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Rough Sets for n-Cycles and n-Paths

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Abstract: In this paper we carry out a research project whose main goal is the study of an undirected graph by means of investigation tools provided by Pawlak rough set theory. Specifically, we determine both the lower and the upper rough approximation functions for an information table induced from a cycle or a path on n vertices. For such graphs we also provide a complete description of the corresponding exact subsets.

Keywords: Graded Lattices, Integer Partitions, Young Diagrams, Sand Piles Models.

1 Introduction

Graphs are objects ubiquitously present in mathematics and computer science. The graph structure has been studied mostly from a geometric point of view, by searching for the analogies with the various types of geometries. In fact, classical problems in graph theory concern the determination of distances, neighborhoods, connectivity and so on. Recently, the impetuous development of computer science has placed new questions about the graph structures. For example, the graphs can be studied in terms of sequential dynamical systems (see [2,3,4,5]), by means of parallel dynamics (see [1]), or also for their analogies with both sequential and parallel dynamics on order structures (see [7,8,9,10,15,17,18,19,20,21]).
In this paper we continue a research project started in [22,23], where a simple undirected graph is studied as a particular type of information system. According to Pawlak (see [40]) an *information system* is a structure $\mathfrak{I} = \langle U, Att, Val, F\rangle$, where $U = \{u_1, u_2, \dots, u_m\}$ is a non-empty finite set called universe set, $Att = \{a_1, a_2, \dots, a_n\}$ is a non-empty finite set called *attribute set* and $F : U \times Att \to Val$, called *information map*, is an application from the direct product $U \times Att$ into the so called *value set* Val. The elements of U are called *objects* and the elements of Att are called *attributes*. In particular, if $Val = \{0,1\}$ we say that $\mathfrak{I}$ is a *Boolean information system*. An information system occurs in all situations in which a huge amount of data needs to be classified in a table according to some criterion of subdivision, therefore it is a structure that is very frequent in various fields of study, both of qualitative and quantitative type. In his seminal works [38,39,40], Pawlak introduced several investigation tools in order to better analyze and reduce the complexity of a generic information system. In this framework, Pawlak proposed the *rough set theory*, abbreviated RST (see also the more recent papers [41,42,43]), that is a useful methodological tool for reasoning about knowledge of objects represented by attributes. Nowadays, RST (and its more general version called *granular computing* [45]) is a well investigated research field [6,12,13,14,27,28,52,53], which has connections with operative research [30], preclusivity spaces [11], machine learning [51], interval analysis [35], formal concept analysis [34,49], database theory [31,46], data mining [32,36,37,50], fuzzy set theory [33,44,54], interactive computing [47,48]. The fundamental assumption of RST is based on the famous *Law of Indiscernibility*, according to which two objects are indiscernible (i.e. similar) if and only if they share the same properties. Formally, if $\mathfrak{I} = \langle U, Att, Val, F\rangle$ is an information system and $A \subseteq Att$, we call *A-indiscernibility* the equivalence relation $\equiv_A$ on the universe set U defined as follows: if $u, u' \in U$ then

$$u \equiv_A u' :\Longleftrightarrow F(u,a) = F(u',a), \forall a \in A. \qquad (1)$$

In rough set theory, there exist different kinds of sets:

– Elementary sets, which are sets of all indiscernible objects;

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–Exact sets, which are unions of elementary sets;
–Rough (in the sense of imprecise) sets, which are not exact.

The objects can be classified as belonging to exact sets or to their complements, but it isn't true in the case of rough sets. Thus the behavior of rough sets is more interesting than that of exact sets. In fact, we can classify objects by means of the knowledge degree we have about them, but in the case of rough sets, there are some objects which belong to a so called *boundary region*, namely those for which it is not possible to say with certainty that belong to the set or to its complement. Hence, rough sets represent vague concepts. The way of studying these sets consists in substituting any vague concept with two precise concepts, called respectively *lower approximation* and *upper approximation*. The lower approximation of an non-empty Y (briefly $\mathbf{l}_A(Y)$) represents the set of objects that *surely* belong to Y, with respect to our knowledge provided by A; the upper approximation of an object subset Y (briefly $\mathbf{u}_A(Y)$) is the set of objects *surely* or *possibly* belonging to Y with respect to our knowledge expressed by A. Furthermore, we say that an object subset is A-exact if its A-lower approximation coincides with its A-upper approximation.
Some natural links of RST with both graph and hypergraph theory have recently been founded. In fact, in [22,23,24,25,26] the idea to study any simple undirected graph G as if it were a Boolean information system was developed (in [16] this idea has been also extended to hypergraph theory). The basic tool to connect graphs and Boolean information systems is the adjacency matrix of G, which in [22,23,24,25,26] has been interpreted as the Boolean table of a particular information system. Specifically, in [22,23,24] some graph families, such as the complete graph K_n or the complete bipartite graph $K_{p,q}$, have been broadly studied in terms of information tables. In particular, in the above papers, the A-lower and the A-upper approximations, the A-positive region of any vertex subset B, the A-attribute dependency function and the rough membership function, where A, B and Y are vertex subsets, have been completely determined both for the complete graph K_n and the complete bipartite graph $K_{p,q}$.
In this paper we apply the RST tools respectively to the cases of the cycle C_n and of the path P_n on n vertices (respectively, n-cycle and n-path). Specifically, we determine the A-lower and the A-upper approximations for both C_n and P_n. Moreover, we also determine all A-exact sets, i.e. those sets for which the A-lower approximation coincide with the A-upper approximation. Although the structure of both C_n and P_n is quite simple, the complete determination of the corresponding A-lower and A-upper approximations was found quite complex, since it was necessary to treat many sub-cases. In fact, the indiscernibility with respect to a vertex subset A gives rise to three sets A', A'' and A''', which provide a partition of the vertex set $V(G)$, for both $G = C_n$ and $G = P_n$. All occurring sub-cases correspond then to all possible relations between the vertex subset Y and the previous three sets. From this, the need to express in detail our results in three tables. Our results show that, also for simple graph structures, the complete determination of both the A-lower and the A-upper approximations can be quite complex. Therefore one expects that the complete computation of these approximations for more complicated families of graphs can be a difficult goal to achieve.

To conclude this introduction we now briefly describe the content of the sections in this paper. In Section 2, we firstly introduce the basic notations that we use in the sequel. Next, we characterize in our graph context the form of two classical RST notions: indiscernibility relation and approximation functions. In Section 3, for any n we completely determine the indiscernibility partition form for both the n-cycle and the n-path. In Section 4 we compute the A-upper approximation and the A-lower approximation for both C_n and P_n. Finally, in Section 5 we use the results obtained in Section 4 in order to find all A-exact subsets for both C_n and P_n.

2 Basic Results

If X is any finite set, we denote by $|X|$ the number of elements in the set X and by $\mathscr{P}(X)$ the power set of X. If $Y \subseteq X$ and X is clear from the context, we write Y^c instead of $X \setminus Y$.
If R is an equivalence relation on X and x is an element of X, we denote by $[x]_R$ the equivalence class of x with respect to the relation R.

We recall now the following classical notion.

Definition 1. *A set-partition π on X is a finite collection of non-empty subsets $B_1,\dots,B_M$ of X such that $B_i \cap B_j = \emptyset$ for all $i \neq j$ and such that $\bigcup_{i=1}^{M} B_i = X$. The subsets $B_1,\dots,B_M$ are called blocks of π and we write $\pi := B_1|\dots|B_M$ to denote that π is a set partition having blocks $B_1,\dots,B_M$.*

In this paper we treat exclusively with finite undirected simple graphs and we refer to [29] for any general notion concerning graph theory. Here we recall only some basic definitions and we fix some notations which we will use in the sequel. We always denote by $G = (V(G),E(G))$ a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1,\dots,v_n\}$ and edge set $E(G)$. If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v,v'\} \in E(G)$ and $v \nsim v'$ otherwise.

Definition 2. *Let $v \in V(G)$. We call neighborhood of v in G the set $N_G(v) := \{w \in V(G) : v \sim w\}$. In particular, if*

$A \subseteq V(G)$ we call neighborhood of A in G the set

$$N_G(A) := \bigcup_{v \in A} N_G(v) \tag{2}$$

In the next definition we see how a graph G becomes a Boolean information system (for details see [22] and [23]).

Definition 3. *We call* information system *of the graph G the Boolean information system*

$$\Im[G] := \langle U(G), Att(G), \{0,1\}, F_G \rangle,$$

where $U(G) := V(G)$, $Att(G) := V(G)$ and

$$F_G(u,v) := \begin{cases} 1 \text{ if } u \sim v \\ 0 \text{ otherwise} \end{cases}$$

Let $A \subseteq V(G)$ a vertex subset. By (1), we can define the A-indiscernibility relation $\equiv_A$ for the information system of the graph G as follows:

$$v \equiv_A v' :\Longleftrightarrow F_G(v,a) = F_G(v',a), \forall a \in A. \tag{3}$$

By the previous definition it follows the next result.

Proposition 1. *Let $v, v' \in G$ and $A \subseteq V(G)$. Then:*
(i) $v \equiv_A v'$ if and only if for all $z \in A$ it results that $v \sim z$ if and only if $v' \sim z$.
(ii) If $v \sim v'$ then $v \not\equiv_A v'$ or $\{v,v'\} \cap A = \emptyset$.
(iii) If $v \equiv_A v'$ and $\{v,v'\} \cap A \neq \emptyset$, then $v \nsim v'$.

Proof. (i) It follows immediately by Definition 3 and by (3).
(ii) We suppose that $v \sim v'$ and $v \equiv_A v'$. We must show that $\{v,v'\} \cap A = \emptyset$. We suppose by contradiction that $\{v,v'\} \cap A \neq \emptyset$. Without loss of generality, we can assume $v \in A$. By (3) we deduce that $F_G(v,v) = F_G(v',v)$, but $F_G(v,v) = 0$ since there are no loops in G while by our assumption $F_G(v',v) = 1$. So the equality $F_G(v,v) = F_G(v',v)$ does not hold, absurd. The case $v' \in A$ is analogous.
On the other hand, a similar argument shows that if $v \sim v'$ and $\{v,v'\} \cap A \neq \emptyset$ then $v \not\equiv_A v'$. In fact we just observe that if $v \in A$, then $F_G(v,v) \neq F_G(v',v)$, hence we conclude by (3). This proves (ii).
(iii) It is the contra-nominal version of (ii).

If $A \subseteq V(G)$ and $v, v' \in V(G)$ it is easy to note, by (i) of Proposition 1, that:

$$v \equiv_A v' :\Longleftrightarrow N_G(v) \cap A = N_G(v') \cap A \tag{4}$$

We denote by $\pi_G(A)$ the set partition of $V(G)$ induced from the equivalence relation $\equiv_A$. If $v \in V(G)$, we denote by $[v]_A$ the equivalence class of the vertex v with respect to $\equiv_A$. Let us also note that $[v]_\emptyset = V(G)$ for all $v \in V(G)$, therefore $\pi_\emptyset(G) = V(G)$.

We recall now the following basic notions of RST.

Definition 4. *Let $\Im = \langle U, Att, Val, F \rangle$ be an information system, $A \subseteq Att$ and $Y \subseteq U$. The A-*lower approximation *of Y is the following subset of U:*

$$\mathbf{l}_A(Y) := \{x \in U : [x]_A \subseteq Y\} = \bigcup \{C \in \pi_A(\Im) : C \subseteq Y\}.$$

The A-upper approximation of Y is defined as:

$$\mathbf{u}_A(Y) := \{x \in U : [x]_A \cap Y \neq \emptyset\} = \bigcup \{C \in \pi_A(\Im) : C \cap Y \neq \emptyset\}.$$

*The subset Y is called A-*exact *if and only if $\mathbf{l}_A(Y) = \mathbf{u}_A(Y)$ and A-*rough *otherwise.*

The lower approximation represents the elements that *certainly*, with respect to our knowledge expressed by A, belongs to Y. On the other hand, the upper approximation is the set of objects *possibly* belonging to Y.

For the A-lower and A-upper approximation functions we obtain the following geometrical interpretation in the simple graph context.

Proposition 2. *Let $G = (V(G), E(G))$ be a simple undirected graph and let $\Im[G]$ be the Boolean information system associated to G. Let A and Y be two subsets of $V(G)$. Then:*
(i) $\mathbf{u}_A(Y) = \{v \in V(G) : \exists u \in Y : N_G(u) \cap A = N_G(v) \cap A\}$. Therefore, $v \in \mathbf{u}_A(Y)$ iff v is an A-symmetric vertex of some $u \in Y$.
(ii) $\mathbf{l}_A(Y) = \{v \in V(G) : (u \in V(G) \wedge N_G(u) \cap A = N_G(v) \cap A) \Longrightarrow u \in Y\}$. Therefore, $v \in \mathbf{l}_A(Y)$ iff all A-symmetric vertices of v are in Y.

Proof. It follows directly by (4) and from the definitions of the approximations.

Hence the lower approximation of a vertex set Y represents a subset of Y such that there are no elements outside Y with the same connections of any vertex in $\mathbf{l}_A(Y)$ (relatively to A). The upper approximation of Y is the set of vertices with the same connections (w.r.t. A) of at least one element in Y. By the previous proposition it is natural to call $\mathbf{l}_A(Y)$ the *A-symmetry kernel* of Y and $\mathbf{u}_A(Y)$ the *A-symmetry closure* of Y.

3 C_n and P_n as Boolean Information Systems

In this section we consider the n-cycle C_n and te n-path P_n. Recall the definitions of the two graphs and introduce some particular vertex subsets which will be used extensively in this paper.

Definition 5. *Let n be a positive integer. The n-*cycle *C_n is the graph having vertex set $V(C_n) = \{v_1, \ldots, v_n\}$ and edge set:*

$$E(C_n) = \{\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}, \{v_n,v_1\}\}.$$

If $A \subseteq V(C_n)$, we set $A^* := \{v_i \in A : v_{i-2} \notin A \wedge v_{i+2} \notin A\}$, $A' := (N_{C_n}(A))^c$, $A'' := N_{C_n}(A^*)$ and $A''' := (A' \cup A'')^c$. Let us note that $V(C_n) = A' \cup A'' \cup A'''$.

Definition 6. *Let n be a positive integer. The n-path P_n is the graph having vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ and edge set:*

$$E(P_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}.$$

If $A \subseteq V(P_n)$, we set $A^* := \{v_i \in A : 3 \le i \le n-2 \wedge v_{i-2} \notin A \wedge v_{i+2} \notin A\}$, $A' := (N_{P_n}(A))^c$, $A'' := N_{P_n}(A^*)$ and $A''' := (A' \cup A'')^c$. Let us note that $V(P_n) = A' \cup A'' \cup A'''$.

In the next result we provide a complete description of the indiscernibility partition of C_n.

Proposition 3. *Let $A = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V := V(C_n) = \{v_1, \ldots, v_n\}$, $A''' = \{v_{s_1}, \ldots, v_{s_l}\}$ and $A^* = \{v_{j_1}, \ldots, v_{j_h}\}$. Then:*

$$\pi_{C_n}(A) = A'|v_{j_1-1}v_{j_1+1}|\cdots|v_{j_h-1}v_{j_h+1}|v_{s_1}|\cdots|v_{s_l},$$

where the index sums are taken mod(n).

Proof. In what follows, all the index sums are taken mod(n). Let $v_i, v_j \in V$, with $i < j$. Then $v_i \equiv_A v_j$ if and only if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \emptyset$ or $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A = \{v_{i+1}\} = \{v_{j-1}\}$. At first, we observe that for each $i \in \{1, \ldots, n\}$, $N_{C_n}(v_i) = \{v_{i-1}, v_{i+1}\}$. The proof follows easily by observing that, since $v_i \neq v_j$, then $|N_{C_n}(v_i) \cap N_{C_n}(v_j)| \le 1$ and the equality holds if and only if $j = i+2$. Hence, if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A$, then $N_{C_n}(v_i) \cap A = (N_{C_n}(v_i) \cap N_{C_n}(v_j)) \cap A \subseteq N_{C_n}(v_i) \cap N_{C_n}(v_j)$. By Proposition 1, $v_i \equiv_A v_j$ if and only if $N_{C_n}(v_i) \cap A = N_{C_n}(v_j) \cap A$. Thus $|N_{C_n}(v_i) \cap A| = |N_{C_n}(v_j) \cap A| \le 1$ and the equality holds if and only if $j = i+2$ and $v_{i+1} = v_{j-1} \in A$. This proves the thesis. In fact, let $v_i, v_j \in V(G)$, with $i < j$ and $v_i \equiv_A v_j$. Then either $N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset$ or $N_G(v_i) \cap A = N_G(v_j) \cap A = \{v_{i+1}\} = \{v_{j-1}\}$. But the first condition is equivalent to say that $v_i, v_j \in A'$, whereas the second is equivalent to say that $\{v_i, v_j\} = N_G(v)$, for some $v \in A^*$. The proposition is thus proved.

Example 1. Let C_{10} be the 10-cycle on the set $V = \{v_1, \ldots, v_{10}\}$. Let $A = \{v_2, v_4, v_7\}$. Then

$$A' = \{v_2, v_4, v_7, v_9, v_{10}\}$$

$$A^* = \{v_7\}$$

$$A'' = \{v_6, v_8\}$$

and

$$A''' = \{v_1, v_3, v_5\}$$

Thus

$$\pi_{C_n}(A) = v_2v_4v_7v_9v_{10}|v_6v_8|v_1|v_3|v_5$$

We give now a complete description of the indiscernibility partition for the graph P_n for any vertex subset $A \subseteq V(P_n)$.

Proposition 4. *Let $A = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V := V(P_n)$, $A''' = \{v_{s_1}, \ldots, v_{s_l}\}$ and $A^* = \{v_{j_1}, \ldots, v_{j_h}\}$. Then:*

$$\pi_{P_n}(A) = A'|v_{j_1-1}v_{j_1+1}|\cdots|v_{j_h-1}v_{j_h+1}|v_{s_1}|\cdots|v_{s_l}.$$

Proof. At first, we observe that for each $i \in \{1, \ldots, n\}$

$$N_{P_n}(v_i) = \begin{cases} \{v_2\} & \text{if } i = 1 \\ \{v_{n-1}\} & \text{if } i = n \\ \{v_{i-1}, v_{i+1}\} & \text{otherwise.} \end{cases}$$

Let $v_i, v_j \in V$, with $i < j$. Then $v_i \equiv_A v_j$ if and only if $N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset$ or $N_G(v_i) \cap A = N_G(v_j) \cap A = \{v_{i+1}\} = \{v_{j-1}\}$. The proof follows easily by observing that, since $v_i \neq v_j$, then $|N_G(v_i) \cap N_G(v_j)| \le 1$ and the equality holds if and only if $j = i+2$. It follows that, if $N_G(v_i) \cap A = N_G(v_j) \cap A$, then $N_G(v_i) \cap A = (N_G(v_i) \cap N_G(v_j)) \cap A \subseteq N_G(v_i) \cap N_G(v_j)$. By Proposition 1, $v_i \equiv_A v_j$ if and only if $N_G(v_i) \cap A = N_G(v_j) \cap A$. Thus $|N_G(v_i) \cap A| = |N_G(v_j) \cap A| \le 1$ and the equality holds if and only if $j = i+2$ and $v_{i+1} = v_{j-1} \in A$. Now, let $v_i, v_j \in V(G)$, with $i < j$ and $v_i \equiv_A v_j$. Then either $N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset$ or $N_G(v_i) \cap A = N_G(v_j) \cap A = \{v_{i+1}\} = \{v_{j-1}\}$. But the first condition is equivalent to say that $v_i, v_j \in A'$, while the second is equivalent to say that $\{v_i, v_j\} = N_G(v)$, for some $v \in A^*$. The proposition is thus proved.

Example 2. Let $G = P_8$ be the 8-path on the set $V = \{v_1, \ldots, v_8\}$. Let $A = \{v_2, v_4, v_7\}$. Then

$$A' = \{v_2, v_4, v_7\}$$

$$A^* = \emptyset$$

$$A'' = \emptyset$$

and

$$A''' = \{v_1, v_3, v_5, v_6, v_8\}$$

Thus

$$\pi_G(A) = v_1|v_3|v_5|v_6|v_8|v_2v_4v_7$$

4 The A-upper and the A-lower Approximations For C_n and P_n

In this section, we compute the $A-$upper and the $A-$lower approximation functions for both the $n-$cycle C_n and the $n-$path P_n. Let us note that it is sufficient to treat only the case of C_n. In fact, by Proposition 4, the A-indiscernibility partitions of P_n and C_n have the same structure and the result proved below depends only by the form of the A-indiscernibility partition. In order to determine the general form of the A-upper approximation function of C_n, we must examine all possible relations between the vertex subset Y and the three subsets A', A'' and A'''. Next, we also show that any possible choice of the vertex subsets A and Y is included in the cases we examined. In what follows we will use the notations introduced in Definition 5.

If A and Y are two vertex subsets of C_n we set

$$Q_A(Y) := \bigcup\{N_{C_n}(v) : v \in A^* \wedge N_{C_n}(v) \cap Y \neq \emptyset\}.$$

Theorem 1. *Let A and Y be two vertex subsets of C_n. The map $\mathbf{u} : (A,Y) \in \mathscr{P}(V(C_n)) \times \mathscr{P}(V(C_n)) \mapsto \mathbf{u}_A(Y) \in \mathscr{P}(V(C_n))$ is completely described from the cases listed in the following table:*

CASE	*CONDITIONS*	$\mathbf{u}_A(Y)$
1)	$Y = V(C_n)$	$V(C_n)$
2)	$Y = \emptyset$	$\emptyset$
3)	$Y \neq \emptyset \wedge Y \subseteq A'$	A'
4)	$Y \neq \emptyset \wedge Y \subseteq A''$	$Q_A(Y)$
5)	$Y \neq \emptyset \wedge Y \subseteq A'''$	Y
6)	$Y \cap A' \neq \emptyset \wedge Y \cap A'' \neq \emptyset \wedge Y \cap A''' = \emptyset$	$Q_A(Y) \cup A'$
7)	$Y \cap A' \neq \emptyset \wedge Y \cap A''' \neq \emptyset \wedge Y \cap A'' = \emptyset$	$A' \cup Y$
8)	$Y \cap A'' \neq \emptyset \wedge Y \cap A''' \neq \emptyset \wedge Y \cap A' = \emptyset$	$Q_A(Y) \cup Y$
9)	$Y \cap A' \neq \emptyset \wedge Y \cap A'' \neq \emptyset \wedge Y \cap A''' \neq \emptyset$	$A' \cup Y \cup Q_A(Y)$

Proof. We recall that for a generic vertex subset A we have

$$\pi_{C_n}(A) = A'|v_{j_1-1}, v_{j_1+1}|\dots|v_{j_h-1}v_{j_h+1}|v_{s_1}|\dots|v_{s_l},$$

where $\{v_{j_1-1}, v_{j_1+1}, \dots, v_{j_h-1}, v_{j_h+1}\} = A''$ and $\{v_{s_1}, \dots, v_{s_l}\} = A'''$.

1): Let $Y = V(C_n)$ and A any vertex subset of $V(C_n)$. Obviously $[v]_A \cap Y \neq \emptyset$ for every vertex v, so $\mathbf{u}_A(V(C_n)) = V(C_n)$.

2): Let $Y = \emptyset$ and A any vertex subset of $V(C_n)$. Obviously $[v]_A \cap \emptyset = \emptyset$ for every vertex v, so $\mathbf{u}_A(\emptyset) = \emptyset$.

3): Let A and Y be two vertex subsets such that $Y \neq \emptyset \wedge Y \subseteq A'$. This means that the indiscernibility block intersecting Y is exactly A', therefore $[v]_A \cap Y \neq \emptyset$ if and only if $v \in A'$.

4): Let A and Y be two vertex subsets such that $Y \neq \emptyset \wedge Y \subseteq A''$. Recalling that $A'' = N_{C_n}(A^*)$, we deduce that Y intersects only the neighbourhoods of some points $v_{j_1}, \dots, v_{j_m} \in A^*$, therefore $Q_A(Y) \neq \emptyset$ and $\mathbf{u}_A(Y) = Q_A(Y)$.

5): Let A and Y be two vertex subsets such that $Y \neq \emptyset \wedge Y \subseteq A'''$. Since the elements of A''' form single blocks in the $A-$indiscernibility partition, we have that $Y \cap [v]_A \neq \emptyset$ if and only if $v \in A''' \cap Y = Y$. Hence $\mathbf{u}_A(Y) = Y$.

6): Let A and Y be two vertex subsets such that $Y \cap A' \neq \emptyset \wedge Y \cap A'' \neq \emptyset \wedge Y \cap A''' = \emptyset$. In other words, Y is transversal only to A' and A''; therefore we have that $[v]_A \cap Y \neq \emptyset$ if and only if $v \in A'$ or $\exists w \in A^* : v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq Q_A(Y)$. Thus $\mathbf{u}_A(Y) = A' \cup Q_A(Y)$.

7): Let A and Y be two vertex subsets such that $Y \cap A' \neq \emptyset \wedge Y \cap A''' \neq \emptyset \wedge Y \cap A'' = \emptyset$. In this case, Y is transversal only to A' and A''', hence $[v]_A \cap Y \neq \emptyset$ if and only if $v \in A'$ or $v \in Y \cap A'''$. Thus $\mathbf{u}_A(Y) = A' \cup (Y \cap A''') = Y \cup A'$.

8): Let A and Y be two vertex subsets such that $Y \cap A'' \neq \emptyset \wedge Y \cap A''' \neq \emptyset \wedge Y \cap A' = \emptyset$. Then $[v]_A \cap Y \neq \emptyset$ if and only if $v \in Y \cap A'''$ or $\exists w \in A^* : v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq Q_A(Y)$, since Y is transversal to both A'' and A'''. So, we conclude that $\mathbf{u}_A(Y) = Q_A(Y) \cup (Y \cap A''') = Q_A(Y) \cup Y$.

9): Let A and Y be two vertex subsets such that $Y \cap A' \neq \emptyset \wedge Y \cap A'' \neq \emptyset \wedge Y \cap A''' \neq \emptyset$. It means that Y is transversal to the three sets. Therefore we have $[v]_A \cap Y \neq \emptyset$ if and only if $v \in A'$ or $v \in Y \cap A'''$ or $\exists w \in A^* : v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq Q_A(Y)$. Therefore, $\mathbf{u}_A(Y) = A' \cup Q_A(Y) \cup (Y \cap A''') = Y \cup A' \cup Q_A(Y)$ and we are done.

At this point, we prove that the previous cases are all disjoint each other and they are all possible cases that can occur. Let Y be a proper vertex subset of $V = V(C_n)$, then since $V(C_n) = A' \cup A'' \cup A'''$, we deduce that Y can be a subset of one of these three sets, as we have said writing down the conditions 3), 4) and 5), or it can be transversal to two of them, without containing none of them and without intersecting the third, as we have said writing down the conditions 6), 7) and 8). Finally, Y can be transversal to every set, without containing none of them, as written in the last condition. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. In this way we have shown the theorem.

Let us compute now the $A-$lower approximation function for the $n-$cycle C_n. By Proposition 4 the A-indiscernibility partitions of P_n and C_n have the same structure and the next result depends only from this partition structure.

Also to compute the A-lower approximation function of C_n we will use the previous proof technique, namely we study all possible relations between the vertex subset Y and the three subsets A', A'' and A''' determining the A-indiscernibility partition. In what follows we will use the notations introduced in Definition 5.

If A and Y are two vertex subsets of C_n we set

$$T_A(Y) := \bigcup_{v \in A^*} \{N_{C_n}(v) : N_{C_n}(v) \subseteq Y\}.$$

Theorem 2. *Let A and Y be two vertex subsets of C_n. The map*
$\mathbf{l} : (A,Y) \in \mathscr{P}(V(C_n)) \times \mathscr{P}(V(C_n)) \mapsto \mathbf{l}_A(Y) \in \mathscr{P}(V(C_n))$ *is completely described from the cases listed in the following table:*

CASE	*CONDITIONS*	$\mathbf{l}_A(Y)$
1)	$Y = V(C_n)$	$V(C_n)$
2)	$Y \subsetneq A'$	$\emptyset$
4)	$Y \supseteq A' \wedge T_A(Y) = \emptyset$	$A' \cup (Y \cap A''')$
5)	$Y \supseteq A' \wedge T_A(Y) \neq \emptyset$	$A' \cup (Y \cap A''') \cup T_A(Y)$
6)	$Y \cap A' \neq \emptyset \wedge A' \nsubseteq Y \wedge T_A(Y) = \emptyset$	$Y \cap A'''$
7)	$Y \cap A' \neq \emptyset \wedge A' \nsubseteq Y \wedge T_A(Y) \neq \emptyset$	$(Y \cap A''') \cup T_A(Y)$

Proof. We recall that by definition $\mathbf{l}_A(Y) = \{v \in V(C_n) : [v]_A \subseteq Y\}$. We also recall that, by

Proposition 3, if A is a generic vertex subset, we have

$$\pi_{C_n}(A) = A'|v_{j_1-1}v_{j_1+1}|\dots|v_{j_h-1}v_{j_h+1}|v_{s_1}|\dots|v_{s_l},$$

where $\{v_{j_1},\dots,v_{j_h}\} = A^*$ and $\{v_{s_1},\dots,v_{s_l}\} = A'''$.
1): Let $Y = V(C_n)$ and A be any vertex subset of $V(C_n)$. Obviously $[v]_A \subseteq V(C_n)$ for every vertex v, so $\mathbf{l}_A(V(C_n)) = V(C_n)$.
2): Let $Y = \emptyset$ and A be any vertex subset of $V(C_n)$. Obviously no indiscernibility class is contained in the empty set, so $\mathbf{l}_A(\emptyset) = \emptyset$.
3): Let A and Y be two vertex subsets such that $Y \subsetneqq A'$. Since A' forms a single block and since Y is disjoint from A'' and A''', we deduce that there is no vertex whose indiscernibility class is contained in Y, thus $\mathbf{l}_A(Y) = \emptyset$.
4): Let A and Y be two vertex subsets such that $Y \supseteq A'$ and $T_A(Y) = \emptyset$. We observe that Y may or not intersect the vertex subset A'' and, in the first case, in such a way that there not exists any vertex $w \in A^*$ whose neighbourhood is contained in Y. Furthermore, Y may or not intersect A'''. Thus we conclude that $[v]_A \subseteq Y$ if and only if $v \in A'$ or, possibly, $v \in Y \cap A'''$, i.e. $\mathbf{l}_A(Y) = A' \cup (Y \cap A''')$.
5): Let A and Y be two vertex subsets such that $Y \supseteq A'$ and $T_A(Y) \neq \emptyset$. We observe that Y may or not intersect the vertex subset A'''. Moreover, there exists at least one vertex $w \in A^*$ whose neighbourhood is contained in Y. Hence we conclude that $[v]_A \subseteq Y$ if and only if $v \in A'$ or $v \in N_{C_n}(w) \subseteq T_A(Y)$ for some $w \in A^*$ or, possibly, $v \in Y \cap A'''$, i.e. $\mathbf{l}_A(Y) = A' \cup T_A(Y) \cup (Y \cap A''')$.
6): Let A and Y be two vertex subsets such that $Y \cap A' \neq \emptyset \wedge A' \not\subseteq Y \wedge T_A(Y) = \emptyset$. Since A' forms a single block, we conclude that $A' \not\subseteq \mathbf{l}_A(Y)$. We also observe that Y may or not intersect the vertex subset A'''. Moreover, there is no vertex $w \in A^*$ such that $v \in N_{C_n}(w)$ and $N_{C_n}(w) \subseteq Y$. This means that $[v]_A \subseteq Y$ if and only if $v \in Y \cap A'''$, i.e. $\mathbf{l}_A(Y) = Y \cap A'''$.
7): Let A and Y be two vertex subsets such that $Y \cap A' \neq \emptyset \wedge A' \not\subseteq Y \wedge T_A(Y) \neq \emptyset$. Since A' forms a single block, we conclude that $A' \not\subseteq \mathbf{l}_A(Y)$. We also observe that Y may or not intersect the vertex subset A'''. Furthermore, there exists at least a vertex $w \in A^*$ such that $v \in N_{C_n}(w)$ and $N_{C_n}(w) \subseteq Y$. In other words, we are saying that $[v]_A \subseteq Y$ if and only if $v \in Y \cap A'''$ or $\exists w \in A^* : v \in N_{C_n}(w) \wedge N_{C_n}(w) \subseteq T_A(Y)$, i.e. $\mathbf{l}_A(Y) = (Y \cap A''') \cup T_A(Y)$.
At this point, we show that we have studied all the occurring cases. Let Y be a proper vertex subset of $V = V(C_n)$. In case 3) we have that $Y \subsetneqq A'$ while in cases 4) and 5) we are requiring that Y contains A' and it may (or not) be transversal to both A'' and A'''. Finally, Y may only transversal to A', without containing it, as we have seen in the last two cases. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. Hence, the theorem is proved.

5 A-exact subsets of C_n and P_n

Let A be a vertex subset of a given graph. By Definition 4, we recall that a vertex subset Y is A-exact if and only if its A-lower approximation of Y coincide with its A-upper approximation. In this section we determine the general form of all $A-$exact subsets of C_n. Also in this case we will obtain an identical form also for the A-exact subsets of P_n, because both the A-lower approximation function and the A-upper approximation function are identical for C_n and P_n. As in the previous sections, in what follows we will use the notations introduced in Definition 5.

Proposition 5. *Let A and Y be two vertex subsets of $V(C_n)$. Then Y is A-exact if and only if one of the following cases holds:*

CASE	*CONDITIONS*
1)	$A = V(C_n)$
2)	$Y = V(C_n)$
3)	$Y = \emptyset$
4)	$Y = A'$
5)	$Y \subseteq A'''$
6)	$Y \subseteq A'' \wedge \exists v_{j_1},\dots,v_{j_k} \in A^* : Y = \bigcup_{k=1}^{m} N_{C_n}(v_{j_k})$
7)	$Y \cap A''' \neq \emptyset \wedge Y \cap A' = \emptyset \wedge A''' \not\subseteq Y \wedge \exists v_{j_1},\dots,v_{j_k} \in A^* : Y \cap A'' = \bigcup_{k=1}^{m} N_{C_n}(v_{j_k}) \wedge A'' \not\subseteq Y$
8)	$Y \supsetneqq A' \wedge \exists v_{j_1},\dots,v_{j_k} \in A^* : Y \cap A'' = \bigcup_{k=1}^{m} N_{C_n}(v_{j_k})$
9)	$Y \supsetneqq A' \wedge Y \cap A'' = \emptyset \wedge Y \cap A''' \neq \emptyset \wedge A''' \not\subseteq Y \wedge A''' \not\subseteq Y$
10)	$Y \supsetneqq A'' \wedge Y \cap A' = \emptyset \wedge Y \cap A''' \neq \emptyset \wedge A''' \not\subseteq Y \wedge A''' \not\subseteq Y$
11)	$Y \supsetneqq A''' \wedge Y \cap A' = \emptyset \wedge \exists v_{j_1},\dots,v_{j_k} \in A^* : Y \cap A'' = \bigcup_{k=1}^{m} N_{C_n}(v_{j_k}) \wedge A'' \not\subseteq Y$
12)	$Y \supsetneqq A' \cup A''' \wedge \exists v_{j_1},\dots,v_{j_k} \in A^* : Y \cap A'' = \bigcup_{k=1}^{m} N_{C_n}(v_{j_k})$
13)	$Y = A' \cup A'''$
14)	$Y = A'' \cup A'''$

Proof. By Theorems 1 and 2, it's easy to show that if A and Y satisfy one of the conditions of the previous table, then Y is $A-$exact. Viceversa, let A and Y be two vertex subsets different from the previous. We will examine all possible cases.
If A and Y are two vertex subsets such that $Y \subsetneqq A'$, we deduce that $\mathbf{l}_A(Y) = \emptyset \neq A' = \mathbf{u}_A(Y)$, therefore Y can't be A-exact.
Suppose that $Y \cap A' \neq \emptyset \wedge Y \cap A'' = \emptyset \wedge A' \not\subseteq Y$. In order to have $\mathbf{l}_A(Y) = \mathbf{u}_A(Y)$, it must result $Y \cap A''' = (Y \cap A''') \cup A'$. In other words, we must require that $A' \subseteq Y \cap A'''$ or $A' = \emptyset$. Both these conclusions are false, since $A' \not\subseteq A'''$ and since the condition $Y \cap A' \neq \emptyset$ ensures that $A' \neq \emptyset$.
In the other cases to analyze, we always have that

$$Y \cap A'' \neq \emptyset \wedge \nexists v_{i_1},\dots,v_{i_k} \in A^* : \, Y \cap A'' = \bigcup_{j=1}^{k} N_{C_n}(v_{i_j}). \quad (5)$$

It's easy to see that whenever (5) holds, it results that $T_A(Y) \subsetneqq Q_A(Y)$. This means that the A-lower approximation and the A-upper approximation of Y must differ each other.
In this way, we have shown that in all possible situations different from those listed in the above table $\mathbf{l}_A(Y) \neq \mathbf{u}_A(Y)$, thus we have completed the proof.

6 Conclusions

This paper is a further contribution to the interpretation of a simple undirected graph in terms of Boolean information table. We have completely determined both the A-lower and A-upper approximation functions for the n-cycles and the n-paths. We have shown that the complete study of these functions can be very laborious, in spite the simplicity of examined graph structure. Our study is part of a research project started in [22,23] and it will be further developed in forthcoming papers relatively to others graph families and by means of others RST investigation tools.

References

[1] J.A. Aledo, S. Martínez, J. C. Valverde, Parallel Dynamical Systems over Graphs and Related Topics: A Survey. *Journal of Applied Mathematics* Volume 2015 (2015), Article ID 594294, 14 pages.

[2] C.L. Barret, C.M. Reidys, Elements of a theory of computer simulation. I. Sequential CA over random graphs, *Appl. Math. Comput.*, **98** (1999), no.2–3, 241–259

[3] C.L. Barret, H.S. Mortveit, C.M. Reidys, Elements of a theory of computer simulation. II. Sequential dynamical systems, *Appl. Math. Comput.*, **107** (2000), no.2–3, 121–136

[4] C.L. Barret, H.S. Mortveit, C.M. Reidys, Elements of a theory of computer simulation. III. Equivalence of SDS, *Appl. Math. Comput.*, **122** (2001), no.3, 325–340

[5] C.L. Barret, H.S. Mortveit, C.M. Reidys, Elements of a theory of computer simulation. IV. Sequential dynamical systems: fixed points, invertibility and equivalence, *Appl. Math. Comput.*, **134** (2003), no.1, 153–171

[6] D. Bianucci, G. Cattaneo, Information Entropy and Granulation Co-Entropy of Partitions and Coverings: a Summary, in: Special Issue on Foundations of Rough Sets, J.F. Peter and A. Skowron Eds., *Transactions on Rough Sets X, Lecture Notes in Computer Science*, vol. 5656, Springer-Verlag, 2009, pp. 15-66.

[7] C. Bisi, G. Chiaselotti, A class of lattices and boolean functions related to the Manickam-Miklös-Singhi conjecture, *Adv. Geom.* **13**, no.1, (2013), 1–27.

[8] C. Bisi, G. Chiaselotti, P.A. Oliverio, Sand Piles Models of Signed Partitions with d piles. *ISRN Combinatorics*, **vol. 2013**, Article ID 615703, 7 pages, 2013. doi: 10.1155/2013/615703

[9] C. Bisi, G. Chiaselotti, G. Marino, P.A. Oliverio, A natural extension of the Young partition lattice. *Adv. Geom.* **15**, no.3, (2015), 263–280.

[10] C. Bisi, G. Chiaselotti, T. Gentile, P.A. Oliverio, Dominance Order on Signed Partitions. *Adv. Geom.* (To Appear).

[11] Cattaneo G, Generalized Rough Sets (Preclusivity Fuzzy-Intuitionistic (BZ) Lattices). *Studia Logica* **01** (1997), pp. 47–77.

[12] Cattaneo, G.: Abstract approximation spaces for rough theories, in: Rough Sets in Knowledge Discovery 1: Methodology and Applications (L. Polkowski, A.Skowron (eds.)). Studies in Fuzziness and Soft Computing, Physica, Heidelberg (1998) 59–98

[13] G. Cattaneo, D. Ciucci, D. Bianucci, Entropy and Co-entropy of Partitions and Coverings with Applications to Roughness Theory, in: *Granular Computing: At the Junction of Rough Sets and Fuzzy sets*. Series: Studies on Fuzziness and Soft Computing. R. Bello, R. Falcon, W Pedrycz and J. Kacprzyk, Eds., vol. 224, Springer-Verlag, 2008, pp. 55-77.

[14] G. Cattaneo, An investigation about rough set theory: some foudational and mathematical aspects, *Fundamenta Informaticae* 108, 2011, pp. 197–221.

[15] G. Cattaneo, G. Chiaselotti, A. Dennunzio, E. Formenti, L. Manzoni, Non Uniform Cellular Automata Description of Signed Partition Versions of Ice and Sand Pile Models. *Cellular Automata. Lecture Notes in Computer Science*. Volume 8751, pp. 115–124, 2014.

[16] G. Cattaneo, G. Chiaselotti, D. Ciucci, T. Gentile, On the connection of Hypergraph Theory with Formal Concept Analysis and Rough Set Theory. *Information Sciences* (To appear).

[17] G. Chiaselotti, G. Marino, P.A. Oliverio, D. Petrassi, A Discrete Dynamical Model of Signed Partitions, *J. Appl. Math*, **2013** (2013), Article ID 973501, 10 pages.

[18] G. Chiaselotti, T. Gentile, G. Marino, P.A. Oliverio, Parallel Rank of Two Sandpile Models of Signed Integer Partitions, *J. Appl. Math*, **2013** (2013), Article ID 292143, 12 pages.

[19] G. Chiaselotti, T. Gentile, P.A. Oliverio, Parallel and sequential dynamics of two discrete models of signed integer partitions. *Applied Mathematics and Computation*, **232**, pp 1249–1261 (2014).

[20] G. Chiaselotti, T. Gentile, P.A. Oliverio, Lattices of Partial Sums. *Algebra and Discrete Mathematics*, Volume 18, Number 2, pp 171–185 (2014).

[21] S. T. Dougherty, B. Yldz, S. Karadeniz,Self-Dual Codes over R_k and Binary Self-Dual Codes, *European Journal of Pure and Applied Mathematics*, **6**, pp 89-106 (2013).

[22] G. Chiaselotti, D. Ciucci, T. Gentile, Simple Graphs in Granular Computing. *Information Sciences* (To appear).

[23] G. Chiaselotti, D. Ciucci, T. Gentile, Simple Undirected Graphs as Formal Contexts. *Proc. ICFCA, Lecture Notes in Computer Science*, Volume 9113, pp. 287–302, Springer, 2015.

[24] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Rough Set Theory Applied to Simple Undirected Graphs. *Proc. RSKT 2015, Lecture Notes in Computer Science*, Vol. 9436, Springer 2015 (To appear).

[25] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Preclusivity and Simple Graphs. *Proc. RSFDGrC 2015, Lecture Notes in Computer Science*, Vol. 9437, Springer 2015 (To appear).

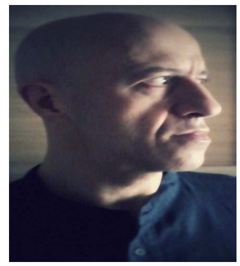
[26] G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Preclusivity and Simple Graphs: the n–cycle and n–path Cases. *Proc. RSFDGrC 2015, Lecture Notes in Computer Science*, Vol. 9437, Springer 2015 (To appear).

[27] D. E. Ciucci, Classification of Dynamics in Rough Sets, RSCTC 2010, *Lecture Notes in Computer Science*, vol. 6086, pp. 257–266, Springer-Verlag 2010.

[28] D. E. Ciucci, Temporal Dynamics in Information Tables, *Fundamenta informaticae* **115** (2012), n. 1, 57–74.

[29] R. Diestel, Graph Theory (4th edition), Graduate Text in Mathematics, Springer 2010.

[30] S. Greco et Al., Rough Sets methodology for sorting problems in presence of multiple attributes and criteria, *European Journal of Operational Research*, 138, pp. 247–259, 2002.
[31] P. Hońko, Relational pattern updating, *Information Sciences*, **189**, 2012, 208–218.
[32] P. Hońko, Association discovery from relational data via granular computing, *Information Sciences*, **10**, 2013, 136–149.
[33] B. Huang, Y. Zhang, H. Li, D. Wei, A Dominance Intuitionistic Fuzzy-Rough Set Approach and Its Applications, Applied Mathematical Modelling, **37**, 2013, 7128–7141.
[34] X. Kang, D. Li, S. Wang, K. Qu, Formal concept analysis based on fuzzy granularity base for different granulations, *Fuzzy Sets and Systems*, **203**, 2012, 33–48.
[35] V Kreinovich, Interval Computation as an Important Part of Granular Computing: An Introduction, in *Handbook of Granular Computing*, Wiley, 2008, 3-32.
[36] T. Y. Lin, Data Mining: Granular Computing Approach, in Methodologies for Knowledge Discovery and Data Mining *Lecture Notes in Computer Science*, **1574**, 1999, 24–33.
[37] T. Y. Lin, Data Mining and Machine Oriented Modeling: A Granular Approach, *Applied Intelligence*, **13**, 2000, 113–124.
[38] Z. Pawlak, Information systems theoretical foundations, *Information Sciences* **6**, Issue 3, 1981, pp. 205–218
[39] Z. Pawlak, Rough Sets, *International Journal of Computer and Information Sciences*, vol. 11, No. 5, 1982, pp. 341–356
[40] Z. Pawlak, Rough sets. Theoretical Aspects of Reasoning about Data. Kluwer Academic Publisher, 1991.
[41] Z. Pawlak, A. Skowron, Rudiments of rough sets, *Information Sciences* **177**, 2007, pp. 3–27
[42] Z. Pawlak, A. Skowron, Rough sets: Some extensions, *Information Sciences* **177**, 2007, pp. 28–40
[43] Z. Pawlak, A. Skowron, Rough sets and Boolean reasoning, *Information Sciences* **177**, 2007, pp. 41–73
[44] W. Pedrycz, K. Hirota, W. Pedrycz, F. Dong, Granular representation and granular computing with fuzzy sets, *Fuzzy Sets and Systems*, **203**, 2012, 17–32.
[45] W. Pedrycz, Granular computing : analysis and design of intelligent systems, CRC Press, 2013.
[46] T. Qiu, X. Chen, Q. Liu, H. Huang, Granular Computing Approach to Finding Association Rules in Relational Database, *International Journal of Intelligent Systems*, **25**, 2010, 165–179.
[47] A. Skowron, P. Wasilewski, Information systems in modeling interactive computations on granules, Theoretical Computer Science, **412**, 2011, 5939–5959.
[48] A. Skowron, P. Wasilewski, Interactive information systems: Toward perception based computing, Theoretical Computer Science, **454**, 2012, 240–260.
[49] W. Z. Wu, Y. Leung, J. S. Mi, Granular Computing and Knowledge Reduction in Formal Contexts, *IEEE Trans. on Knowledge and Data Engineering*, **21**, 2009, 1461–1474
[50] Y. Y. Yao, On modeling data mining with granular computing, COMPSAC 2001. IEEE, 2001, pp. 638–643.
[51] J. T. Yao, Y. Y. Yao, A Granular Computing Approach to Machine Learning, in Proceedings of the 1st International Conference on Fuzzy Systems and Knowledge Discovery, 2002, 732–736
[52] Y. Yao, Y. Zhao, Discernibility matrix simplification for constructing attribute reducts, *Information Sciences*, **179**, 2009, 867–882.
[53] Y. Yao, B. X. Yao, Covering based rough set approximations, *Information Sciences*, **200**, 2012, 91–107.
[54] L. A. Zadeh, Towards a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, **19**, 1997, 111–127.



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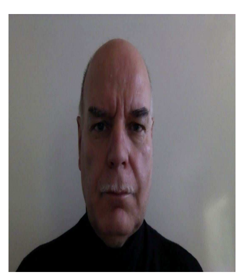

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