic structures): $\boldsymbol{v}_g \neq \boldsymbol{v}_p$.

The distinction between phase and group velocities is crucial for accurately interpreting experimental measurements and numerical simulations of longitudinal waves. While phase velocity governs the oscillatory behavior of individual wave components, group velocity determines signal arrival times, energy flow, and pulse distortion. Consequently, both velocities must be carefully analyzed in studies involving broadband excitation, transient wave propagation, and dispersive layered structures. A rigorous understanding of phase and group velocities provides essential insight into wave dispersion, energy transmission, and dynamic response in complex media. This understanding forms the theoretical foundation for advanced applications such as ultrasonic testing of layered materials, seismic wave analysis in stratified layers, and high-resolution medical imaging techniques. Figure 3, Representative geometry used for determining a longitudinal wave packet illustrating the distinction between phase velocity and group velocity. The oscillatory carrier wave propagates at the phase velocity, while the envelope governing energy transport moves at the group velocity.

3.9 Dispersion, Attenuation, and Scattering of a longitudinal wave

3.9.1 Dispersion of a longitudinal wave

Dispersion is a defining feature of wave propagation in layered media. It arises from the interaction between wavelength and layer thickness, as well as from guided wave behavior. Dispersive effects cause wave velocity to depend on frequency, leading to waveform distortion during propagation. The dispersion relation $\omega(k)$ depends on material contrast, layer thickness ratios, and periodicity [32, 52]. Dispersion analysis provides insight into phase and group velocity (equations (16) and (17)).

$$v_g = \frac{d\omega}{dk} \tag{26}$$

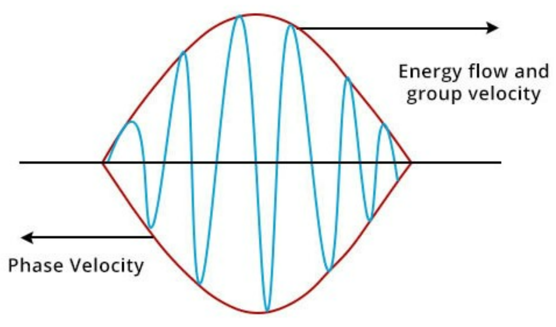

Fig. 3: Representative geometry used for determining a longitudinal wave packet illustrating the distinction between phase velocity and group velocity.

3.9.2 Dispersion in Complex Media

When longitudinal waves travel through layered media, their speed does depend on frequency, leading to dispersion. The equation of motion of longitudinal wave becomes more intricate. For the complex layered media, the equation of motion simplifies to:

$$\rho_i \frac{\partial^2 u_i}{\partial x^2} = (\lambda_i + 2\mu_i)\nabla u_i \tag{27}$$

where λ_i, μ_i are the Lame parameters.
Then the frequency equation for complex media can be written in the following form:

$$\Omega_i = \frac{2\pi k\omega}{\sqrt{\frac{E_i}{\rho_i}}} \tag{28}$$

where $k = k_{\text{real}} + ik_{\text{Imag}}$ is the non-dimensional wave number.

Fig. 3 shows the computed and plotted dispersion curves for longitudinal waves propagating in complex mediums. The true part of the dispersion relation is represented by the solid lines, which show up as several branches of passband modes of wave propagation. The imaginary part of the dispersion relation is represented by the dashed lines, which also show up as several branches of stopband modes. Waves are ”allowed” to move through a medium at passband frequencies. The effective transfer of energy across the layers is ”forbidden” at stopband frequencies because incident waves are confined and dampened in space.

3.9.3 Attenuation of a longitudinal wave

Longitudinal wave attenuation in layered media results from material damping, scattering at interfaces, and radiation losses. Viscoelastic models are commonly used to represent energy dissipation in real materials. Accurate attenuation modeling is essential for realistic prediction of wave amplitudes and signal interpretation. Wave attenuation in layered media results from material damping, scattering at interfaces, and radiation losses. Viscoelastic models are often employed to represent energy dissipation in real materials. Attenuation analysis is critical for realistic prediction of wave amplitudes and signal interpretation. Energy loss in layered media arises from intrinsic material damping, scattering at interfaces, and mode conversion [53]. Attenuation is typically modeled using complex moduli or quality factors and plays a critical role in seismic and ultrasonic applications [54].

3.9.4 Scattering of a longitudinal wave

Scattering and mode conversion occur when waves interact with layer interfaces or defects. Incident waves

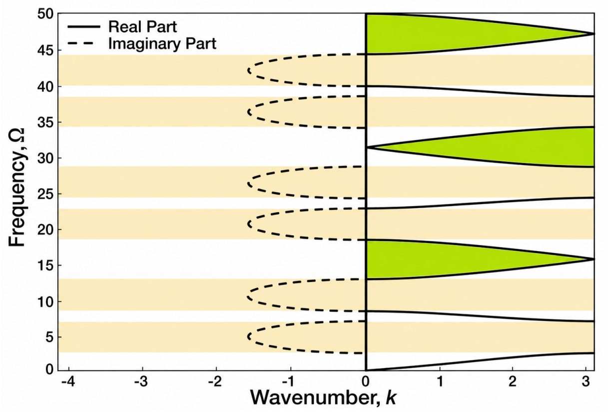

Fig. 4: Dispersion curves for longitudinal wave propagation in the infinite layered media

may split into multiple reflected and transmitted modes, significantly enriching the wave field. Layer interfaces act as scattering centers, causing incident waves to split into multiple reflected and transmitted modes. Mode conversion between P- and S-waves, or between symmetric and antisymmetric guided modes, plays a key role in seismic wave fields and ultrasonic inspections.

The equation for longitudinal wave scattering is not a single universal formula but rather a set of governing equations (like Biot's poroelasticity equations or elastic wave equations). The governing equations solved with boundary conditions for a specific scatterer (crack, cavity, and sphere). This solutions use a potential functions and spherical harmonics, yielding scattered displacement fields (e.g., using Hankel/Bessel functions) that describe how waves diverge and change form (longitudinal/transverse) after interaction, often resulting in complex integral equations for numerical solutions.

Roberts et al. [55] obtained a precise governing equation and its solution for the scattering of longitudinal elastic waves from an anisotropic sphere with transversely orthotropic characteristics (five independent elastic constants). These formulas were expressed in terms of displacement. Legendre polynomials, which are independent of material qualities, satisfied the angular equations, much like in the case of scattering from an isotropic sphere. In contrast to the isotropic situation, spherical Bessel functions did not satisfy the radial equations; instead, the Frobenius approach produced accurate series solutions. They discover that when elastic constants are isotropic and provide the proper formulations, the iterative process for determining series' coefficients needs to be handled differently.

4 Analytical Modeling Approaches

Analytical methods play a crucial role in understanding longitudinal wave propagation in layered media. The transfer matrix method is one of the most widely used techniques, in which each layer is represented by a matrix relating wave field variables at its boundaries. By multiplying the matrices of individual layers, the global response of the system is obtained.

Despite its conceptual simplicity, the transfer matrix method may suffer from numerical instability for systems with many layers or at high frequencies. To address this issue, global matrix and stiffness matrix methods have been developed. These approaches assemble all boundary and interface conditions into a single system of equations, improving numerical stability and robustness. Analytical dispersion relations derived using these methods provide valuable insight into wave modes, cut-off frequencies, and phase and group velocities. Although often implicit and transcendental, such relations form the basis for many theoretical and experimental studies.

4.1 Matrix formalisms and variants

While classical derivations often considered few layers and simple properties, practical applications involve many layers (tens to hundreds), high frequency content, anisotropy, attenuation, etc. To treat this complexity, matrix-based methods have become dominant.

4.2 Transfer-Matrix / Propagator-Matrix

In the propagator or transfer-matrix (TMM) method, the continuity conditions at each interface are encoded in a 2 × 2 (or larger) matrix that connects wave amplitudes (or state vectors) from one boundary of a layer to the next. By successive multiplication of layer matrices one obtains the overall response of the stack. This is standard in seismic [12, 25] and optical thin-film theory. For example, Luan and Ye [56] studied acoustic wave propagation in a one-dimensional layered system via transfer matrices, calculating band-structures for periodic arrangement and localization for random cases. The advantages of transfer matrix method are good physical transparency, ease of coding, direct calculation of reflectivity, transmission, numerical instability when many layers, high frequencies, or large impedance contrasts produce large exponents or ill-conditioned matrices.

4.3 Scattering / S-Matrix and other stabilized forms

To mitigate instabilities from direct transfer matrix multiplication, scattering-matrix (or S-matrix) formalisms are often used where one works with reflected and transmitted wave amplitudes rather than state vectors, thereby keeping magnitudes bounded. In optics, this is standard for multilayer optical stacks [57]. In seismic and acoustics, alternative forms such as global reflectivity methods compute reflectivity series layer by layer and avoid direct state-vector propagation. For example, Zhao et al. [58] extend reflectivity for diffusive–viscous waves in dip-layered media.

4.4 Homogenization / Effective-Medium / High-Order Approximation

When the number of layers is very large, or when periodic or random layering is present, one can resort to effective-medium theory or homogenization. For instance, Quezada de Luna and Ketcheson [59] studied 2-D wave propagation in layered periodic media and used high-order homogenization to derive a dispersive anisotropic effective medium from microscopically periodic layering. Such methods enable approximate modeling of wave dispersion and attenuation from layering without tracking each interface explicitly—useful for upscaling and multi-scale problems.

5 Numerical Modeling Techniques

Numerical methods are indispensable for modeling wave propagation in layered media with complex geometries, material anisotropy, and damping. The finite element method (FEM) is widely used due to its flexibility and ability to handle arbitrary geometries and boundary conditions. Both time-domain and frequency-domain FEM formulations are employed in wave analysis [60]. Finite difference time-domain (FDTD) methods are particularly popular in seismology for large-scale simulations of wave propagation in layered Earth models. Spectral and pseudo-spectral methods offer high accuracy for problems involving periodic or smoothly varying layered structures [61]. The semi-analytical finite element (SAFE) method combines analytical solutions in the propagation direction with finite element discretization across the thickness. SAFE is highly efficient for guided wave analysis in layered plates and has become a standard tool in ultrasonic non-destructive testing [62]. Beyond analytic/matrix methods, a variety of full-wave numerical simulation techniques have been applied to layered media, especially when geometry, heterogeneity, anisotropy or large bandwidth makes the analytic approach unwieldy. Also, Wang et al. [63] developed a Fast Multipole Method (FMM) for the 3-D Helmholtz equation in layered media using Green's functions separated into free-space and reaction components. These numerical methods are critical when: (i) lateral heterogeneity is present (layer properties vary horizontally), (ii) non-planar interfaces or curvature appear, or (iii) nonlinearities, attenuation or complex boundary conditions are involved.

5.1 Finite difference and finite element simulations

While matrix methods provide exact solutions for horizontally stratified media, more general configurations—such as dipping layers, lateral heterogeneity, or topographic variations—require numerical modeling. Finite-difference (FD) methods [64, 65] and spectral-element methods [66, 67] have become standard tools for simulating full wave fields in complex geological settings. These methods discretize the wave equation on a grid and accommodate variations in both vertical and horizontal directions.

5.2 Semi-analytical hybrid techniques

Hybrid techniques combine the efficiency of analytical formulations with the flexibility of numerical solvers. For example, Luco and Apsel [68] developed a semi-analytical method for wave propagation in layered elastic half-spaces using Green's functions, enabling efficient analysis of soil–structure interaction. Similarly, Kausel and Roesset [69] formulated dynamic stiffness matrices for layered soils, leading to the well-known cone model used in civil-engineering site-response analysis.

5.3 Spectral methods and mode summation

Spectral and modal summation methods represent another powerful class of approaches. Aki and Richards [4] describe how wave fields can be decomposed into normal modes of a layered half-space, each with specific phase velocity and attenuation. These methods are widely used to model surface-wave dispersion and resonance in shallow soil layers, forming the theoretical basis for site characterization via micro tremor or H/V spectral ratio techniques [70, 71].

6 Applications

Longitudinal wave propagation analysis in layered media has numerous applications in science and engineering. In geophysics, layered Earth models are fundamental to seismic exploration, earthquake engineering, and site response analysis. Surface wave methods exploit dispersion characteristics to estimate subsurface shear-wave velocity profiles. In civil engineering, layered models are used to evaluate pavements, railway tracks, and layered foundations. Non-destructive techniques based on surface and guided waves provide valuable information on layer thickness, stiffness, and damage.

In materials science and engineering, guided waves are used to detect defects such as delamination and deboning in composite materials. Layered metamaterials and phonic crystals enable advanced wave manipulation such as bandgap formation and vibration isolation. Layered-media approaches also underpin non-destructive testing and ultrasonic evaluation of pavements and concrete structures, where guided Lamb and Rayleigh waves in thin layers provide information on material integrity [55, 72]. In geotechnical design, equivalent-linear and nonlinear analyses rely on the same theoretical basis, illustrating the broad applicability of the layered-media paradigm. In geophysics, the Earth is commonly modeled as a layered medium. Seismic wave propagation analysis underpins earthquake engineering, site response analysis, and hydrocarbon exploration. Surface wave methods exploit dispersion characteristics to estimate subsurface shear-wave velocity profiles.

7 Modern advances and research directions

7.1 Modern advances

Longitudinal wave propagation in layered media has long been a fundamental topic in elastodynamics and acoustics; however, recent research has significantly expanded its scope due to advances in material engineering, numerical methods, and experimental diagnostics. Layered structures are ubiquitous in engineering and natural systems, including composite laminates, multilayered pipes, geological strata, phononic crystals, and acoustic metamaterials, making accurate modeling of longitudinal wave behavior essential for both theoretical understanding and practical applications such as nondestructive testing, structural health monitoring, and seismic exploration.

In recent years, longitudinal wave propagation in layered media has attracted renewed attention due to its importance in ultrasonic, nondestructive testing (NDT), seismology, phononic crystals, and metamaterials. Advances in material engineering, computational power, and experimental techniques have enabled researchers to move beyond classical elastic models toward multiphysics, multiscale, and wave-control-oriented frameworks. Contemporary studies emphasize dispersion, interface effects, coupling phenomena, and inverse problems in increasingly complex layered systems.

Recent studies have emphasized the dispersive nature of longitudinal waves in stratified media, even in purely elastic and isotropic layers. Multiple reflections, impedance mismatches, and periodicity lead to frequency-dependent phase and group velocities, which are critical in the interpretation of guided wave signals and ultrasonic measurements [1, 73]. Analytical and semi-analytical formulations based on transfer matrix, propagator matrix, and eigenvalue approaches continue to be refined to improve numerical stability and accuracy in highly layered or high-contrast systems. In parallel, significant progress has been made in the study of engineered layered media and elastic metamaterials, where layering is intentionally designed to manipulate longitudinal wave propagation. Recent research demonstrates the formation of longitudinal bandgaps, wave localization, and tailored dispersion characteristics in periodic and locally resonant layered structures [18, 74].

These developments enable unprecedented control of wave transmission, with promising applications in vibration isolation, acoustic filtering, and energy attenuation. Another important direction in recent research is the incorporation of multiphysics effects, including thermos-elasticity, poroelasticity, and viscoelastic damping. Studies show that thermal relaxation, fluid–solid interaction, and internal dissipation mechanisms significantly influence longitudinal wave attenuation and velocity dispersion in layered solids, particularly in geophysical and high-temperature environments [75, 76]. Such models provide a more realistic description of wave behavior in complex materials compared to classical elastic theory. Furthermore, advances in numerical simulation techniques, such as high-order finite element methods, spectral element methods, and hybrid analytical–numerical schemes, have enabled efficient and accurate modeling of high-frequency longitudinal waves in multilayered structures [77–79]. These computational developments are increasingly complemented by data-driven and machine-learning-assisted approaches for inverse problems, allowing the extraction of layer properties and defect characteristics from measured wave responses. Overall, recent research on longitudinal wave propagation in layered media reflects a clear transition from classical formulations toward multiscale, multiphysics, and wave-control-oriented frameworks, positioning this field at the intersection of fundamental wave physics and advanced engineering applications. Modern research in wave propagation in layered media increasingly focuses on:

- Functionally graded and anisotropic layers
- Smart materials and adaptive layered systems
- Topological wave phenomena in layered structures
- Inverse problems and machine-learning-based parameter identification

7.1.1 Full-waveform modeling and inversion

Recent advances in computing have enabled full-waveform inversion (FWI) in layered and laterally heterogeneous media [80]. By iteratively minimizing the misfit between observed and synthetic seismograms, FWI retrieves high-resolution velocity and attenuation profiles [81]. Layered-media theory still underpins the forward modeling and sensitivity kernels used in these algorithms.

7.1.2 Time-dependent and nonlinear layering

Emerging research has begun exploring time-varying or nonlinear layered systems. In dynamic soil–structure problems, for example, stiffness degradation and modulus reduction under cyclic loading can lead to transient layering effects [82]. Such models combine nonlinear constitutive behavior with traditional layering frameworks.

7.1.3 Metamaterials and wave control

Inspired by elastic metamaterials, researchers have investigated engineered layered composites that manipulate seismic wave propagation. Periodic layering with high contrast in stiffness and density can generate band gaps that attenuate certain frequency ranges, analogous to photonic or phononic crystals [83]. These concepts have potential applications in seismic isolation and vibration control in civil structures.

7.1.4 Machine learning and data-driven modeling

Another emerging trend is the integration of machine learning with classical layered-media theory. Neural networks trained on synthetic seismograms from layered models are increasingly used to invert for subsurface parameters or classify site conditions [84]. Such hybrid approaches blend physics-based modeling with data-driven inference, opening new avenues for high-resolution subsurface imaging. The integration of advanced computational methods and data-driven techniques is reshaping the field.

7.2 Research Directions

Recent research direction of the longitudinal wave propagation in layered media increasingly focuses on functionally graded and anisotropic layers, smart materials, and adaptive structures. Advances in computational power have enabled high-fidelity simulations of complex layered systems. Recent advances in layered metamaterials and phononic crystals have opened new possibilities for wave manipulation. Periodic layered structures can exhibit bandgaps where wave propagation is forbidden, enabling vibration isolation and wave filtering. Machine learning and data-driven approaches are being explored for inverse problems, such as identifying layer properties from measured wave responses. These methods offer new opportunities for rapid and robust characterization of layered media. Several research directions have emerged in recent years:

7.2.1 Temporal or time-varying layered media:

In addition to spatial layering, materials whose properties vary in time (temporal metamaterials) produce non-reciprocal wave phenomena. Matrix methods are being adapted (temporal transfer-matrices) in electromagnetics and acoustics.

7.2.2 Integration of multiphysics layered systems:

For example, piezoelectric layered media (electric-mechanical coupling), thermos-elastic layers, poro-viscoelastic layering, etc. Matrix formalisms have been extended to multiple coupled fields.

7.2.3 Many-layer high-frequency stability:

The classic propagator matrices become numerically unstable for large numbers of layers or high frequencies. Research has focused on orthonormalization, block-diagonalization, and robust scattering-matrix methods (especially in seismic and photonics) to maintain numerical stability.

7.2.4 Inverse problems and parameter estimation:

Instead of forward modelling, layered-media theories are being used for inversion: estimating layer thicknesses, impedances, anisotropy, attenuation, etc. from measured dispersion, reflectivity or guided-wave data.

7.2.5 Layered metamaterials and wave control:

As mentioned above, engineered layered composite systems allow tailored dispersion (band-gaps, negative refraction, local resonances) and dynamic wave control (via deformation, tuning, external stimuli).

7.2.6 Lateral heterogeneity and 3-D layering:

Although much of the theory assumes horizontal stratification, real media often have lateral variations, curved interfaces, dipping layers, and 3D heterogeneity. Wenzel et al. [85] addressed 2-D wave propagation through layered media with curved interfaces.

8 Current Challenges and Future Directions

Despite significant progress, several challenges remain. These include accurate modeling of nonlinear behavior, strong heterogeneity, and uncertainty in material properties. Inverse problems associated with reconstructing layered structures from wave measurements remain ill-posed and computationally demanding. Future research is expected to focus on multiscale modeling, real-time wave monitoring, and the design of engineered layered materials with tailored wave properties.

Key Challenges & Future Research Directions:

1. Multiscale Modeling: The need for robust multiscale frameworks that seamlessly link microstructure, interface physics, and macroscopic propagation — especially in anisotropic and heterogeneous layered systems.
2. Nonlinearity & Large Amplitudes: Classical linear theory breaks down at large strains or nonlinear materials. New theories must incorporate nonlinear elastic behavior and coupling with mean stress states.
3. Time-Varying & Active Media: Interest is growing in time-modulated layered structures that can break reciprocity, create one-way waveguides, or dynamically control wave propagation.
4. Machine Learning for Inversion & Prediction: Integrating physics-based models with AI for real-time inversion, prediction, and control of wave propagation in layered media.
5. Integrated Metamaterial Approaches: Designing layered metamaterials that exploit bandgaps, local resonances, and novel dispersion effects for energy harvesting, vibration isolation, and acoustic cloaking.

9 Conclusion

Longitudinal Wave propagation in layered media remains a vibrant and evolving research area. From classical analytical theory to modern numerical and experimental techniques, the study of layered systems continues to provide essential insights into longitudinal wave phenomena. Ongoing developments in computational methods, smart materials, and data-driven analysis are expected to further expand the scope and impact of research in this field. In addition, the longitudinal wave propagation in layered media is a mature yet rapidly evolving research field with profound theoretical and practical importance. From classical elasticity theory to modern computational and experimental techniques, the study of layered systems continues to provide critical insights into wave phenomena. Advances in this area will further enhance applications in geophysics, engineering, and materials science. Modern trends including metamaterials, time-varying layers, and machine learning inversion continue to expand the scope of this classical field. Despite these advances, challenges remain in bridging scales (from laboratory to crustal), integrating lateral heterogeneity, and quantifying uncertainty in complex layered systems. In summary, layered-media theory remains a cornerstone of seismic and geotechnical wave analysis. It provides not only a powerful forward modeling framework but also a conceptual foundation for understanding how Earth's stratification shapes observed seismic phenomena.

Ethical approval:

This article does not contain any studies with human participants performed by the authors.

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Data Availability Statement

Data sharing is not applicable to this article as no data sets were generated during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

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