Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100318

Some Integrals Involving Generalized Hypergeometric functions and Srivastava polynomials

Praveen Agarwal[1,*], *Jaekeun Park*[2], *Mehar Chand*[3] *and Shilpi Jain*[4]

[1] Department of Mathematics, Anand International College of Engineering, Jaipur-303012, India
[2] Department of Mathematics, Hanseo University, Chungnam-do, Seosan-si 356-706, Republic of Korea
[3] Department of Mathematics, Fateh College for Women, Bathinda-151001, India
[4] Department of Mathematics, Poornima College of Engineering, Jaipur, India

Received: 17 Oct. 2015, Revised: 17 Jan. 2016, Accepted: 18 Jan. 2016
Published online: 1 May 2016

Abstract: We aim to establish certain (presumably) new and (potentially) useful integral results involving the generalized Gauss hypergeometric function and the Srivastava polynomial. Next, we obtain certain new integrals and expansion formulas by the application of our theorems. Some interesting special cases of our main result are also considered and shown to be connected with certain known ones.

Keywords: Special function, generalized Gauss hypergeomtric functions, Srivastava polynomials.

1 Introduction and definitions

Recently, Özergin *et al.* [6] introduced and studied some fundamental properties and characteristics of the generalized Beta type function $B_p^{(\alpha,\beta)}(x,y)$ in their paper and defined by (see, *e.g.*, [6, p. 4602, Eq.(4)]; see also, [5, p.32, Chapter 4.]):

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{1}$$

$$(\Re(p)\geq 0; \min(\Re(x),\Re(y),\Re(\alpha),\Re(\beta))>0 \text{ and } B_0^{(\alpha,\beta)}(x,y)=B(x,y)),$$

where $B(x,y)$ is a well known Euler's Beta function defined by:

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt \ (\Re(x)>0, \Re(y)>0). \tag{2}$$

Along with, generalized Beta function (1), Özergin *et al.* introduced and studied a family of the following potentially useful generalized Gauss hypergeometric functions defined as follows (see, *e.g.*, [6, p. 4606, Section 3.]; see also, [5, p.39, Chapter 4.]):

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty} a_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (|z|<1), \tag{3}$$

where $\min(\Re(\alpha),\Re(\beta))>0; \Re(c)>\Re(b)>0$ and $\Re(p)\geq 0$.

Indeed, in their special case when $p=0$, the function $F_p^{(\alpha,\beta)}(a,b;c;z)$ would reduce immediately to the extensively-investigated Gauss hypergeometric function ${}_2F_1(.)$. The ${}_2F_1(.)$ is special case of the well known generalized hypergeometric series ${}_pF_q(.)$ defined by:

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!} = {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z), \tag{4}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda\in\mathbb{C}$) by:

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbb{N}) \end{cases}$$

* Corresponding author e-mail: goyal.praveen2011@gmail.com

$$= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{5}$$

and $\mathbb{Z}_0^-$ denotes the set of Non-positive integers.

The concept of the Hadamard is very useful in our investigation. Let us consider the function ${}_pF_{q+r}^{(\alpha,\beta;\rho,\lambda)}[z;b]$. Its decomposition is illustrative. That is [4, p. 633]:

$$\begin{aligned} {}_pF_{q+r}^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\dots,x_p \\ y_1,\dots,y_{q+r} \end{matrix};z;b\right] &= {}_1F_r\left[\begin{matrix} 1 \\ y_1,\dots,y_r \end{matrix};z;b\right] \\ &* {}_pF_q^{(\alpha,\beta;\rho,\lambda)}\left[\begin{matrix} x_1,\dots,x_p \\ y_{1+r},\dots,y_{q+r} \end{matrix};z;b\right] \end{aligned} \tag{6}$$

In 1972, Srivastava [9] introduce the following family of polynomials:

$$S_n^m(x) := \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; m \in \mathbb{N}), \tag{7}$$

where $\mathbb{N}$ is the set of positive integers, the coefficients $A_{n,k}(n,k \geq 0)$ are arbitrary constants, real or complex. $S_n^m(x)$ yields a number of known polynomials as its special cases. These includes, among other, the Jacobi polynomials, the Bessel Polynomials, the Lagurre Polynomials, the Brafman Polynomials and several others [10, p. 158-161].

The following formulas [7, p. 77, Eqn. 3.1, 3.2 and 3.3] will be required in our investigation:

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1} dx = \frac{\sqrt{\pi}}{2a(4ab+c)^{p+1/2}} \frac{\Gamma(p+1/2)}{\Gamma(p+1)}, \quad (a>0; b\geq 0; c+4ab>0; \Re(p)+1/2>0). \tag{8}$$

$$\int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1} dx = \frac{\sqrt{\pi}}{2b(4ab+c)^{p+1/2}} \frac{\Gamma(p+1/2)}{\Gamma(p+1)}, \quad (a\geq 0; b>0; c+4ab>0; \Re(p)+1/2>0) \tag{9}$$

$$\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1} dx = \frac{\sqrt{\pi}}{(4ab+c)^{p+1/2}} \frac{\Gamma(p+1/2)}{\Gamma(p+1)}, \quad (a>0; b>0; c+4ab>0; \Re(p)+1/2>0). \tag{10}$$

2 Main Results

The following Orr's relation connecting products of hypergeometric series is also needed (see, *e.g.*, [8, p. 75]): If

$$(1-y)^{\alpha+\beta-\gamma}{}_2F_1(2\alpha,2\beta;2\gamma;y) = \sum_{k=0}^{\infty} A_k y^k,$$

then

$${}_2F_1\left(\alpha,\beta;\gamma+\frac{1}{2};y\right){}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma+\frac{1}{2};y\right) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\left(\gamma+\frac{1}{2}\right)_k} A_k y^k. \tag{11}$$

Theorem 1. *Let $a>0,\ b\geq 0;\ c+4ab>0;\ \mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0,\ -\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2},\ m,r\in\mathbb{N}$ and coefficients $a_r, A_{n,l}, (n,l\in\mathbb{N}_0)$ are arbitrary (real or complex) constants. Then we have*

$$\begin{aligned} &\int_0^\infty X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X) \\ &\quad\times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X) \\ &\quad\times S_n^m\left[yX^{-\mu}\right] F_p^{(\sigma,\rho)}(a,b;c;t/X)dx \\ &= \frac{\sqrt{\pi}}{2a(4ab+c)^{\lambda+1/2}} \sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]} a_r \frac{(\gamma)_r}{(\gamma+1/2)_r} \\ &\quad\times \frac{(-n)_{ml}}{l!} A_{n,l} y^l \frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} (4ab+c)^{r+\mu l} \\ &\quad\times {}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2; c,\lambda-r-\mu l+1; \frac{t}{4ab+c}\right]. \end{aligned} \tag{12}$$

Theorem 2. *Let $a>0,\ b\geq 0;\ c+4ab>0;\ \mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0,\ -\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2},\ m,r\in\mathbb{N}$ and coefficients $a_r, A_{n,l}, (n,l\in\mathbb{N}_0)$ are arbitrary (real or complex) constants. Then we have*

$$\begin{aligned} &\int_0^\infty \frac{1}{x^2} X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X) \\ &\quad\times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X) \\ &\quad\times S_n^m\left[yX^{-\mu}\right] F_p^{(\sigma,\rho)}(a,b;c;t/X)dx \\ &= \frac{\sqrt{\pi}}{2b(4ab+c)^{\lambda+1/2}} \sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]} a_r \frac{(\gamma)_r}{(\gamma+1/2)_r} \\ &\quad\times \frac{(-n)_{ml}}{l!} A_{n,l} y^l \frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} (4ab+c)^{r+\mu l} \\ &\quad\times {}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2; c,\lambda-r-\mu l+1; \frac{t}{4ab+c}\right]. \end{aligned} \tag{13}$$

Theorem 3. *Let $a > 0,\ b \geq 0;\ c + 4ab > 0;\ \mu, \lambda \in \mathbb{C}$, $\Re(\lambda) + 1/2 > 0,\ -\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2},\ m, r \in \mathbb{N}$ and coefficients $a_r, A_{n,l}, (n, l \in \mathbb{N}_0)$ are arbitrary (real or complex) constants. Then we have*

$$\int_0^\infty \left(a + \frac{b}{x^2}\right) X^{-\lambda-1} {}_2F_1(\alpha, \beta; \gamma + 1/2; X) \times {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma + 1/2; X) \times S_n^m\left[yX^{-\mu}\right] F_p^{(\sigma,\rho)}(a, b; c; t/X) dx$$
$$= \frac{\sqrt{\pi}}{(4ab + c)^{\lambda+1/2}} \sum_{r=0}^{\infty} \sum_{l=0}^{[n/m]} a_r \frac{(\gamma)_r}{(\gamma + 1/2)_r} \times \frac{(-n)_{ml}}{l!} A_{n,l} y^l \frac{\Gamma(\lambda - r - \mu l + 1/2)}{\Gamma(\lambda - r - \mu l + 1)} (4ab + c)^{r+\mu l} \times {}_1F_{p,1}^{(\sigma,\rho)}\left[a, b, \lambda - r - \mu l + 1/2; c, \lambda - r - \mu l + 1; \frac{t}{4ab + c}\right]. \tag{14}$$

Proof: To prove the Theorem 1, first using the result given by equation (11) and express Srivastava polynomials $S_n^m(x)$ in series form with the help of equation (7) and generalized Gauss hyper geometric function given by equation(3), then interchanging the order of integration and summation we get

$$\int_0^\infty X^{-\lambda-1} {}_2F_1(\alpha, \beta; \gamma + 1/2; X) \times {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma + 1/2; X) \times S_n^m\left[yX^{-\mu}\right] F_p^{(\sigma,\rho)}(a, b; c; tX^{-1}) dx$$
$$= \sum_{r=0}^{\infty} \sum_{l=0}^{[n/m]} a_r \frac{(\gamma)_r}{(\gamma + 1/2)_r} \frac{(-n)_{ml}}{l!} A_{n,l} y^l \sum_{k=0}^{\infty} \frac{a_k B_p^{(\sigma,\rho)}(b + k, c - b)}{B(b, c - b)} \frac{t^k}{k!} \int_0^\infty X^{-\lambda+r+\mu l-k-1} dx, \tag{15}$$

then using the formula given in equation (8), the above equation (15) reduced to the following form

$$= \sum_{r=0}^{\infty} \sum_{l=0}^{[n/m]} a_r \frac{(\gamma)_r}{(\gamma + 1/2)_r} \frac{(-n)_{ml}}{l!} A_{n,l} y^l \times \sum_{k=0}^{\infty} \frac{a_k B_p^{(\sigma,\rho)}(b + k, c - b)}{B(b, c - b)} \frac{t^k}{k!} \times \frac{\sqrt{\pi}}{2a(4ab + c)^{\lambda-r-\mu l+k+1/2}} \times \frac{\Gamma(\lambda - r - \mu l + k + 1/2)}{\Gamma(\lambda - r - \mu l + k + 1)}$$

$$= \frac{\sqrt{\pi}}{2a(4ab + c)^{\lambda+1/2}} \sum_{r=0}^{\infty} \sum_{l=0}^{[n/m]} a_r \frac{(\gamma)_r}{(\gamma + 1/2)_r} \times \frac{(-n)_{ml}}{l!} A_{n,l} y^l \frac{\Gamma(\lambda - r - \mu l + 1/2)}{\Gamma(\lambda - r - \mu l + 1)} \times (4ab + c)^{r+\mu l} \sum_{k=0}^{\infty} \frac{a_k B_p^{(\sigma,\rho)}(b + k, c - b)}{B(b, c - b)} \times \frac{(\lambda - r - \mu l + 1/2)_k}{(\lambda - r - \mu l + 1)_k} \left[\frac{t}{4ab + c}\right]^k \frac{1}{k!}$$

$$= \frac{\sqrt{\pi}}{2a(4ab + c)^{\lambda+1/2}} \sum_{r=0}^{\infty} \sum_{l=0}^{[n/m]} a_r \frac{(\gamma)_r}{(\gamma + 1/2)_r} \frac{(-n)_{ml}}{l!} A_{n,l} y^l \times \frac{\Gamma(\lambda - r - \mu l + 1/2)}{\Gamma(\lambda - r - \mu l + 1)} (4ab + c)^{r+\mu l} \times F_p^{(\sigma,\rho)}\left[a, b; c; \frac{t}{4ab + c}\right] * {}_1F_1\left[\begin{matrix}\lambda - r - \mu l + 1/2 \\ \lambda - r - \mu l + 1\end{matrix}\right]. \tag{16}$$

Applying the concept of Hadamard given by equation(6) in the above equation (16), we have the required result (12). Proceeding on same parallel lines, theorems second and third given by equations (13) and (14) can be obtained by using the results(9) and(10) respectively.

3 Special Cases and Applications

We conclude present investigation by remarking that the integral formulas used in Theorem 2.1 to 2.3 are unified in nature. Moreover, the integrals involving the generalized Gauss hypergeometric function and the Srivastava polynomial in Theorem 2.1 to 2.3 reduce to numbers of integrals involving a large spectrum of well known special functions functions. Thus, we can further obtain various integral formulas involving a number of simpler special functions. In addition, the generalized Gauss hypergeometric function *i. e.* $F_p^{(\alpha,\beta)}(a, b; c; z)$ and the Srivastava polynomial $S_n^m(x)$ occurring in Theorems 2.1 to 2.3 can be suitably specialized to a extremely wide variety of useful functions which are expressible in terms of the Hermite polynomials and Lagurre polynomials function respectively.

For example:

1. By applying our results given in(12),(13) and(14) to the case of Hermite polynomials [11,12] by setting $S_n^2(x) \to x^{n/2} H_n\left[\frac{1}{2\sqrt{x}}\right]$ in which $m = 2, A_{n,l} = (-1)^l$, we have the following results:

Corollary 1. *Let $a > 0,\ b \geq 0;\ c + 4ab > 0;\ \mu, \lambda \in \mathbb{C}$, $\Re(\lambda) + 1/2 > 0,\ -\frac{1}{2} < \alpha - \beta - \gamma < \frac{1}{2},\ r \in \mathbb{N}$ and* *coefficients a_r is arbitrary (real or complex) constant. Then we have*

$$\int_0^\infty X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)$$
$$\times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\left[yX^{-\mu}\right]^{n/2}$$
$$\times H_n\left[\frac{1}{2\sqrt{yX^{-\mu}}}\right]F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$=\frac{\sqrt{\pi}}{2a(4ab+c)^{\lambda+1/2}}\sum_{r=0}^{\infty}\sum_{l=0}^{[n/2]}a_r\frac{(\gamma)_r}{(\gamma+1/2)_r} \tag{17}$$
$$\times\frac{(-n)_{2l}}{l!}(-y)^l\frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)}(4ab+c)^{r+\mu l}$$
$$\times {}_1F_{p,1}^{(\sigma,\rho)}[a,b,\lambda-r-\mu l+1/2;$$
$$\times c,\lambda-r-\mu l+1;\frac{t}{4ab+c}\Big].$$

Corollary 2.*Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $r\in\mathbb{N}$ and coefficients a_r is arbitrary (real or complex) constant. Then we have*

$$\int_0^\infty \frac{1}{x^2}X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)$$
$$\times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\left[yX^{-\mu}\right]^{n/2}$$
$$\times H_n\left[\frac{1}{2\sqrt{yX^{-\mu}}}\right]F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$=\frac{\sqrt{\pi}}{2b(4ab+c)^{\lambda+1/2}}\sum_{r=0}^{\infty}\sum_{l=0}^{[n/2]}a_r\frac{(\gamma)_r}{(\gamma+1/2)_r} \tag{18}$$
$$\times\frac{(-n)_{2l}}{l!}(-y)^l\frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)}(4ab+c)^{r+\mu l}$$
$$\times {}_1F_{p,1}^{(\sigma,\rho)}[a,b,\lambda-r-\mu l+1/2;$$
$$c,\lambda-r-\mu l+1;\frac{t}{4ab+c}\Big].$$

Corollary 3.*Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $r\in\mathbb{N}$ and coefficients a_r is arbitrary (real or complex) constants. Then we have*

$$\int_0^\infty\left(a+\frac{b}{x^2}\right)X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)$$
$$\times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)\left[yX^{-\mu}\right]^{n/2}$$
$$\times H_n\left[\frac{1}{2\sqrt{yX^{-\mu}}}\right]F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$=\frac{\sqrt{\pi}}{(4ab+c)^{\lambda+1/2}}\sum_{r=0}^{\infty}\sum_{l=0}^{[n/2]}a_r\frac{(\gamma)_r}{(\gamma+1/2)_r} \tag{19}$$
$$\times\frac{(-n)_{2l}}{l!}(-y)^l\frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)}$$
$$\times(4ab+c)^{r+\mu l}$$
$$\times {}_1F_{p,1}^{(\sigma,\rho)}[a,b,\lambda-r-\mu l+1/2;$$
$$c,\lambda-r-\mu l+1;\frac{t}{4ab+c}\Big].$$

2.By applying our results given in (12),(13) and(14) to the case of Lagurre polynomials [11,12] by setting $S_n^2(x)\rightarrow L_n^{(\alpha')}[x]$ in which $m=2, A_{n,l}=\binom{n+\alpha'}{n}\frac{1}{\alpha'+1}$, we have the following results:

Corollary 4.*Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $r\in\mathbb{N}$ and coefficients a_r is arbitrary (real or complex) constant. Then we have*

$$\int_0^\infty X^{-\lambda-1}{}_2F_1(\alpha,\beta;\gamma+1/2;X)$$
$$\times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X)L_n^{(\alpha')}\left[yX^{-\mu}\right]$$
$$\times F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$=\frac{\sqrt{\pi}}{2a(4ab+c)^{\lambda+1/2}}\sum_{r=0}^{\infty}\sum_{l=0}^{[n/2]}a_r\frac{(\gamma)_r}{(\gamma+1/2)_r}$$
$$\times\frac{(-n)_{2l}}{l!}\binom{n+\alpha'}{n}\frac{y^l}{\alpha'+1}\frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} \tag{20}$$
$$\times(4ab+c)^{r+\mu l}$$
$$\times {}_1F_{p,1}^{(\sigma,\rho)}[a,b,\lambda-r-\mu l+1/2;$$
$$c,\lambda-r-\mu l+1;\frac{t}{4ab+c}\Big].$$

Corollary 5.*Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $r\in\mathbb{N}$ and coefficients a_r is arbitrary (real or complex) constant. Then we have*

$$\int_0^\infty \frac{1}{x^2} X^{-\lambda-1} {}_2F_1(\alpha,\beta;\gamma+1/2;X) \times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X) L_n^{(\alpha')}\left[yX^{-\mu}\right] \times F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$= \frac{\sqrt{\pi}}{2b(4ab+c)^{\lambda+1/2}} \sum_{r=0}^{\infty}\sum_{l=0}^{[n/2]} a_r \frac{(\gamma)_r}{(\gamma+1/2)_r} \frac{(-n)_{2l}}{l!} \times \binom{n+\alpha'}{n} \frac{y^l}{\alpha'+1} \frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} \times (4ab+c)^{r+\mu l} {}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2; c,\lambda-r-\mu l+1; \frac{t}{4ab+c}\right]. \tag{21}$$

Corollary 6. *Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $r\in\mathbb{N}$ and coefficients a_r is arbitrary (real or complex) constant. Then we have*

$$\int_0^\infty \left(a+\frac{b}{x^2}\right) X^{-\lambda-1} {}_2F_1(\alpha,\beta;\gamma+1/2;X) \times {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma+1/2;X) L_n^{(\alpha')}\left[yX^{-\mu}\right] \times F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$= \frac{\sqrt{\pi}}{(4ab+c)^{\lambda+1/2}} \sum_{r=0}^{\infty}\sum_{l=0}^{[n/2]} a_r \frac{(\gamma)_r}{(\gamma+1/2)_r} \frac{(-n)_{2l}}{l!} \times \binom{n+\alpha'}{n} \frac{y^l}{\alpha'+1} \frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} \times (4ab+c)^{r+\mu l} {}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2; c,\lambda-r-\mu l+1; \frac{t}{4ab+c}\right]. \tag{22}$$

3. If we put $\alpha=\gamma$, in the main theorem,the value of a_r comes out to be equal to $\frac{\beta_r}{r!}$ and the result(12),(13) and(14) gives the following results:

Corollary 7. *Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $m,r\in\mathbb{N}$ and coefficients $A_{n,l},(n,l\in\mathbb{N}_0)$ is arbitrary (real or complex) constants. Then we have*

$$\int_0^\infty X^{-\lambda-1} {}_2F_1(\alpha,\beta;\gamma+1/2;X) \times S_n^m\left[yX^{-\mu}\right] F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$= \frac{\sqrt{\pi}}{2a(4ab+c)^{\lambda+1/2}} \sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]} \frac{(\alpha)_r(\beta)_r}{(\alpha+1/2)_r r!} \times \frac{(-n)_{ml}}{l!} A_{n,l} y^l \frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} \times (4ab+c)^{r+\mu l} {}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2; c,\lambda-r-\mu l+1; \frac{t}{4ab+c}\right]. \tag{23}$$

Corollary 8. *Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $m,r\in\mathbb{N}$ and coefficients $A_{n,l},(n,l\in\mathbb{N}_0)$ is arbitrary (real or complex) constants. Then we have*

$$\int_0^\infty \frac{1}{x^2} X^{-\lambda-1} {}_2F_1(\alpha,\beta;\gamma+1/2;X) \times S_n^m\left[yX^{-\mu}\right] F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$= \frac{\sqrt{\pi}}{2b(4ab+c)^{\lambda+1/2}} \sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]} \frac{(\alpha)_r(\beta)_r}{(\alpha+1/2)_r r!} \times \frac{(-n)_{ml}}{l!} A_{n,l} y^l \frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} \times (4ab+c)^{r+\mu l} {}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2; c,\lambda-r-\mu l+1; \frac{t}{4ab+c}\right]. \tag{24}$$

Corollary 9. *Let $a>0$, $b\geq 0$; $c+4ab>0$; $\mu,\lambda\in\mathbb{C}$, $\Re(\lambda)+1/2>0$, $-\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2}$, $m,r\in\mathbb{N}$ and coefficients $A_{n,l},(n,l\in\mathbb{N}_0)$ is arbitrary (real or complex) constants. Then we have*

$$\int_0^\infty \left(a+\frac{b}{x^2}\right) X^{-\lambda-1} {}_2F_1(\alpha,\beta;\gamma+1/2;X) \times S_n^m\left[yX^{-\mu}\right] F_p^{(\sigma,\rho)}(a,b;c;t/X)dx$$
$$= \frac{\sqrt{\pi}}{(4ab+c)^{\lambda+1/2}} \sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]} \frac{(\alpha)_r(\beta)_r}{(\alpha+1/2)_r r!} \times \frac{(-n)_{ml}}{l!} A_{n,l} y^l \frac{\Gamma(\lambda-r-\mu l+1/2)}{\Gamma(\lambda-r-\mu l+1)} \times (4ab+c)^{r+\mu l} {}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2; c,\lambda-r-\mu l+1; \frac{t}{4ab+c}\right]. \tag{25}$$

4. If we put $\beta=\alpha+\frac{1}{2}$ and $\alpha=-f$ (f is non negative integer) in (23), (24) and (25), we have:

Corollary 10. *Let $a>0,\ b\geq 0;\ c+4ab>0;\ \mu,\lambda\in\mathbb{C},\ \Re(\lambda)+1/2>0,\ -\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2},\ m,r\in\mathbb{N}$ and coefficients $A_{n,l},(n,l\in\mathbb{N}_0)$ is arbitrary (real or complex) constants. Then we have*

$$\begin{aligned}&\int_0^\infty X^{-\lambda-1}(1-X)^f S_n^m\left[yX^{-\mu}\right]\\&\qquad\times F_p^{(\sigma,\rho)}(a,b;c;t/X)dx\\&=\frac{\sqrt{\pi}}{2a\,(4ab+c)^{\lambda+1/2}}\sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]}\frac{(-f)_r}{r!}\frac{(-n)_{ml}}{l!}A_{n,l}y^l\\&\times\frac{\Gamma\,(\lambda-r-\mu l+1/2)}{\Gamma\,(\lambda-r-\mu l+1)}(4ab+c)^{r+\mu l}\\&\quad\times{}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2;\right.\\&\qquad\left.c,\lambda-r-\mu l+1;\frac{t}{4ab+c}\right].\end{aligned}\tag{26}$$

Corollary 11. *Let $a>0,\ b\geq 0;\ c+4ab>0;\ \mu,\lambda\in\mathbb{C},\ \Re(\lambda)+1/2>0,\ -\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2},\ m,r\in\mathbb{N}$ and coefficients $A_{n,l},(n,l\in\mathbb{N}_0)$ is arbitrary (real or complex) constants. Then we have*

$$\begin{aligned}&\int_0^\infty \frac{1}{x^2}X^{-\lambda-1}(1-X)^f S_n^m\left[yX^{-\mu}\right]F_p^{(\sigma,\rho)}(a,b;c;t/X)dx\\&=\frac{\sqrt{\pi}}{2b\,(4ab+c)^{\lambda+1/2}}\sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]}\frac{(-f)_r}{r!}\frac{(-n)_{ml}}{l!}A_{n,l}y^l\\&\quad\times\frac{\Gamma\,(\lambda-r-\mu l+1/2)}{\Gamma\,(\lambda-r-\mu l+1)}(4ab+c)^{r+\mu l}\\&\quad\times{}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2;\right.\\&\qquad\left.c,\lambda-r-\mu l+1;\frac{t}{4ab+c}\right].\end{aligned}\tag{27}$$

Corollary 12. *Let $a>0,\ b\geq 0;\ c+4ab>0;\ \mu,\lambda\in\mathbb{C},\ \Re(\lambda)+1/2>0,\ -\frac{1}{2}<\alpha-\beta-\gamma<\frac{1}{2},\ m,r\in\mathbb{N}$ and coefficients $A_{n,l},(n,l\in\mathbb{N}_0)$ is arbitrary (real or complex) constants. Then we have*

$$\begin{aligned}&\int_0^\infty\left(a+\frac{b}{x^2}\right)X^{-\lambda-1}(1-X)^f S_n^m\left[yX^{-\mu}\right]\\&\times F_p^{(\sigma,\rho)}(a,b;c;t/X)dx\\&=\frac{\sqrt{\pi}}{(4ab+c)^{\lambda+1/2}}\sum_{r=0}^{\infty}\sum_{l=0}^{[n/m]}\frac{(-f)_r}{r!}\frac{(-n)_{ml}}{l!}A_{n,l}y^l\\&\times\frac{\Gamma\,(\lambda-r-\mu l+1/2)}{\Gamma\,(\lambda-r-\mu l+1)}(4ab+c)^{r+\mu l}\\&\quad\times{}_1F_{p,1}^{(\sigma,\rho)}\left[a,b,\lambda-r-\mu l+1/2;\right.\\&\qquad\left.c,\lambda-r-\mu l+1;\frac{t}{4ab+c}\right].\end{aligned}\tag{28}$$

Furthermore, if we put $p=0$, all the results established in Section 2 gives new formulas involving ${}_2F_1(.)$, which is special case of generalized hypergeometric function and by changing the parameters suitably, the results in equations (12), (13) and (14) can be reduced to the work of Agarwal [1], Agarwal and chand [2] and Chand [3], respectively.

4 Conclusion

Finally, it is noted that the results derived in this paper are general in character and give some contributions to the theory of integral equations and Special functions. Therefore, the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable para-metric replacements. We are also trying to find certain possible applications of those results presented here to some other research areas like random walk and boundary value problems.

References

[1] P. Agarwal, On a new theorem involving the generalized Mellin-Barnes type of contour integrals and Srivastava polynomials, *JAMSI* 8(2) (2012).

[2] P. Agarwal and M. Chand, New theorems involving the generalized Mellin-Barnes type of contour integrals and general class of polynomials,*Global Journal of Scieince Frontier Research Mathematics & Decision Sciences* 12(3) (2012).

[3] M. Chand, New theorems involving the *I*-function and general class of polynomials,*Global Journal of Scieince Frontier Research Mathematics & Decision Sciences* 12(4) (2012).

[4] M-J. Luo,G. V. Milovanovic and P. Agarwal, Some results on the extended beta and extended hypergeometric functions, *Appl. Math. Comput.* 248 (2014) 631651.

[5] E.Özergin, Some properties of hypergeometric functions, Ph.D. Thesis, Eastern Mediterranean University, North Cyprus, February 2011.

[6] E. Özergin, M. A. Özarslan and A. Altin, Extension of gamma, beta and hypergeometric functions, *J. Comput. Appl. Math.* **235**(2011) 4601–4610.

[7] M. I. Qureshi, K. A. Quraishi, R. Pal, Some definite integrals of Gradshteyn-Ryzhil and other integrals, *Global Journal of Scieince Frontier Research Mathematics & Decision Sciences* 11(4) (2011) 75–80.

[8] L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966.

[9] H.M. Srivastava, A contour integral involving Foxs *H*-function. *Indian J. Math.* 14 (1972)1–6.

[10] H.M. Srivastava and N.P. Singh, The integration of certain products of the multivariable H-function with a general class of polynomials, *Rend. Circ. Mat. Palermo* 2(32) (1983) 157-187.

[11] C. Szego, Orthogonal polynomials, *Amer. Math. Soc. Colloq. Publ.* 23 Fourth edition, Amer. Math. Soc. Providence, Rhode Island (1975).
[12] E.M. Wright, The asymptotic expansion of the generalized Bessel Function. *Proc. London Math. Soc.* (Ser.2) 38(1935) 257–260.

Praveen Agarwal is the Full Professor of Mathematics at Anand International College of Engineering, Jaipur, Rajasthan, INDIA. He received the B. Sc (1998), M. Sc (2000) and M. Phil (2003) in mathematics from University of Rajasthan, Jaipur and PhD (2006) in Special Function from University of Rajasthan. His research interests center around Special functions, fractional calculus, differential and fractional differential equations as well as their applications to economics, finance, biology, physics, and engineering. He is the author of four textbooks and more than 120 publications, Editor-in-Chief of one international journal, and Associate Editor for more than 60 international journals. His work has been cited more than 345 times in the literature. He was visited many Universities and Research Centres.

Jaekeun Park received his BS and MS degrees from Seoul National University and his Ph.D degree in Choongang University. Since 1977 he is a professor in the Faculty of Department of Mathematics in Korea Air Force Academy and Hanseo University, Korea. His research interests focus on the function spaces of ultradistribution, convex analysis, partial differential equations and Fuzzy sets.

Mehar Chand obtained his M.Sc. degree from the Faculty of Science at S.G.N. Khalsa College, Sri Ganganagar affiliated to University of Bikaner (Rajasthan). He doing Ph. D. from Singhania University, Pacheri Kalan, Rajasthan, India under the Supervision of Dr. Praveen Agarwal. Currently, he is Assistant Professor and Head at the Department of Mathematics, Fateh College for Women, Bathinda, India. His research interest includes special functions, fractional calculus, integral transforms, basic hypergeometric series and mathematical physics.

Shilpi Jain is the Associate Professor of Mathematics at Poornima College of Engineering, Jaipur, Rajasthan (INDIA). She received the PhD (2006) in Relativity Theory from University of Rajasthan. Her research interests centre on the Relativity Theory, Mathematical Physics, Special functions and fractional calculus as well as their applications to economics, finance, biology, physics, and engineering. She is the author of two textbooks and more than 50 publications and Associate Editor of many international journals.