Rough Sets for $n$-Cycles and $n$-Paths


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Abstract: In this paper we carry out a research project whose main goal is the study of an undirected graph by means of investigation tools provided by Pawlak rough set theory. Specifically, we determine both the lower and the upper rough approximation functions for an information table induced from a cycle or a path on $n$ vertices. For such graphs we also provide a complete description of the corresponding exact subsets.

Keywords: Graded Lattices, Integer Partitions, Young Diagrams, Sand Piles Models.

1 Introduction

Graphs are objects ubiquitously present in mathematics and computer science. The graph structure has been studied mostly from a geometric point of view, by searching for the analogies with the various types of geometries. In fact, classical problems in graph theory concern the determination of distances, neighborhoods, connectivity and so on. Recently, the impetuous development of computer science has placed new questions about the graph structures. For example, the graphs can be studied in terms of sequential dynamical systems (see [2,3,4,5]), by means of parallel dynamics (see [1]), or also for their analogies with both sequential and parallel dynamics on order structures (see [7,8,9,10,15,17,18,19,20,21]).

In this paper we continue a research project started in [22,23], where a simple undirected graph is studied as a particular type of information system. According to Pawlak (see [40]) an information system is a structure $\mathcal{I} = \langle U, \text{Att}, \text{Val}, F \rangle$, where $U = \{u_1, u_2, \ldots, u_m\}$ is a non-empty finite set called universe set, $\text{Att} = \{a_1, a_2, \ldots, a_n\}$ is a non-empty finite set called attribute set and $F : U \times \text{Att} \rightarrow \text{Val}$, called information map, is an application from the direct product $U \times \text{Att}$ into the so called value set $\text{Val}$. The elements of $U$ are called objects and the elements of $\text{Att}$ are called attributes. In particular, if $\text{Val} = \{0, 1\}$ we say that $\mathcal{I}$ is a Boolean information system. An information system occurs in all situations in which a huge amount of data needs to be classified in a table according to some criterion of subdivision, therefore it is a structure that is very frequent in various fields of study, both of qualitative and quantitative type. In his seminal works [38,39,40], Pawlak introduced several investigation tools in order to better analyze and reduce the complexity of a generic information system. In this framework, Pawlak proposed the rough set theory, abbreviated RST (see also the more recent papers [41,42,43]), that is a useful methodological tool for reasoning about knowledge of objects represented by attributes. Nowadays, RST (and its more general version called granular computing [45]) is a well investigated research field [6,12,13,14,27,28,52,53], which has connections with operative research [30], preclusivity spaces [11], machine learning [51], interval analysis [35], formal concept analysis [34,49], database theory [31,46], data mining [32,36,37,50], fuzzy set theory [33,44,54], interactive computing [47,48].

The fundamental assumption of RST is based on the famous Law of Indiscernibility, according to which two objects are indiscernible (i.e. similar) if and only if they share the same properties. Formally, if $\mathcal{I} = \langle U, \text{Att}, \text{Val}, F \rangle$ is an information system and $A \subseteq \text{Att}$, we call $A$-indiscernibility the equivalence relation $\equiv_A$ on the universe set $U$ defined as follows: if $u, u' \in U$ then

$$u \equiv_A u' : \iff F(u, a) = F(u', a), \forall a \in A.$$ (1)

In rough set theory, there exist different kinds of sets:

–Elementary sets, which are sets of all indiscernible objects;

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Exact sets, which are unions of elementary sets;

–Rough (in the sense of imprecise) sets, which are not exact.

The objects can be classified as belonging to exact sets or to their complements, but it isn’t true in the case of rough sets. Thus the behavior of rough sets is more interesting than that of exact sets. In fact, we can classify objects by means of the knowledge degree we have about them, but in the case of rough sets, there are some objects which belong to a so called boundary region, namely those for which it is not possible to say with certainty that belong to the set or to its complement. Hence, rough sets represent vague concepts. The way of studying these sets consists in substituting any vague concept with two precise concepts, called respectively lower approximation and upper approximation. The lower approximation of an non-empty $Y$ (briefly $l(Y)$) represents the set of objects that surely belong to $Y$, with respect to our knowledge provided by $A$; the upper approximation of an object subset $Y$ (briefly $u(Y)$) is the set of objects surely or possibly belonging to $Y$ with respect to our knowledge expressed by $A$. Furthermore, we say that an object subset is $A$-exact if its $A$-lower approximation coincides with its $A$-upper approximation.

Some natural links of RST with both graph and hypergraph theory have recently been found. In fact, in [22, 23, 24, 25, 26] the idea to study any simple undirected graph $G$ as if it were a Boolean information system was developed (in [16] this idea has been also extended to hypergraph theory). The basic tool to connect graphs and Boolean information systems is the adjacency matrix of $G$, which in [22, 23, 24, 25, 26] has been interpreted as the Boolean table of a particular information system. Specifically, in [22, 23, 24] some graph families, such as the complete graph $K_n$ or the complete bipartite graph $K_{p,q}$, have been broadly studied in terms of information tables. In particular, in the above papers, the $A$-lower and the $A$-upper approximations, the $A$-positive region of any vertex subset $B$, the $A$-attribute dependency function and the rough membership function, where $A$, $B$ and $Y$ are vertex subsets, have been completely determined both for the complete graph $K_n$ and the complete bipartite graph $K_{p,q}$.

In this paper we apply the RST tools respectively to the cases of the cycle $C_n$ and of the path $P_n$ on $n$ vertices (respectively, $n$-cycle and $n$-path). Specifically, we determine the $A$-lower and the $A$-upper approximations for both $C_n$ and $P_n$. Moreover, we also determine all $A$-exact sets, i.e. those sets for which the $A$-lower approximation coincide with the $A$-upper approximation. Although the structure of both $C_n$ and $P_n$ is quite simple, the complete determination of the corresponding $A$-lower and $A$-upper approximations was found quite complex, since it was necessary to treat many sub-cases. In fact, the indiscernibility with respect to a vertex subset $A$ gives rise to three sets $A'$, $A''$ and $A'''$, which provide a partition of the vertex set $V(G)$, for both $G = C_n$ and $G = P_n$. All occurring sub-cases correspond then to all possible relations between the vertex subset $Y$ and the previous three sets. From this, the need to express in detail our results in three tables. Our results show that, also for simple graph structures, the complete determination of both the $A$-lower and the $A$-upper approximations can be quite complex. Therefore one expects that the complete computation of these approximations for more complicated families of graphs can be a difficult goal to achieve.

To conclude this introduction we now briefly describe the content of the sections in this paper. In Section 2, we firstly introduce the basic notations that we use in the sequel. Next, we characterize in our graph context the form of two classical RST notions: indiscernibility relation and approximation functions. In Section 3, for any $n$ we completely determine the indiscernibility partition form for both the $n$-cycle and the $n$-path. In Section 4 we compute the $A$-upper approximation and the $A$-lower approximation for both $C_n$ and $P_n$. Finally, in Section 5 we use the results obtained in Section 4 in order to find all $A$-exact subsets for both $C_n$ and $P_n$.

2 Basic Results

If $X$ is any finite set, we denote by $|X|$ the number of elements in the set $X$ and by $\mathcal{P}(X)$ the power set of $X$. If $Y \subseteq X$ and $X$ is clear from the context, we write $Y'$ instead of $X \setminus Y$.

If $R$ is an equivalence relation on $X$ and $x$ is an element of $X$, we denote by $[x]_R$ the equivalence class of $x$ with respect to the relation $R$.

We recall now the following classical notion.

**Definition 1.** A set-partition $\pi$ on $X$ is a finite collection of non-empty subsets $B_1, \ldots, B_M$ of $X$ such that $B_i \cap B_j = \emptyset$ for all $i \neq j$ and such that $\bigcup_{i=1}^M B_i = X$. The subsets $B_1, \ldots, B_M$ are called blocks of $\pi$ and we write $\pi := B_1 | \ldots | B_M$ to denote that $\pi$ is a set partition having blocks $B_1, \ldots, B_M$.

In this paper we treat exclusively with finite undirected simple graphs and we refer to [29] for any general notion concerning graph theory. Here we recall only some basic definitions and we fix some notations which we will use in the sequel. We always denote by $G = (V(G), E(G))$ a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v, v'\} \in E(G)$ and $v \not\sim v'$ otherwise.

**Definition 2.** Let $v \in V(G)$. We call neighborhood of $v$ in $G$ the set $N_G(v) := \{w \in V(G) : v \sim w\}$. In particular, if
A ⊆ V(G) we call neighborhood of A in G the set
\[ N_G(A) := \bigcup_{v \in A} N_G(v) \]  
(2)

In the next definition we see how a graph G becomes a Boolean information system (for details see [22] and [23]).

Definition 3. We call information system of the graph G the Boolean information system
\[ \mathcal{I}[G] := \langle U(G), Att(G), \{0, 1\}, F_G \rangle, \]
where \( U(G) := V(G), \) Att(G) := V(G) and
\[ F_G(u,v) := \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases} \]

Let \( A \subseteq V(G) \) a vertex subset. By (1), we can define the A-indiscernibility relation \( \equiv_A \) for the information system of the graph G as follows:
\[ v \equiv_A v' :\iff F_G(v,a) = F_G(v',a), \forall a \in A. \]  
(3)

By the previous definition it follows the next result.

Proposition 1. Let \( v, v' \in G \) and \( A \subseteq V(G) \). Then:
(i) \( v \equiv_A v' \) if and only if for all \( z \in A \) it results that \( v \sim z \) if and only if \( v' \sim z \).
(ii) If \( v \sim v' \) then \( v \equiv_A v' \) or \( \{v,v'\} \cap A = \emptyset \).
(iii) If \( v \equiv_A v' \) and \( \{v,v'\} \cap A \neq \emptyset \), then \( v \sim v' \).

Proof. (i) It follows immediately by Definition 3 and by (3).
(ii) We suppose that \( v \sim v' \) and \( v \equiv_A v' \). We must show that \( \{v,v'\} \cap A = \emptyset \). Without loss of generality, we can assume \( v \in A \). By (3) we deduce that \( F_G(v,v) = F_G(v',v) \), but \( F_G(v,v) = 0 \) since there are no loops in G while by our assumption \( F_G(v',v) = 1 \). So the equality \( F_G(v,v) = F_G(v',v) \) does not hold, absurd. The case \( v' \in A \) is analogous.
On the other hand, a similar argument shows that if \( v \sim v' \) and \( \{v,v'\} \cap A \neq \emptyset \) then \( v \equiv_A v' \). In fact we just observe that if \( v \in A \), then \( F_G(v,v) = F_G(v',v) \), hence we conclude by (3). This proves (ii).
(iii) It is the contra-nominal version of (ii).

If \( A \subseteq V(G) \) and \( v,v' \in V(G) \) it is easy to note, by (i) of Proposition 1, that:
\[ v \equiv_A v' :\iff N_G(v) \cap A = N_G(v') \cap A \]  
(4)

We denote by \( \pi_G(A) \) the set partition of \( V(G) \) induced from the equivalence relation \( \equiv_A \). If \( v \in V(G) \), we denote by \([v]_A \) the equivalence class of the vertex v with respect to \( \equiv_A \). Let us also note that \([v]_\emptyset = V(G)\) for all \( v \in V(G) \), therefore \( \pi_G(V) = V(G) \).

We recall now the following basic notions of RST.

Definition 4. Let \( \mathcal{I} = \langle U, Attr, Val, F \rangle \) be an information system, \( A \subseteq Attr \) and \( Y \subseteq U \). The A-lower approximation of \( Y \) is the following subset of \( U \):
\[ l_A(Y) := \{x \in U : [x]_A \subseteq Y \} = \bigcup\{C \in \pi_A(\mathcal{I}) : C \subseteq Y \}. \]
The A-upper approximation of \( Y \) is defined as:
\[ u_A(Y) := \{x \in U : [x]_A \cap Y \neq \emptyset \} = \bigcup\{C \in \pi_A(\mathcal{I}) : C \cap Y \neq \emptyset \}. \]
The subset \( Y \) is called A-exact if and only if \( l_A(Y) = u_A(Y) \) and A-rough otherwise.

The lower approximation represents the elements that certainly, with respect to our knowledge expressed by \( A \), belong to \( Y \). On the other hand, the upper approximation is the set of objects possibly belonging to \( Y \).

For the A-lower and A-upper approximation functions we obtain the following geometrical interpretation in the simple graph context.

Proposition 2. Let \( G = (V(G), E(G)) \) be a simple undirected graph and let \( \mathcal{I}[G] \) be the Boolean information system associated to \( G \). Let \( A \) and \( Y \) be two subsets of \( V(G) \). Then:
(i) \( u_A(Y) = \{v \in V(G) : \exists u \in Y : N_G(u) \cap A = N_G(v) \cap A\} \).
(ii) \( l_A(Y) = \{v \in V(G) : (\forall u \in Y : N_G(u) \cap A = N_G(v) \cap A) \implies u \in Y\} \).
Therefore, \( v \in u_A(Y) \) iff \( v \) is an A-symmetric vertex of some \( u \in Y \).

Proof. It follows directly by (4) and from the definitions of the approximations.

Hence the lower approximation of a vertex set \( Y \) represents a subset of \( Y \) such that there are no elements outside \( Y \) with the same connections of any vertex in \( l_A(Y) \) (relative to \( A \)). The upper approximation of \( Y \) is the set of vertices with the same connections (w.r.t. \( A \)) of at least one element in \( Y \). By the previous proposition it is natural to call \( l_A(Y) \) the A-symmetry kernel of \( Y \) and \( u_A(Y) \) the A-symmetry closure of \( Y \).

3 Cn and Pn as Boolean Information Systems

In this section we consider the \( n \)-cycle \( C_n \) and \( n \)-path \( P_n \). Recall the definitions of the two graphs and introduce some particular vertex subsets which will be used extensively in this paper.

Definition 5. Let \( n \) be a positive integer. The \( n \)-cycle \( C_n \) is the graph having vertex set \( V(C_n) = \{v_1, \ldots, v_n\} \) and edge set:
\[ E(C_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}. \]

If \( A \subseteq V(C_n) \), we set \( A^* := \{v_i \in A : v_{i-2} \notin A \land v_{i+2} \notin A\} \), \( A' := (N_G(A))^c \), \( A'' := (N_G(A'))^c \) and \( A''' := (A' \cup A'')^c \). Let us note that \( V(C_n) = A' \cup A'' \cup A''' \).
Definition 6. Let \( n \) be a positive integer. The \( n \)-path \( P_n \) is the graph having vertex set \( V(P_n) = \{v_1, \ldots, v_n\} \) and edge set:

\[
E(P_n) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}.
\]

If \( A \subseteq V(P_n) \), we set \( A' := \{v_i \in A : 3 < i \leq n - 2 \wedge v_{i-2} \notin A \wedge v_{i+2} \notin A\} \), \( A'' := (N_G(A))^c \), \( A''' := N_P(A) \) and \( A'''' := (A' \cup A'')^c \). Let us note that \( V(P_n) = A' \cup A'' \cup A''' \).

In the next result we provide a complete description of the indiscernibility partition of \( C_n \).

Proposition 3. Let \( A = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V := V(C_n) = \{v_1, \ldots, v_n\} \), \( A'''' = \{v_{i_1}, \ldots, v_{i_k}\} \) and \( A' = \{v_{j_1}, \ldots, v_{j_k}\} \). Then:

\[
\pi_n(A) = A'[v_{j_1 - 1}v_{j_1 + 1}][\cdots][v_{j_k - 1}v_{j_k + 1}][v_{i_1}][\cdots][v_{i_k}],
\]

where the index sums are taken \( \text{mod}(n) \).

Proof. In what follows, all the index sums are taken \( \text{mod}(n) \). Let \( v_i, v_j \in V \), with \( i < j \). Then \( v_i \equiv_A v_j \) if and only if \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \) or \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \) and \( v_i \equiv_A v_j \). Hence, if \( N_G(v_i) \cap A = N_G(v_j) \cap A \), then \( N_G(v_i) \cap N_G(v_j) \leq 1 \) and the equality holds if and only if \( j = i + 2 \). This proves the thesis. In fact, let \( v_i, v_j \in V(G) \), with \( i < j \) and \( v_i \equiv_A v_j \). Then either \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \) or \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \). The first condition is equivalent to say that \( v_i, v_j \in A' \), whereas the second is equivalent to say that \( v_i, v_j \in A'' \). The proposition is thus proved.

Example 1. Let \( C_{10} \) be the 10-cycle on the set \( V = \{v_1, \ldots, v_{10}\} \). Let \( A = \{v_2, v_4, v_7\} \). Then:

\[
A' = \{v_2, v_4, v_7, v_9, v_{10}\}
\]

\[
A'' = \{v_7\}
\]

\[
A''' = \{v_6, v_8\}
\]

and

\[
A'''' = \{v_1, v_3, v_5\}
\]

Thus

\[
\pi_{C_{10}}(A) = v_2v_4v_7v_9v_{10}[v_6v_8][v_1][v_3][v_5]
\]

We give now a complete description of the indiscernibility partition for the graph \( P_n \) for any vertex subset \( A \subseteq V(P_n) \).

Proposition 4. Let \( A = \{v_{i_1}, \ldots, v_{i_k}\} \subseteq V := V(P_n) = \{v_{i_1}, \ldots, v_{i_k}\} \) and \( A' = \{v_{j_1}, \ldots, v_{j_k}\} \). Then:

\[
\pi_n(A) = A'[v_{j_1 - 1}v_{j_1 + 1}][\cdots][v_{j_k - 1}v_{j_k + 1}][v_{i_1}][\cdots][v_{i_k}]
\]

Proof. At first, we observe that for each \( i \in \{1, \ldots, n\} \)

\[
N_{P_n}(v_i) = \begin{cases} 
  \{v_2\} & \text{if } i = 1 \\
  \{v_{i-1}\} & \text{if } i = n \\
  \{v_{i-1}, v_{i+1}\} & \text{otherwise}.
\end{cases}
\]

Let \( v_i, v_j \in V \), with \( i < j \). Then \( v_i \equiv_A v_j \) if and only if \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \) or \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \). The proof follows easily by observing that, since \( v_i \neq v_j \), then \( |N_G(v_i) \cap N_G(v_j)| \leq 1 \) and the equality holds if and only if \( j = i + 2 \). It follows that, if \( N_G(v_i) \cap A = N_G(v_j) \cap A \), then \( N_G(v_i) \cap A = \{N_G(v_i) \cap N_G(v_j)\} \cap A \subseteq N_G(v_j) \cap N_G(v_j) \). By Proposition 1, \( v_i \equiv_A v_j \) if and only if \( N_G(v_i) \cap A = N_G(v_j) \cap A \). Thus \( |N_G(v_i) \cap A| = |N_G(v_j) \cap A| \leq 1 \) and the equality holds if and only if \( j = i + 2 \) and \( v_{i+1} = v_{j-1} \in A \). Now, let \( v_i, v_j \in V(G) \), with \( i < j \) and \( v_i \equiv_A v_j \). Then either \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \) or \( N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset \). But the first condition is equivalent to say that \( v_i, v_j \in A' \), whereas the second is equivalent to say that \( v_i, v_j \in A'' \). The proposition is thus proved.

Example 2. Let \( G = P_8 \) be the 8-path on the set \( V = \{v_1, \ldots, v_8\} \). Let \( A = \{v_2, v_4, v_7\} \).

\[
A' = \{v_2, v_4, v_7\}
\]

\[
A'' = \emptyset
\]

\[
A''' = \emptyset
\]

and

\[
A'''' = \{v_1, v_3, v_5, v_6, v_8\}
\]

Thus

\[
\pi_G(A) = v_1[v_3v_5v_6v_8v_2v_4v_7]
\]

4 The \( A \)-upper and the \( A \)-lower Approximations For \( C_n \) and \( P_n \)

In this section, we compute the \( A \)-upper and the \( A \)-lower approximation functions for both the \( n \)-cycle \( C_n \) and the \( n \)-path \( P_n \). Let us note that it is sufficient to treat only the case of \( C_n \). In fact, by Proposition 4, the \( A \)-indiscernibility partitions of \( P_n \) and \( C_n \) have the same structure and the result proved below depends only by the form of the \( A \)-indiscernibility partition. In order to determine the general form of the \( A \)-upper approximation function of \( C_n \), we must examine all possible relations between the vertex subset \( Y \) and the three subsets \( A', A'' \) and \( A''' \). Next, we also show that any possible choice of the vertex subsets \( A \) and \( Y \) is included in the cases we
examined. In what follows we will use the notations introduced in Definition 5.

If A and Y are two vertex subsets of \( C_n \) we set

\[ Q_A(Y) := \bigcup \{ N_{C_n}(v) : v \in A^* \cap N_{C_n}(v) \cap Y \neq \emptyset \}. \]

**Theorem 1.** Let A and Y be two vertex subsets of \( C_n \). The map \( u : (A, Y) \in \mathcal{P}(V(C_n)) \times \mathcal{P}(V(C_n)) \rightarrow u_A(Y) \in \mathcal{P}(V(C_n)) \) is completely described from the cases listed in the following table:

<table>
<thead>
<tr>
<th>CASE</th>
<th>CONDITIONS</th>
<th>( u_A(Y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>( Y = V(C_n) )</td>
<td>( V(C_n) )</td>
</tr>
<tr>
<td>2)</td>
<td>( Y = \emptyset )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>3)</td>
<td>( Y \neq \emptyset \cap Y \subseteq A^* )</td>
<td>( A^* )</td>
</tr>
<tr>
<td>4)</td>
<td>( Y \neq \emptyset \cap Y \subseteq A^* )</td>
<td>( Q_A(Y) )</td>
</tr>
<tr>
<td>5)</td>
<td>( Y \neq \emptyset \cap Y \subseteq A^* )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>6)</td>
<td>( Y \cap A^* \neq \emptyset \cap Y \cap A^* \neq \emptyset \cap Y \cap A^* = \emptyset )</td>
<td>( Q_A(Y) \cup A^* )</td>
</tr>
<tr>
<td>7)</td>
<td>( Y \cap A^* \neq \emptyset \cap Y \cap A^* \neq \emptyset \cap Y \cap A^* \neq \emptyset \cap Y \cap A^* = \emptyset )</td>
<td>( A^* \cup Q_A(Y) )</td>
</tr>
<tr>
<td>8)</td>
<td>( Y \cap A^* \neq \emptyset \cap Y \cap A^* \neq \emptyset \cap Y \cap A^* = \emptyset \cap Y \cap A^* = \emptyset )</td>
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<td>9)</td>
<td>( Y \cap A^* \neq \emptyset \cap Y \cap A^* \neq \emptyset \cap Y \cap A^* = \emptyset \cap Y \cap A^* = \emptyset \cap Y \cap A^* = \emptyset )</td>
<td>( Q_A(Y) )</td>
</tr>
</tbody>
</table>

**Proof.** We recall that for a generic vertex subset \( A \) we have

\[ \pi_{C_n}(A) = A^\prime | \{ v_{j_1-1}, v_{j_1+1}, \ldots, v_{j_{m-1}}, v_{j_m} \} = A'' \]  

where \( \{ v_{j_1-1}, v_{j_1+1}, \ldots, v_{j_{m-1}}, v_{j_m} \} = A'' \), and \( \{ v_{j_1}, \ldots, v_{j_m} \} = A^\prime \).

1): Let \( Y = V(C_n) \) and A any vertex subset of \( V(C_n) \). Obviously \( [v]_A \cap Y \neq \emptyset \) for every vertex \( v \), so \( u_A(V(C_n)) = V(C_n) \).

2): Let \( Y = \emptyset \) and A any vertex subset of \( V(C_n) \). Obviously \( [v]_A \cap \emptyset = \emptyset \) for every vertex \( v \), so \( u_A(\emptyset) = \emptyset \).

3): Let A and Y be two vertex subsets such that \( Y \neq \emptyset \cap Y \subseteq A^* \). This means that the indiscernibility block intersecting \( Y \) is exactly \( A^* \), therefore \( [v]_A \cap Y \neq \emptyset \) if and only if \( v \in A^* \).

4): Let A and Y be two vertex subsets such that \( Y \neq \emptyset \cap Y \subseteq A^* \). Recalling that \( A'' = N_{C_n}(A^* \cap A'' \neq \emptyset \cap Y \cap A'' \neq \emptyset \cap Y \cap A'' = \emptyset \), we deduce that \( Y \cap A'' \) intersects only the neighbourhoods of some points \( v_{j_1}, \ldots, v_{j_m} \in A^*, \) therefore \( Q_A(Y) \cap \emptyset \) and \( u_A(Y) = Q_A(Y) \).

5): Let A and Y be two vertex subsets such that \( Y \neq \emptyset \cap Y \subseteq A'' \). Since the elements of \( A'' \) form single blocks in the A-indiscernibility partition, we have that \( Y \cap [v]_A \neq \emptyset \) if and only if \( v \in A'' \cap Y = Y \). Hence \( u_A(Y) = Y \).

6): Let A and Y be two vertex subsets such that \( Y \cap A' \neq \emptyset \cap Y \cap A'' \neq \emptyset \cap Y \cap A'' = \emptyset \). In other words, Y is transversal only to \( A' \) and \( A'' \); therefore we have that \( [v]_A \cap Y \neq \emptyset \), and only if \( v \in A' \) or \( \exists w \in A'' : v \in N_{C_n}(w) \cap N_{C_n}(w) \subseteq Q_A(Y) \). Thus \( u_A(Y) = A' \cup Q_A(Y) \).

7): Let A and Y be two vertex subsets such that \( Y \cap A' \neq \emptyset \cap Y \cap A'' \neq \emptyset \cap Y \cap A'' = \emptyset \). In this case, Y is transversal only to \( A' \) and \( A'' \), hence \( [v]_A \cap Y \neq \emptyset \) if and only if \( v \in A' \) or \( v \in Y \cap A'' \). Thus \( u_A(Y) = A' \cup (Y \cap A'') \).

8): Let A and Y be two vertex subsets such that \( Y \cap A' \neq \emptyset \cap Y \cap A'' \neq \emptyset \cap Y \cap A'' = \emptyset \). Then \( [v]_A \cap Y \neq \emptyset \) if and only if \( v \in Y \cap A'' \) or \( \exists w \in A'' : v \in N_{C_n}(w) \cap N_{C_n}(w) \subseteq Q_A(Y) \), since Y is transversal to both \( A' \) and \( A'' \). So, we conclude that \( u_A(Y) = Q_A(Y) \cap (Y \cap A'') = Q_A(Y) \cup Y \).

9): Let A and Y be two vertex subsets such that \( Y \cap A' \neq \emptyset \cap Y \cap A'' \neq \emptyset \cap Y \cap A'' = \emptyset \). It means that Y is transversal to the three sets. Therefore we have \( [v]_A \cap Y \neq \emptyset \), and only if \( v \in A' \) or \( v \in Y \cap A'' \) or \( \exists w \in A'' : v \in N_{C_n}(w) \cap N_{C_n}(w) \subseteq Q_A(Y) \). Therefore, \( u_A(Y) = A' \cup Q_A(Y) \cap (Y \cap A'') = Y \cup A' \cup Q_A(Y) \) and we are done.

At this point, we prove that the previous cases are all disjoint each other and they are all possible cases that can occur. Let Y be a proper vertex subset of \( V = V(C_n) \), then \( V(C_n) = A' \cup A'' \cup A''' \), we deduce that Y can be a subset of one of these three sets, as we have said writing down the conditions 3, 4 and 5), or it can be transversal to two of them, without containing none of them and without intersecting the third, as we have said writing down the conditions 6, 7 and 8). Finally, Y can be transversal to every set, without containing none of them, as written in the last condition. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. In this way we have shown the theorem.

Let us compute now the A-indiscernibility approximation function for the n-cycle \( C_n \). By Proposition 4 the A-indiscernibility partitions of \( P_n \) and \( C_n \) have the same structure and the next result depends only from this partition structure.

Also to compute the A-indiscernibility function of \( C_n \), we will use the previous proof technique, namely we study all possible relations between the vertex subset \( Y \) and the three subsets \( A', A'' \) and \( A''' \) determining the A-indiscernibility partition. In what follows we will use the notations introduced in Definition 5.

If A and Y are two vertex subsets of \( C_n \) we set

\[ T_A(Y) := \bigcup \{ N_{C_n}(v) : N_{C_n}(v) \cap Y \neq \emptyset \}. \]

**Theorem 2.** Let A and Y be two vertex subsets of \( C_n \). The map \( I_A(Y) : (A, Y) \in \mathcal{P}(V(C_n)) \times \mathcal{P}(V(C_n)) \rightarrow I_A(Y) \in \mathcal{P}(V(C_n)) \) is completely described from the cases listed in the following table:

<table>
<thead>
<tr>
<th>CASE</th>
<th>CONDITIONS</th>
<th>( I_A(Y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>( Y = V(C_n) )</td>
<td>( V(C_n) )</td>
</tr>
<tr>
<td>2)</td>
<td>( Y \subseteq A' )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>3)</td>
<td>( Y \subseteq A'' )</td>
<td>( A' \cap A'' )</td>
</tr>
<tr>
<td>4)</td>
<td>( Y \subseteq A' \cap T_A(Y) )</td>
<td>( A' \cap A'' \cup T_A(Y) )</td>
</tr>
<tr>
<td>5)</td>
<td>( Y \subseteq A'' \cap T_A(Y) )</td>
<td>( A' \cap A'' \cup T_A(Y) )</td>
</tr>
<tr>
<td>6)</td>
<td>( Y \cap A' \neq \emptyset \cap Y \cap A'' \neq \emptyset \cap Y \cap A'' = \emptyset )</td>
<td>( Y \cap A' )</td>
</tr>
<tr>
<td>7)</td>
<td>( Y \cap A' \neq \emptyset \cap Y \cap A'' \neq \emptyset \cap Y \cap A'' = \emptyset )</td>
<td>( Y \cap A'' \cup T_A(Y) )</td>
</tr>
</tbody>
</table>

**Proof.** We recall that by definition \( I_A(Y) = \{ v \in V(C_n) : [v]_A \subseteq Y \} \). We also recall that, by
Proposition 3, if $A$ is a generic vertex subset, we have

$$\pi_{A}^{c}(A) = A'\left|v_{j_{1}} - 1v_{j_{i}} + 1\ldots v_{j_{h}} - 1v_{j_{i}} + 1\ldots v_{j_{l}}\right|,$$

where $\{v_{j_{1}},\ldots,v_{j_{l}}\} = A'$ and $\{v_{j_{1}},\ldots,v_{j_{i}}\} = A''$.

1): Let $Y = V(C_{n})$ and $A$ be any vertex subset of $V(C_{n})$. Obviously $[v]_{A} \subseteq V(C_{n})$ for every vertex $v$, so $l_{A}(V(C_{n})) = V(C_{n})$.

2): Let $Y = \emptyset$ and $A$ be any vertex subset of $V(C_{n})$. Obviously no indiscernibility class is contained in the empty set, so $l_{A}(\emptyset) = \emptyset$.

3): Let $A$ and $Y$ be two vertex subsets such that $Y \subseteq A'$. Since $A'$ forms a single block and since $Y$ is disjoint from $A''$ and $A'''$, we deduce that there is no vertex whose indiscernibility class is contained in $Y$, thus $l_{A}(Y) = \emptyset$.

4): Let $A$ and $Y$ be two vertex subsets such that $Y \supseteq A'$ and $T_{A}(Y) = \emptyset$. We observe that $Y$ may or not intersect the vertex subset $A''$ and, in the first case, in such a way that there not exists any vertex $w \in A'$ whose neighbourhood is contained in $Y$. Furthermore, $Y$ may or not intersect $A'''$. Thus we conclude that $[v]_{A} \subseteq Y$ if and only if $v \in A'$ or, possibly, $v \in Y \cap A''$. Hence $l_{A}(Y) = A' \cup (Y \cap A'')$.

5): Let $A$ and $Y$ be two vertex subsets such that $Y \supseteq A'$ and $T_{A}(Y) \neq \emptyset$. We observe that $Y$ may or not intersect the vertex subset $A'''$. Moreover, there exists one vertex $w \in A'$ whose neighbourhood is contained in $Y$. Hence we conclude that $[v]_{A} \subseteq Y$ if and only if $v \in A'$ or $v \in N_{C_{n}}(w) \subseteq T_{A}(Y)$ for some $w \in A'$ or, possibly, $v \in Y \cap A''$. Hence $l_{A}(Y) = A' \cap T_{A}(Y) \cap (Y \cap A'')$.

6): Let $A$ and $Y$ be two vertex subsets such that $Y \cap A' \neq \emptyset$, then $Y \cap A'' \subseteq Y$. Since $A'$ forms a single block, we conclude that $A' \subseteq l_{A}(Y)$. We also observe that $Y$ may or not intersect the vertex subset $A'''$. Moreover, there is no vertex $w \in A''$ such that $v \in N_{C_{n}}(w)$ and $N_{C_{n}}(w) \subseteq Y$. This means that $[v]_{A} \subseteq Y$ if and only if $v \in Y \cap A''$. Hence $l_{A}(Y) = Y \cap A''$.

7): Let $A$ and $Y$ be two vertex subsets such that $Y \cap A' \neq \emptyset$ and $Y \cap T_{A}(Y) \neq \emptyset$. Since $A'$ forms a single block, we conclude that $A' \subseteq l_{A}(Y)$. We also observe that $Y$ may or not intersect the vertex subset $A'''$. Furthermore, there exists at least a vertex $w \in A''$ such that $v \in N_{C_{n}}(w)$ and $N_{C_{n}}(w) \subseteq Y$. In other words, we are saying that $l_{A}(Y) = Y \cap A''$. At this point, we show that we have studied all the occurring cases. Let $Y$ be a proper vertex subset of $V = V(C_{n})$. In case 3) we have that $Y \supseteq A'$ while in cases 4) and 5) we are requiring that $Y$ contains $A'$ and it may (or not) be transversal to both $A''$ and $A'''$. Finally, $Y$ may only transversal to $A'$, without containing it, as we have seen in the last two cases. So, the cases discussed above are disjoint one another and, above all, describe all possible occurring situations. Hence, the theorem is proved.

5 A-exact subsets of $C_{n}$ and $P_{n}$

Let $A$ be a vertex subset of a given graph. By Definition 4, we recall that a vertex subset $Y$ is $A$-exact if and only if its $A$-lower approximation of $Y$ coincides with its $A$-upper approximation. In this section we determine the general form of all $A$-exact subsets of $C_{n}$. Also in this case we will obtain an identical form also for the $A$-exact subsets of $P_{n}$, because both the $A$-lower approximation function and the $A$-upper approximation function are identical for $C_{n}$ and $P_{n}$. As in the previous sections, in what follows we will use the notations introduced in Definition 5.

Proposition 5 Let $A$ and $Y$ be two vertex subsets of $V(C_{n})$. Then $Y$ is $A$-exact if and only if one of the following cases holds:

**Proof.** By Theorems 1 and 2, it’s easy to show that if $A$ and $Y$ satisfy one of the conditions of the previous table, then $Y$ is $A$-exact. Viceversa, let $A$ and $Y$ be two vertex subsets different from the previous. We will examine all possible cases.

If $A$ and $Y$ are two vertex subsets such that $Y \subseteq A'$, we deduce that $l_{A}(Y) = \emptyset \neq A' = u_{A}(Y)$, therefore $Y$ can’t be $A$-exact.

Suppose that $Y \cap A' \neq \emptyset$ and $Y \cap A'' \neq \emptyset$. In order to have $l_{A}(Y) = u_{A}(Y)$, it must result $Y \cap A'' = (Y \cap A'' \cup A')$. In other words, we must require that $A' \subseteq Y \cap A''$ or $A' = \emptyset$. Both these conclusions are false, since $A' \not\subseteq A''$ and since the condition $Y \cap A' \neq \emptyset$ ensures that $A' \neq \emptyset$.

In the other cases to analyze, we always have that

$$Y \cap A'' \neq \emptyset \cap \bigcup_{j=1}^{k} N_{C_{n}}(v_{j}).$$

It’s easy to see that whenever (5) holds, it results that $T_{A}(Y) \subseteq g_{A}(Y)$. This means that the $A$-lower approximation and the $A$-upper approximation of $Y$ must differ each other.

In this way, we have shown that in all possible situations different from those listed in the above table $l_{A}(Y) \neq u_{A}(Y)$, thus we have completed the proof.

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6 Conclusions

This paper is a further contribution to the interpretation of a simple undirected graph in terms of Boolean information table. We have completely determined both the A-lower and A-upper approximation functions for the $n$-cycles and the $n$-paths. We have shown that the complete study of these functions can be very laborious, in spite the simplicity of examined graph structure. Our study is part of a research project started in [22, 23] and it will be further developed in forthcoming papers relatively to others graph families and by means of others RST investigation tools.

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