Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100228

An Interactive Method for Solving Multiple Objective Linear Fractional Programming Problem Based on Linear Optimization Models

Tahere Hossein-Abadi and Ali Payan∗

Department of Mathematics, Zahedan Branch, Islamic Azad University, Zahedan, Iran

Received: 25 Aug. 2015, Revised: 12 Nov. 2015, Accepted: 13 Nov. 2015
Published online: 1 Mar. 2016

Abstract: This study aims to present an interactive method in order to solve multiple objective linear fractional programming problem (MOLFPP) according to the linear optimization models. Based on the nonlinear nature of the MOLFPP, presenting interactive methods to solve this problem usually involves long, massive, and complex calculation processes. In this paper, the MOLFPP turns into a linear programming problem (LPP) by disregarding some facts (by considering some negligence) through a linearization process. Then, the answer obtained from the new LPP is tested by an LPP in order to determine whether it is efficient or not. Finally, an efficient or dominant solution to the obtained solution is presented to the decision maker (DM). When the obtained solution does not coincide with the comments of the DM, another LPP is solved based on the judgments of the DM to reach a satisfying solution. Due to the linearity of the models applied, this method is easy to understand and use. A numerical example is given to illustrate the proposed method.

Keywords: linear programming, multiple objective linear fractional programming, efficient solution, interactive method, decision maker

1 Introduction

Multiple objective linear fractional programming problem (MOLFPP) is a special form of multiple objective fractional problems in which the numerator and the denominator of the fractions of the intended functions are linear. MOLFPP has many applications in different branches of sciences. For example, Ravi and Reddy [13] modelled chemical process plant operations planning in an oil refinery as MOLFPP. In order to generate common weights in data envelopment analysis, an MOLFPP whose purposes its objective functions include the efficiency of the DMUs should be solved [8]. In a study done by Duran Tuksari [5], two applications of MOLFPP including production planning and financial planning were presented. The objective functions in production planning included the maximization for both the profitability of the owned employed capital and inventory turnover ratio, and debt ratio, turnover ratio and total ratio of debt and turnover were considered as three fractional objective functions in financial planning. In a similar study, Peric and Balic [11] surveyed a financial planning as MOLFPP whose objective functions were as follows (1) minimization of the current ratio, (2) minimization of the debt ratio, (3) maximization of the turnover ratio, and (4) maximization of the profitability ratio. To see more application of MOLFPP, refer to Frenk and Schaible [6].

MOLFPP has been one of the centers of interest to the researchers. In 1960, a goal programming method was offered by Charnes and Cooper [2] in order to solve the problem. Kornbluth and Steuer [9,10] presented two different procedures for MOLFPP. Stancu-Minasian [15] introduced 386 recorded cases in a directory. Besides, Pramanik and Dey [12] as well as Duran Toksari [5] used the Taylor series in order to transform MOLFPP to a multiple objective linear programming problem. Finally, a linearization approach was suggested by Hosseinzadeh Lofti et al. [7] to check the efficiency of MOLFPP.

One of the proposed methods to deal with MOLFPP is the interactive method. Sakawa and Yano [14] suggested a satisfactory interactive fuzzy procedure to solve MOLFPP. To solve their proposed model, a bisection method and phase one of simplex method must be consecutively solved. Moreover, Costa [4] presented an

∗ Corresponding author e-mail: payan_iauz@yahoo.com, a.payan@iauzah.ac.ir

interactive procedure for solving an MOLFPP, based on pay-off table information. Interactional procedures are complex due to extensive calculations. On the other hand, the calculations might be more complicated because of the unfamiliarity of the decision maker (DM) with the procedure in solving the problem or the existence of nonlinear programs. In this direction, this article intends to present an interactive process to solve MOLFPP based on the linear programming problem (LPP). The proposed method is to convert MOLFPP into an LPP, and another linear procedure is carried out for testing the efficiency, and based on that an LPP is presented for interaction with the DM in order to meet its judgments. When an efficient solution is available, in case the judgments of the DM is not met, the DM is required to divide the goals and purposes of the objective functions into three groups including ones that intend to increase, decrease and also some that the DM has no idea of changing in them, and then an LPP is solved according to this classification. Therefore, all the mentioned procedures are LPPs which help to obtain an easy understanding of the solving procedure. Furthermore, in comparison to the other interactive methods for solving MOLFPP such as Sakawa and Yano's method [14], the computational bulk will decline.

Under these circumstances, the present study consists of the following sections. In the second section the definitions regarding the MOLFPP are being discussed. Existing interactive algorithms for solving MOLFPP are briefly discussed in the next section. In the fourth section based an interaction with DM a linear process for solving MOLFPP is suggested. The subsequent section compares the interactive algorithms in the literature with the proposed algorithm in this paper. The next section includes a numerical example to illustrate and expound the proposed method. The paper presents a conclusion in the last section.

2 Primary definitions

Consider the following MOLFPP:

$$\begin{aligned} &max \quad z_k(x) = \frac{c_k^T x + \alpha_k}{d_k^T x + \beta_k}, \quad k = 1, \dots, K, \\ &s.t \quad x \in S = \{x|\ Ax \le b,\ x \ge 0\}, \end{aligned} \tag{1}$$

so that $\alpha_k,\ \beta_k\ (k = 1,\dots,K)$ are scalar, S is a nonempty as well as a bounded set, b is an u dimensional vector, $c_k,\ d_k$ $(k = 1,\dots,K)$ are w dimensional vectors, $x \in R^w$, and A is a $u \times w$ matrix, $\forall x \in S,\ d_k^T x + \beta_k > 0,\ c_k^T x + \alpha_k > 0$ and z_k is the kth $(k = 1,\dots,K)$ objective function, and $z = (z_1,\dots,z_K)$ is called the objective vector or a criterion vector. Positivity assumption of numerator of the objective functions in MOLFPP (1) is not a usual supposition. But by adding a suitable large positive number to the objectives, they are transformed to objectives with positive numerators. This has been explained in [7].

Definition 1. $\bar{x} \in S$ is called a weakly efficient solution if and only if there is no x; such that $x \in S,\ x \neq \bar{x}$ and $z_k(x) > z_k(\bar{x})\ (k = 1,\dots,K)$.

Definition 2. $\bar{x} \in S$ is called a strongly efficient solution if and only if there is no x; such that $x \in S,\ x \neq \bar{x}$ and $z_k(x) \ge z_k(\bar{x})\ (k = 1,\dots,K)$ and at least for one k, $z_k(x) > z_k(\bar{x})$.

Theorem 1. The set of the strongly efficient solutions is the subset of the set of the weakly efficient solutions. [3]

Finding the strongly efficient solutions is difficult because the set of such solutions is not always closed [3]. On the other hand, the set of the weakly efficient solutions is closed and, as a result, its calculation and generation is easier compared to the strongly efficient solutions.

3 Interactive algorithms for MOLFPP

Sakawa and Yano [14] proposed an interactive method for solving MOLFPP. In this method, reference values of DM are firstly determined and a strongly efficient solution, which is as closer as possible to the reference values, is then acquired. Algorithm has many details which the aim of this paper is not to survey all of them. A summary of this algorithm is presented as follows:

Algorithm 1:

[Step 0.]
Obtain the individual minimum and maximum of each objective function on the feasible region. Determine the membership function $\mu_{z_k}(x)$ for each objective function $z_k(x)$.

[Step 1.]
In this step DM selects initial reference values $\bar{\mu}_{z_k}\ (k = 1,\dots,K)$ and set $s = 0$.

[Step 2.]
Solve the following problem by phase one of simplex method and bisection method to obtain the optimal value v:

$$\begin{aligned} &min \quad v, \\ &s.t \quad c_k^T x + \alpha_k \ge \mu_{z_k}^{-1}(\bar{\mu}_{z_k} - v)(d_k^T x + \beta_k), k = 1,\dots,K, \\ &\qquad x \in S. \end{aligned}$$

[Step 3.]
Determine the appropriate standing objective of DM as $z_h\ (h \in \{1,\dots,K\})$ from among the objective functions and consider x^* as the optimal solution of following linear fractional program as:

$$\begin{aligned} &max \quad z_h(x), \\ &s.t \quad z_k(x) \ge \mu_{z_k}^{-1}(\bar{\mu}_{z_k} - v^*), \quad k = 1,\dots,K, k \neq h, \\ &\qquad x \in S. \end{aligned}$$

[Step 4.]

Perform the efficiency test on x^* by solving following LPP as:

$$\begin{aligned} max \ & \sum_{k=1}^{K} \varepsilon_k, \\ s.t \ & c_k^T x + \alpha_k - \varepsilon_k = z_k(x^*)(d_k^T x + \beta_k), \quad k = 1, \ldots, K, \\ & x \in S. \end{aligned}$$

Let $(\bar{x}, \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_K)$ be an optimal solution for this problem with optimal value $\bar{\varepsilon}$. If $\bar{\varepsilon} = 0$ or $s = K$, then x^* is an efficient solution. Else
[Step 5.]
Run following linear fractional problem as:

$$\begin{aligned} max \ & z_j(x), \\ s.t \ & z_k(x) = z_k(\bar{x}), \quad k \in \{j | \bar{\varepsilon}_j = 0\}, \\ & z_k(x) \geq z_k(\bar{x}), \quad k \in \{j | \bar{\varepsilon}_j > 0\}, \\ & x \in S, \end{aligned}$$

where j is an index in which $\bar{\varepsilon}_j > 0$. Consider x^* as an optimal solution for this problem. Put $s = s+1$ and go to step 4.
[Step 6.] If the DM is satisfied with the current efficient solution, the algorithm is finished. Otherwise, DM modifies reference values $\bar{\mu}_{z_k}$ $(k = 1, \ldots, K)$ by using the trade-off rate between objective functions $\Pi_k = \frac{-\partial \mu_{z_h}(x)}{\partial \mu_{z_k}(x)}$, based on the optimal solution of the model in step 3, and go to step 2.

Another interactive algorithm to MOLFPP was suggested by Costa [4] in 2007. Costa [4] used judgments of DM to divide feasible region to several sub-regions, and for each sub-region pay-off table and middle point were constructed. In this method, feasible region based on the DM's point of views is searched to find efficient solutions. The algorithm is as follows:

Algorithm 2:
[Step 0.]
Consider $S(1^1) = S$. $S(n^i)$ $(i \in I)$ is sub-region of nth iteration on the ith objective function. I is determined in step 4.
[Step 1.]
Calculate pay-off table for each sub-region $S(n^i)$ by solving following linear fractional programming problem, for each k, as:

$$\begin{aligned} max \ & z_k(x), \\ s.t \ & x \in S(n^i). \end{aligned}$$

[Step 2.]
Produce middle point for each sub-region $S(n^i)$ by solving following linear fractional programming problem as:

$$\begin{aligned} max \ & z_v(x), \\ s.t \ & x \in S(n^i), \\ & z_r(x) \geq z_{vr} + \frac{1}{2}\Delta z_r, \end{aligned}$$

where

$$\begin{aligned} \Delta z_r &= \max_{k=1,\ldots,K} \{\Delta z_k = (z_{kk} - min_{s=1,\ldots,K, s\neq k}\{z_{sk}\})\}, \\ z_{vr} &= \min_{k=1,\ldots,K} \{z_{kr}\}. \end{aligned}$$

[Step 3.]
DM selects a solution from among the obtained efficient solutions in steps 1 and 2. Let the selected solution be a member of sub-region $S(n^h)$ and the value of the kth objective function for the solution is z_k^* $(k = 1, ..., K)$.
If the DM is satisfied with the solution, algorithm is stopped. Otherwise. DM determines the expected value for improvement objectives as $\gamma(n^h)$.
[Step 4.]
Determine new sub-regions as:

$$S((n+1)^i) = S(n^h) \cap \{x \in R^n | z_i \geq z_i^* + \gamma(n^h)\}, i \in I,$$

where $I = \{i | z_{ii} \geq z_i^* + \gamma(n^h)\}$. If $I = \emptyset$, we return to step 3 and decease the value of $\gamma(n^h)$ or stop.
[Step 5.]
Return to step 1 with $n = n+1$, $i \in I$.

4 Suggested interactive method

4.1 Converting MOLFPP into a linear model

Scalarization is a method to find efficient solutions in multiple objective programming problem in which all objective functions are combined to obtain a single objective function. Weighted maximin method [1] is one of the scalarization methods which use maximin operator to convert multiple objective programming problem to a single optimization problem. Thus, the weighted maximin method is defined as:

$$\begin{aligned} max \ & \min_{k=1,\ldots,K} \{w_k z_k(x)\}, \\ s.t \ & x \in S, \end{aligned} \tag{2}$$

where $w_k \geq 0$ $(k = 1, ..., K)$. In this paper, we are going to consider equal weights for objective functions. The proposed method in the following can be applied on a problem with different weights. Therefore, we use the weighted maximin version of MOLFPP (1), in which weights are equal, as follows:

$$\begin{aligned} max \ & \min_{k=1,\ldots,K} \{z_k(x) = \frac{c_k^T x + \alpha_k}{d_k^T x + \beta_k}\}, \\ s.t \ & x \in S. \end{aligned} \tag{3}$$

If $v(x)=\min_{k=1,\dots,K}\{z_k(x)=\frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k}\}$, we come up with the following model:

$$\begin{aligned} &max\ \ v,\\ &s.t\ \ v\le \frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k},\quad k=1,\dots,K,\\ &\qquad x\in S. \end{aligned} \tag{4}$$

Problem (4) is a nonlinear programming model. Thus, we propose a process to present an LPP in order to solve problem (1) instead of solving problem (4). The linear function $f_k(.)$ is going to be found in a way such that $v\le f_k(.)$ $(k=1,\dots,K)$ and also $f_k(.)\le z_k(x)=\frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k}$ $(k=1,\dots,K)$. In doing so, the following process should be followed:

$$t=\min_{k=1,\dots,K}\{\frac{1}{d_k^Tx+\beta_k}\}\Rightarrow t\le\frac{1}{d_k^Tx+\beta_k},\quad k=1,\dots,K.$$

So, we have:

$$(c_k^Tx+\alpha_k)t\le\frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k},\quad k=1,\dots,K$$

$$\Rightarrow\ c_k^Txt+\alpha_kt\le\frac{c_k^Txt+\alpha_kt}{d_k^Txt+\beta_kt},\quad k=1,\dots,K,$$

and we define $y=xt$; so that

$$c_k^Ty+\alpha_kt\le\frac{c_k^Ty+\alpha_kt}{d_k^Ty+\beta_kt}=\frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k},\quad k=1,\dots,K.$$

In this situation, the function $f_k(.)=c_k^Ty+\alpha_kt$ $(k=1,\dots,K)$ possesses the property of $f_k(.)\le\frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k}$ $(k=1,\dots,K)$. Now, considering $v=\min_{k=1,\dots,K}\{f_k(.)=c_k^Ty+\alpha_kt\}$, we will have $v\le f_k(.)=c_k^Ty+\alpha_kt$. According to the above mentioned issues, the following LPP is recommended as an alternative to nonlinear problem (4), as:

$$\begin{aligned} &max\ \ v,\\ &s.t\ \ v\le c_k^Ty+\alpha_kt,\quad k=1,\dots,K,\\ &\qquad d_k^Ty+\beta_kt\le 1,\quad k=1,\dots,K,\\ &\qquad Ay-bt\le 0,\\ &\qquad y\ge 0, t\ge 0. \end{aligned} \tag{5}$$

The above mentioned problem is an LPP which can be solved based on the simplex method. If $(\bar{y},\bar{t})$ is the optimal solution of (5), $\bar{x}=\frac{\bar{y}}{\bar{t}}$ will be a feasible solution for problem (4). Regarding the nature of problem (5); i.e., regarding the fact that $v\le f_k(.)$ and $f_k(.)\le\frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k}$ $(k=1,\dots,K)$, the increase in v leads to the increase in $\frac{c_k^Tx+\alpha_k}{d_k^Tx+\beta_k}$ therefore, it is expected that (4) and (5) models have similar behaviors except for the fact that model (5) is linear. However, feasible solution $\bar{x}=\frac{\bar{y}}{\bar{t}}$ may not be efficient, and so we need a method to test efficiency for a solution such as $\bar{x}$. In the following section, we review an efficiency test method for MOLFPP (1) which has been suggested by Hosseinzadeh Lotfi et al. [7].

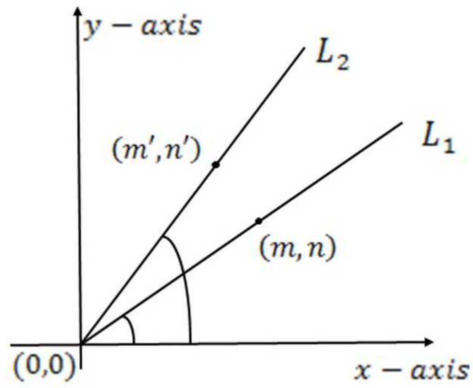

Fig. 1: Slope line of L_1 is less than that of L_2

4.2 Efficiency test

Let $\bar{x}$ is a feasible solution of MOLFPP (1) obtained from solving the problem (5). Consider the following linear fractional programming problem:

$$\begin{aligned} &max\ \ \frac{c^Tx+\alpha}{d^Tx+\beta},\\ &s.t\ \ x\in S=\{x|\ Ax\le b,\ x\ge 0\}, \end{aligned} \tag{6}$$

where A is a $u\times w$ matrix , $b\in R^u$, $x,\ c,\ d\in R^w$, S is a nonempty as well as a bounded set, and $\forall x\in S, d^Tx+\beta>0,\ c^Tx+\alpha>0$.

Suppose that $(m,n)\in R^2,\ m>0,\ n>0,\ y=\frac{n}{m}x$ is a linear equation which crosses $(m,\ n)$ points and as well as the origin. Point $(m',\ n')\in R^2$, $m'>0,\ n'>0$ is located above the line $y=\frac{n}{m}x$ if and only if the linear slope which crosses the $(m',\ n')$ point and the origin is more than $\frac{n}{m}$. In other words, $\frac{n'}{m'}>\frac{n}{m}$. This is illustrated in the following Figure 1.

Theorem 2. If $n'>0,\ m'>0,\ m>0,\ n>0$ are real numbers, then $\frac{n'}{m'}>\frac{n}{m}$ if and only if $\exists\theta\in R^+,\ d^-,\ d^+\in R^{\ge0}$, such that $n'-d^+=n\theta,\ m'+d^-=m\theta,\ d^-+d^+>0$.[7]

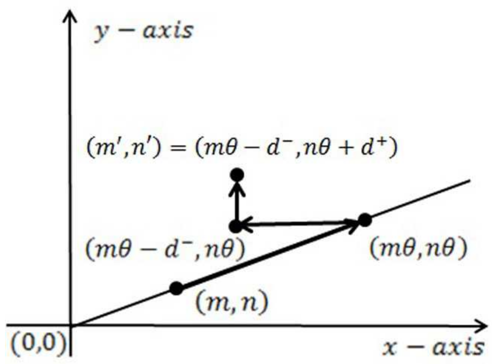

Fig. 2: The path moving from (m,n) to $(m^{'},n^{'})$

Therefore, $\frac{n^{'}}{m^{'}} > \frac{n}{m}$ is equivalent to the fact that it is possible to move with the step length of $\theta > 0$ along (m,n) in order to reach $(m\theta, n\theta)$ and move along $(-1,0)$ with the step length of $d^- \geq 0$. So one can get to the point of $(m\theta - d^-, n\theta)$, move along $(0,1)$ with the step length of $d^+ \geq 0$ to get to the point of $(m^{'},n^{'}) = (m\theta - d^-, n\theta + d^+)$. This is illustrated in Figure 2.

The above mentioned geometrical interpretation can be applied for the optimization test of the linear fractional programming problem (6). The ordered pair $(d^T x + \beta, c^T x + \alpha) \in R^2$ will be defined for each $x \in S$. Therefore, x is an optimal solution if and only if another feasible solution does not exist; so that the corresponding ordered pair is situated above the line crossing the origin and the point $(d^T x + \beta, c^T x + \alpha) \in R^2$.

Based on this, the following theorem was presented in [7] for the optimization test of the linear fractional programming problem (6).

Theorem 3. $\bar{x} \in S$ is an optimal solution for the linear fractional programming problem (6) if and only if the optimal value of the following problem is equal to zero.

$$\begin{aligned} f_{To}^* = max \;\; & d^- + d^+, \\ s.t \;\; & c^T x + \alpha - d^+ = n\theta, \\ & d^T x + \beta + d^- = m\theta, \\ & x \in S, \\ & d^+ \geq 0,\ d^- \geq 0,\ \theta \geq 0. \end{aligned} \tag{7}$$

It must be noted that $n = c^T\bar{x} + \alpha$ and $m = d^T\bar{x} + \beta$.[7] To generalize the above discussion, the following theorem has been stated in [7] for testing the weak efficiency in MOLFPP.

Theorem 4. $\bar{x} \in S$ is a weakly efficient solution for MOLFPP (1) if and only if the following optimal value equals zero.

$$\begin{aligned} f_{Tw}^* = max \;\; & t, \\ s.t \;\; & t \leq d_k^- + d_k^+, \quad k = 1,\ldots,K, \\ & c_k^T x + \alpha_k - d_k^+ = n_k\theta_k, \quad k = 1,\ldots,K, \\ & d_k^T x + \beta_k + d_k^- = m_k\theta_k, \quad k = 1,\ldots,K, \\ & x \in S, \\ & d_k^- \geq 0,\ d_k^+ \geq 0,\ \theta_k \geq 0, \quad k = 1,\ldots,K. \end{aligned} \tag{8}$$

Also, $n_k = c_k^T\bar{x} + \alpha_k$ and $m_k = d_k^T\bar{x} + \beta_k$ $(k = 1,\ldots,K)$.[7]

In the following, a revised version of theorem 4 in [7] is presented, while this theorem does not need weakly efficient solution assumption for feasible solution $\bar{x}$.

Theorem 5. $\bar{x} \in S$ is a strongly efficient solution in MOLFPP (1) if and only if the optimal value of the following problem equals zero.

$$\begin{aligned} f_{Ts}^* = max \;\; & \sum_{k=1}^{K} (d_k^- + d_k^+), \\ s.t \;\; & c_k^T x + \alpha_k - d_k^+ = n_k\theta_k, \quad k = 1,\ldots,K, \\ & d_k^T x + \beta_k + d_k^- = m_k\theta_k, \quad k = 1,\ldots,K, \\ & x \in S, \\ & d_k^- \geq 0,\ d_k^+ \geq 0,\ \theta_k \geq 0, \quad k = 1,\ldots,K. \end{aligned} \tag{9}$$

Proof. Let $\bar{x} \in S$ be not a strongly efficient solution. Based on definition 2, there exists another feasible solution as $\bar{\bar{x}} \in S$; such that $\frac{c_k^T\bar{\bar{x}}+\alpha_k}{d_k^T\bar{\bar{x}}+\beta_k} \geq \frac{c_k^T\bar{x}+\alpha_k}{d_k^T\bar{x}+\beta_k}$ $(k = 1,\ldots,K)$ and strict inequality is held for at least one $k = k_1$. In this situation, we have $\frac{c_{k_1}^T\bar{\bar{x}}+\alpha_{k_1}}{d_{k_1}^T\bar{\bar{x}}+\beta_{k_1}} > \frac{c_{k_1}^T\bar{x}+\alpha_{k_1}}{d_{k_1}^T\bar{x}+\beta_{k_1}}$. This implies that $\exists\theta_{k_1} \in R^+, d_{k_1}^-, d_{k_1}^+ \in R^{\geq 0}$ such that $c_{k_1}^T\bar{\bar{x}}+\alpha_{k_1}-d_{k_1}^+ = n_{k_1}\theta_{k_1}$, $d_{k_1}^T\bar{\bar{x}}+\beta_{k_1}+d_{k_1}^- = m_{k_1}\theta_{k_1}$, $d_{k_1}^+ + d_{k_1}^- > 0$. This shows that there is a feasible solution for problem (9) in which its objective function is positive. As a result, the optimal value of model (9) is more than zero. **Converse**: Consider the optimal value of the objective function of model (9) is positive. Thus, there is a $k = k_1$ in which $d_{k_1}^+ + d_{k_1}^- > 0$ in optimality. Without losing generality, assume $d_{k_1}^- > 0$. So, based on the constraints of model (9), we have $c_{k_1}^T\bar{\bar{x}} + \alpha_{k_1} \geq n_{k_1}\theta_{k_1}$, $d_{k_1}^T\bar{\bar{x}} + \beta_{k_1} < m_{k_1}\theta_{k_1}$, for $\bar{x} \neq \bar{\bar{x}} \in S$ which concludes $\frac{c_{k_1}^T\bar{\bar{x}}+\alpha_{k_1}}{d_{k_1}^T\bar{\bar{x}}+\beta_{k_1}} > \frac{c_{k_1}^T\bar{x}+\alpha_{k_1}}{d_{k_1}^T\bar{x}+\beta_{k_1}}$. We have also $\frac{c_k^T\bar{\bar{x}}+\alpha_k}{d_k^T\bar{\bar{x}}+\beta_k} \geq \frac{c_k^T\bar{x}+\alpha_k}{d_k^T\bar{x}+\beta_k}$ $(k = 1,\ldots,K, k \neq k_1)$ from the constraints of model (9). Therefore, $\bar{x} \in S$ is not a strongly efficient solution.

Now, suppose problem (5) is solved and $(\bar{y},\bar{t})$ is its optimal solution. Model (8) check whether point $\bar{x} = \frac{\bar{y}}{\bar{t}}$ is a weakly efficient solution or not. It is tested by model (9) whether the point $\bar{x}$ is a strongly efficient solution or not? If the optimal value of problem (8) (or (9)) is zero, then $\bar{x}$ is a weakly (strongly) efficient solution. Otherwise, let $(x^*, {\theta_1}^*, \ldots, {\theta_K}^*, {d_1^-}^*, \ldots, {d_K^-}^*, {d_1^+}^*, \ldots, {d_K^+}^*)$ be optimal solution of model (8) (or (9)). Although, x^* is a dominant solution for $\bar{x}$, there is no guarantee that x^* is a weakly (strongly) efficient solution. Stanojevic and Stanojevic [16] proposed to use problem (8) (or (9)) for efficiency test of x^*. By repeating this procedure, a sequence of dominant solution for $\bar{x}$ is constructed. Stanojevic and Stanojevic [16] proved this sequence is convergent to a weakly (strongly) efficient solution.

4.3 Interaction with the DM

However, x^* as a weakly (strongly) efficient solution is presented to DM. If the solution is acceptable to the DM, the problem is solved. If not, we ask the DM to determine what objectives have more priority for increasing than the other ones.

Suppose that K_1 is an indexical objective set and, as a result, the DM intends to increase them. Moreover, we'd like the DM to determine the reduction in which the objective function is less harmful. Suppose that K_2 is the set of indices of functions that are acceptable to decrease in quantity. Therefore,

$$\frac{c_k^T x+\alpha_k}{d_k^T x+\beta_k} \leq \frac{c_k^T x^*+\alpha_k}{d_k^T x^*+\beta_k}, \quad k \in K_2,$$

According to theorem 2, we have $\forall k \in K_2, \ \exists \theta_k \in R^+, \ d_k^-, d_k^+ \in R^{\geq 0}$ such that $c_k^T x+\alpha_k+d_k^+ = n_k\theta_k, \ d_k^T x+\beta_k-d_k^- = m_k\theta_k$, where $n_k = c_k^T x^*+\alpha_k$ and $m_k = d_k^T x^*+\beta_k$.

On the other hand, some objectives must be fixed in the resent values. Suppose K_3 is an index set of such objectives and so we have $\dfrac{c_k^T x+\alpha_k}{d_k^T x+\beta_k} = \dfrac{c_k^T x^*+\alpha_k}{d_k^T x^*+\beta_k}$ $(k \in K_3)$. Therefore, $\forall k \in K_3$, we have θ_k, such that $c_k^T x+\alpha_k = n_k\theta_k, \ d_k^T x+\beta_k = m_k\theta_k$, where $n_k = c_k^T x^*+\alpha_k$ and $m_k = d_k^T x^*+\beta_k$. Based on the classification indices of objectives into three sets K_1, K_2 and K_3, the following problem is so suggested to survey the idea of the DM:

$$\begin{aligned}
f_{Inw}^* = max \ \ & t, \\
s.t \ \ & t \leq d_k^- + d_k^+, \quad k \in K_1, \\
& c_k^T x+\alpha_k-d_k^+ = n_k\theta_k, \quad k \in K_1, \\
& d_k^T x+\beta_k+d_k^- = m_k\theta_k, \quad k \in K_1, \\
& c_k^T x+\alpha_k+d_k^+ = n_k\theta_k, \quad k \in K_2, \\
& d_k^T x+\beta_k-d_k^- = m_k\theta_k, \quad k \in K_2, \\
& c_k^T x+\alpha_k = n_k\theta_k, \quad k \in K_3, \\
& d_k^T x+\beta_k = m_k\theta_k, \quad k \in K_3, \\
& x \in S, \\
& d_k^- \geq 0, \ d_k^+ \geq 0, \quad k \in K_1 \cup K_2 \\
& \theta_k \geq 0, \quad k = 1,\ldots,K.
\end{aligned} \tag{10}$$

Theorem 6. The optimal value for problem (10) equals zero if and only if there is no feasible solution of MOLFPP (1) to meet the decision maker's judgments.

Proof. Suppose $f_{Inw}^* = 0$ and $x^* \in S$ is a given feasible solution for MOLFPP (1) and according to the supposition by contradiction that a point such as $\tilde{x} \in S$ is better that x^* so that it meets all the needs of the DM. Therefore

$$\frac{c_k^T \tilde{x}+\alpha_k}{d_k^T \tilde{x}+\beta_k} > \frac{c_k^T x^*+\alpha_k}{d_k^T x^*+\beta_k}, \quad k \in K_1,$$

$$\frac{c_k^T \tilde{x}+\alpha_k}{d_k^T \tilde{x}+\beta_k} \leq \frac{c_k^T x^*+\alpha_k}{d_k^T x^*+\beta_k}, \quad k \in K_2,$$

$$\frac{c_k^T \tilde{x}+\alpha_k}{d_k^T \tilde{x}+\beta_k} = \frac{c_k^T x^*+\alpha_k}{d_k^T x^*+\beta_k}, \quad k \in K_3.$$

According to theorem 2, we have $\forall k \in K_1, \exists \widetilde{\theta_k} \in R^+, \ \widetilde{d_k^-}, \ \widetilde{d_k^+} \in R^{\geq 0}$, such that $c_k^T \tilde{x}+\alpha_k-\widetilde{d_k^+} = n_k\widetilde{\theta_k}$, $d_k^T \tilde{x}+\beta_k+\widetilde{d_k^-} = m_k\widetilde{\theta_k}$, $\widetilde{d_k^+}+\widetilde{d_k^-} > 0$ (a_1), $\forall k \in K_2, \exists \widetilde{\theta_k} \in R^+, \ \widetilde{d_k^-}, \ \widetilde{d_k^+} \in R^{\geq 0}$, such that $c_k^T \tilde{x}+\alpha_k+\widetilde{d_k^+} = n_k\widetilde{\theta_k}$, $d_k^T \tilde{x}+\beta_k-\widetilde{d_k^-} = m_k\widetilde{\theta_k}$, $\widetilde{d_k^+}+\widetilde{d_k^-} \geq 0$ (b_1), and $\forall k \in K_3, \ \exists \widetilde{\theta_k} \in R^+, \ \widetilde{d_k^-}, \ \widetilde{d_k^+} \in R^{\geq 0}$ such that $c_k^T \tilde{x}+\alpha_k-\widetilde{d_k^+} = n_k\widetilde{\theta_k}$, $d_k^T \tilde{x}+\beta_k+\widetilde{d_k^-} = m_k\theta_k$, $\widetilde{d_k^+}+\widetilde{d_k^-} = 0$ (c_1) where $n_k = c_k^T x^*+\alpha_k$ and $m_k = d_k^T x^*+\beta_k$. Thus $d_k^-+d_k^+ > 0$ $(k \in K_1)$.

Consequently, $(\tilde{x}, \widetilde{\theta_1}, \ldots, \widetilde{\theta_K}, \widetilde{d_1^-}, \ldots, \widetilde{d_K^-}, \widetilde{d_1^+}, \ldots, \widetilde{d_K^+})$ is a feasible solution for model (10) so that the value of its objective function is more than zero, and it contradicts hypothesis ($f_{Inw}^* = 0$). **Converse**: Suppose there is no solution better than x^*. We will then prove that the optimal value of problem (10) is zero. We consider it with the supposition by contradiction that the optimal value is not zero i.e. $f_{Inw}^* > 0$. Let $(\tilde{x}, \widetilde{\theta_1}, \ldots, \widetilde{\theta_K}, \widetilde{d_1^-}, \ldots, \widetilde{d_K^-}, \widetilde{d_1^+}, \ldots, \widetilde{d_K^+})$ is the optimal solution of problem (10). Because $f_{Inw}^* > 0$, then $\exists k \in K_1$, such that $\widetilde{d_k^-}+\widetilde{d_k^+} > 0$.

Without reducing the integrity and the generality of the problem, we suppose that $\widetilde{d_k^+} > 0$.
So we have:

$$c_k^T \tilde{x}+\alpha_k > n_k\widetilde{\theta}_k, \quad d_k^T \tilde{x}+\beta_k = m_k\widetilde{\theta}_k, \quad k \in K_1 \quad (a_2)$$

$$c_k^T \tilde{x}+\alpha_k \leq n_k\widetilde{\theta}_k, \quad d_k^T \tilde{x}+\beta_k \geq m_k\widetilde{\theta}_k, \quad k \in K_2 \quad (b_2)$$

$$c_k^T \tilde{x}+\alpha_k = n_k\widetilde{\theta}_k, \quad d_k^T \tilde{x}+\beta_k = m_k\widetilde{\theta}_k, \quad k \in K_3 \quad (c_2)$$

Suppose $\widetilde{\theta}_k = 0$ $(k = 1,\ldots,K)$, then

$$(a_2) \Rightarrow d_k^T \tilde{x}+\beta_k = 0, \quad k \in K_1$$

$$(b_2) \Rightarrow c_k^T \tilde{x}+\alpha_k \leq 0, \quad k \in K_2$$

$$(c_2)\ \Rightarrow\ c_k^T\tilde{x}+\alpha_k=0,\ d_k^T\tilde{x}+\beta_k=0\quad k\in K_3$$

The above relationships contradict the suppositions of the problem (1). So $\widetilde{\theta}_k>0$ $(k=1,\ldots,K)$ and we have

$$(a_2)\ \Rightarrow\ c_k^T\tilde{x}+\alpha_k>n_k\widetilde{\theta}_k>0,\ \frac{1}{d_k^T\tilde{x}+\beta_k}=\frac{1}{m_k\widetilde{\theta}_k}>0$$
$$\Rightarrow\frac{c_k^T\tilde{x}+\alpha_k}{d_k^T\tilde{x}+\beta_k}>\frac{c_k^Tx^*+\alpha_k}{d_k^Tx^*+\beta_k},\quad k\in K_1\ (a_3)$$

$$(b_2)\ \Rightarrow\ 0<c_k^T\tilde{x}+\alpha_k\leq n_k\widetilde{\theta}_k,\ 0<\frac{1}{d_k^T\tilde{x}+\beta_k}\leq\frac{1}{m_k\widetilde{\theta}_k}$$
$$\Rightarrow\frac{c_k^T\tilde{x}+\alpha_k}{d_k^T\tilde{x}+\beta_k}\leq\frac{c_k^Tx^*+\alpha_k}{d_k^Tx^*+\beta_k},\quad k\in K_2\ (b_3)$$

$$(c_2)\ \Rightarrow\ c_k^T\tilde{x}+\alpha_k=n_k\widetilde{\theta}_k>0,\ \frac{1}{d_k^T\tilde{x}+\beta_k}=\frac{1}{m_k\widetilde{\theta}_k}>0$$
$$\Rightarrow\frac{c_k^T\tilde{x}+\alpha_k}{d_k^T\tilde{x}+\beta_k}=\frac{c_k^Tx^*+\alpha_k}{d_k^Tx^*+\beta_k},\quad k\in K_3\ (c_3)$$

It can be concluded from (a_3), (b_3) and (c_3) that $\widetilde{x}\in S$ is a point that meets all of the judgments of the DM and contradicts the supposition.

Suppose that $\tilde{x}$ is a part of the optimal solution for problem (10). If the DM is satisfied with the obtained solution, then the problem is over. If $\tilde{x}$ is not in accordance with the judgment of the DM, so then K_1, K_2 and K_3 are modified in a way to meet the decision maker's judgments and, consequently, problem (10) is solved again and this process continues so that the DM judgments and opinions are met.

As a similar manner, consider x^* as the optimal solution of (9). The following model can modify the solution based on the judgments of the DM in order to obtain a strongly efficient solution.

Theorem 7. The optimal value of the problem (11) equals zero if and only if there is no feasible solution of MOLFPP (1) to meet the decision maker's judgments.

$$\begin{aligned}
f_{Ins}^*=max\ &\sum_{k\in K_1}\left(d_k^++d_k^-\right),\\
s.t\ \ &c_k^Tx+\alpha_k-d_k^+=n_k\theta_k,\quad k\in K_1,\\
&d_k^Tx+\beta_k+d_k^-=m_k\theta_k,\quad k\in K_1,\\
&c_k^Tx+\alpha_k+d_k^+=n_k\theta_k,\quad k\in K_2,\\
&d_k^Tx+\beta_k-d_k^-=m_k\theta_k,\quad k\in K_2,\\
&c_k^Tx+\alpha_k=n_k\theta_k,\quad k\in K_3,\\
&d_k^Tx+\beta_k=m_k\theta_k,\quad k\in K_3,\\
&x\in S,\\
&d_k^-\geq0,\ d_k^+\geq0,\quad k\in K_1\cup K_2,\\
&\theta_k\geq0,\quad k=1,\ldots K.
\end{aligned}\tag{11}$$

Proof. Proof is similar to theorem 6.

Model (10) (or (11)) is thus solved to improve x^*, based on the judgments of DM. If the optimal value of the problem (10) (or (11)) equals zero, improvement is impossible. In such situation, DM accepts x^* or changes its judgments and solve model (10) (or (11)), again. Otherwise, consider $(\tilde{x},\widetilde{\theta_1},\ldots,\widetilde{\theta_K},\widetilde{d_1^-},\ldots,\widetilde{d_K^-},\widetilde{d_1^+},\ldots,\widetilde{d_K^+})$ as an optimal solution for problem (10) (or (11)). Then $\tilde{x}$ is a feasible solution of (1) which meet the judgments of DM. Now, we solve model (8) (or (9)) for efficiency test of $\tilde{x}$. If $\tilde{x}$ is efficient, we have a weakly (strongly) efficient solution which meet DM's judgments. Otherwise ($\tilde{x}$ is not efficient), by applying the mentioned procedure in the last paragraph of subsection (4.2) we reach a weakly (strongly) efficient solution which dominant $\tilde{x}$ and also meet the judgments of DM.

4.4 Interactive algorithm

Based on the above discussion, the following algorithm is provided as an interactive method for solving MOLFPP (1) in order to obtain a satisfactory solution for the DM.

Algorithm 3:
[Step 0.] (Primal feasible solution)
Solve the LPP (5), consider $(\overline{y},\overline{t})$ as its optimal solution, and put $\overline{x}=\dfrac{\overline{y}}{\overline{t}}$.
[Step 1.] (Efficiency test)
Solve the LPP (8) (or (9)) for $\overline{x}$ and consider $(x^*,{\theta_1}^*,\ldots,{\theta_K}^*,{d_1^-}^*,\ldots,{d_K^-}^*,{d_1^+}^*,\ldots,{d_K^+}^*)$ as its optimal solution.
[Step 2.] (Find efficient solution)
If $f_{Tw}^*=0$ (or $f_{Ts}^*=0$), then go to step 3, otherwise consider $\overline{x}=x^*$ and go to step 1.
[Step 3.] (Terminate condition)
If the DM satisfies with $\overline{x}$, then stop. Otherwise, go to step 4.
[Step 4.](Interaction with DM)
Determine K_1, K_2 and K_3 based on the judgments of DM and solve problem (10) (or (11)) and consider its optimal solution as $(\tilde{x},\widetilde{\theta_1},\ldots,\widetilde{\theta_K},\widetilde{d_1^-},\ldots,\widetilde{d_K^-},\widetilde{d_1^+},\ldots,\widetilde{d_K^+})$.
[Step 5.] (Return step)
If $f_{Inw}^*=0$ (or $f_{Ins}^*=0$), then go to step 3, otherwise put $\overline{x}=\tilde{x}$ and go to step 1.

5 Comparison among interactive methods

Although interactive algorithms (Sakawa and Yano [14], Costa [4] and method in this paper) use LPPs for solving MOLFPP, they have differences which are compared with three directions in this section.

Computation Algorithm 1 applies a nonlinear programming problem in step 2 and solves this problem using a sequence of linear programming problems. Two LPPs are utilized in steps 3 and 4. A sequence of LPPs are solved consecutively in steps 4 and 5. As a result algorithm 1 includes high performance computing.

Solving $K+1$ LPPs for each sub-region is required referring to steps 1 and 2 of algorithm 2, though algorithm 2 needs less computing than algorithm 1, its computations are heavier in comparison with the recommended method in this article that just requires solving 2 LPPs in steps 1 and 5 plus a sequence of LPPs in steps 1, 2 from the first iteration, and one LPP in in step 5 along with a sequence of LPPs in steps 1 and 2 in the other iterations.

Presented solution to DM Suppose we have m sub-region in nth iteration of algorithm 2; Therefore, the DM should choose a solution from among $m\ (k+1)$ obtained solutions while one solution is persented to the DM in algorithms 1 and 3, by solving optimization models. The presented solution for the DM in algorithm 1 is created by solving at least two LPPs while such a solution in algorithm 3 is just obtained by one LPP.

Judgments of DM If the DM is not satisfied with the obtained result, he should change the reference values according to algorithm 1 and should determine the minimum improvement value of the objective functions using algorithm 2. Performing such numerical changes is usually difficult task for the DM while the qualitative determination of the changes is much easier for the decision maker. Algorithm 3 suggests that the changes in objective functions be done qualitatively. This kind of changes includes the decline or lack of change in objective functions. It is noteworthy that trade-off rate among objective functions in step 6 of algorithm 1 contributes to the DM in order to determine more new reference values while such a guide does not exist in other two algorithms.

6 Example

Take the following MOLFPP in to account:

$$\begin{aligned}
\max\ \ z_1 &= \frac{x_1+x_2+2}{x_1+2x_2+5},\\
\max\ \ z_2 &= \frac{x_1+x_2+1}{5x_1+x_2-1},\\
\max\ \ z_3 &= \frac{x_1+x_2-1}{3x_1+2x_2-1},\\
s.t\ \ & 3x_1+2x_2 \geq 6,\\
& x_1 \leq 3,\\
& x_2 \leq 3,\\
& x_1, x_2 \geq 0. \qquad (12)
\end{aligned}$$

We consider two cases: in a case, let there exists a feasible solution for the system, then the judgments of DM in order to reach a satisfactory solution is applied on models (10) and (11). In another case, we illustrate the proposed algorithm in this paper by solving problem (12). At last, we compare our numerical results with the suggested method of Pramanik and Dey [12].

One can easily show that the numerators and denominators of the objective functions in (12) are positive on the feasible region. The feasible solution $(3, 1.2222)$ is a strongly efficient solution of (12) and the values of the objective functions for this solution are:

$$z_1 = 0.5957, \quad z_2 = 0.3430, \quad z_3 = 0.3085.$$

Table 1 shows six cases of the conditions that DM can consider for repairing this solution. Notations $\uparrow$ and $\downarrow$ are used in order to determine the increase as well as the decrease in the values of objective functions. Notation $\|$ is considered when DM is satisfied with the value of an objective function. Although DM can define many different cases for the objectives functions, especially when the number of objective functions is large, we need to solve only one LPP in each case. Moreover, when all of the results of different cases are presented to DM, he can determines the effect of each objective function in comparison to another. This help the DM for making a correct decision. For example, the first row of Table 1 shows that there is no feasible solution; in which, the first, second, and third objectives are greater than those of the solution $(3, 1.2222)$. In other words, simultaneously, increasing the objective functions is impossible. Also, in the last row of Table 1, a feasible solution is reported in which the first objective function shows an increase in its value, while the second and the third objective functions show a decrease in their values. The solution is $(3,0)$, which, is a weakly and strongly efficient solution (Based on the optimal values of the objective functions in problems (8) and (9)). An overview of the results of Table 1 shows that improvement in the first objective function is impossible unless it is contrary to decrease in the second and third objective functions. Therefore, if DM tends to increase the first objective function, he must accept to decrease the second and third objective functions.

Now, we solve problem (12) by the proposed algorithm in this paper.

[Step 0.] Primal feasible solution

Solve the problem (12) by the linear model (5).

Define

$$t = min\{\frac{1}{x_1+2x_2+5}, \frac{1}{5x_1+x_2-1}, \frac{1}{3x_1+2x_2-1}\}$$

. Therefore, we have

$$t \leq \frac{1}{x_1+2x_2+5}, \quad t \leq \frac{1}{5x_1+x_2-1}, \quad t \leq \frac{1}{3x_1+2x_2-1}.$$

In this situation,

$$(x_1+x_2+2)t \leq \frac{x_1+x_2+2}{x_1+2x_2+5},$$

$$(x_1+x_2+1)t \leq \frac{x_1+x_2+1}{5x_1+x_2-1},$$

Table 1: Several cases for objective functions

Goals			Interactive test		$\overline{x}$	Efficiency test		The values of objective functions		
z_1	z_2	z_3	f^*_{Inw}	f^*_{Ins}		f^*_{Tw}	f^*_{Ts}	z_1	z_2	z_3
↑	↑	↑	0	0	(3,1.2222)	0	0	0.5957	0.3430	0.3085
↑	↑	‖	0	0	(3,1.2222)	0	0	0.5957	0.3430	0.3085
↑	↑	↓	0	0	(3,1.2222)	0	0	0.5957	0.3430	0.3085
↑	↓	↑	0	0	(3,1.2222)	0	0	0.5957	0.3430	0.3085
↑	↓	‖	0	0	(3,1.2222)	0	0	0.5957	0.3430	0.3085
↑	↓	↓	0.3928	0.3928	(3,0)	0	0	0.6250	0.2857	0.2500

$$(x_1+x_2-1)t \le \frac{x_1+x_2-1}{3x_1+2x_2-1}.$$

Consider $y_1 = x_1 t,\ y_2 = x_2 t$; and therefore,

$$y_1+y_2+2t \le \frac{y_1+y_2+2t}{y_1+2y_2+5t},$$

$$y_1+y_2+t \le \frac{y_1+y_2+t}{5y_1+y_2-t},$$

$$y_1+y_2-t \le \frac{y_1+y_2-t}{3y_1+2y_2-t}.$$

Thus, the following linear model is solved as:

$$\begin{aligned} Max\ \ & v,\\ s.t\ \ & v \le y_1+y_2+2t,\\ & v \le y_1+y_2+t,\\ & v \le y_1+y_2-t,\\ & y_1+2y_2+5t \le 1,\\ & 5y_1+y_2-t \le 1,\\ & 3y_1+2y_2-t \le 1,\\ & 3y_1+2y_2-6t \ge 0,\\ & y_1-3t \le 0,\\ & y_2-3t \le 0,\\ & y_1,y_2,t \ge 0. \end{aligned} \tag{13}$$

Point $(\overline{y}_1,\overline{y}_2,\overline{t}) = (0.1698, 0.2264, 0.0754)$ is the optimal solution of the problem (13). Therefore $(\overline{x}_1,\overline{x}_2) = (2.25,3)$ is a feasible solution of MOLFPP (12).

[Step 1.] Efficiency test

To survey the weak efficiency of $(\overline{x}_1,\overline{x}_2) = (2.25,3)$, the following LPP is provided:

$$\begin{aligned} f^*_{Tw} = max\ \ & t,\\ s.t\ \ & t \le d^-_k + d^+_k,\quad k=1,2,3\\ & x_1+x_2+2-d^+_1 = 7.25\theta_1,\\ & x_1+2x_2+5+d^-_1 = 13.25\theta_1,\\ & x_1+x_2+1-d^+_2 = 6.25\theta_2,\\ & 5x_1+x_2-1+d^-_2 = 13.25\theta_2,\\ & x_1+x_2-1-d^+_3 = 4.25\theta_3,\\ & 3x_1+2x_2-1+d^-_3 = 11.75\theta_3,\\ & 3x_1+2x_2 \ge 6,\\ & x_1 \le 3,\\ & x_2 \le 3,\\ & x_1,\ x_2 \ge 0,\\ & d^-_k \ge 0,\ d^+_k \ge 0,\ \theta_k \ge 0,\quad k=1,2,3. \end{aligned} \tag{14}$$

[Step 2.] Find efficient solution

The optimal value of this problem is zero; therefore, the solution $(\overline{x}_1,\overline{x}_2) = (2.25,3)$ is a weakly efficient one. Solving model (9), one realizes that it is a strongly efficient solution as well.

[Step 3.] Terminate condition

If DM is satisfied with $(2.25,3)$, then the process is finished, otherwise we go to step 4.

[Step 4.] Interaction with DM

The values of the objective functions for $(\overline{x}_1,\overline{x}_2) = (2.25,3)$ are:

$$\overline{z}_1 = 0.5471,\quad \overline{z}_2 = 0.4717,\quad \overline{z}_3 = 0.3617.$$

Consider that the DM thinks the amount of the second objective function should increase, and no changes in the third objective function, while a decrease in the first objective can be done. Therefore, we have the following

Table 2: Interactive algorithm for solving model (12).

Iteration	Step	Discussions
1	0	$\bar{x}=(2.25,\ 3)$
	1	$x^*=(2.25,3)$, $z_1=0.5471$,$z_2=0.4717$,$z_3=0.3617$
	2	$f_{Tw}=0$. $\bar{x}=(2.25,\ 3)$ is efficient.
	3	$\bar{x}=(2.25,3)$ is not desirable in the DM's view.
	4	$K_1=\{2\},K_2=\{1\},K_3=\{3\}$, $\tilde{x}=(0.3829,2.4255)$, $z_1=0.4698$,$z_2=1.1402$,z_3=0.3617
	5	$f_{Inw}\neq 0$. Put $\bar{x}=\tilde{x}=(0.3829,2.4255)$ and go to step 1 in the next iteration.
2	1	$x^*=(0.3564,3)$, $z_1=0.4716$,$z_2=1.1518$,z_3=0.3882
	2	$f_{Tw}\neq 0$. $\bar{x}=(0.3829,2.4255)$ is not efficient. Put $\bar{x}=x^*=(0.3564,3)$ and go to step1
	1	$x^*=(0.3564,3)$, $z_1=0.5471$,$z_2=0.4717$,$z_3=0.3617$
	2	$f_{Tw}=0$. $\bar{x}=(0.3564,3)$ is efficient.
	3	$\bar{x}=(0.3564,3)$ is not desirable in the DM's view.
	4	$K_1=\{1,3\},K_2=\{2\},K_3=\{\}$, $\tilde{x}=(0.3564,\ 3)$, $z_1=0.4716$,$z_2=1.1518$,z_3=0.3882
	5	$f_{Inw}=0$ and go to step 3 in the next iteration.
3	3	$\bar{x}=(0.3564,3)$ is not desirable in the DM's view.
	4	$K_1=\{3\},K_2=\{1,2\},K_3=\{\}$,$\tilde{x}=(0.3564,\ 3)$, $z_1=0.4716$,$z_2=1.1518$,z_3=0.3882
	5	$f_{Inw}=0$. go to step 3 in the next iteration.
4	3	$\bar{x}=(0.3564,3)$ is not desirable in the DM's view.
	4	$K_1=\{2,3\},K_2=\{1\},K_3=\{\}$, $\tilde{x}=(0,3)$, $z_1=0.4545$,$z_2=2.00$,z_3=0.4
	4	$f_{Inw}\neq 0$. Put $\bar{x}=\tilde{x}=(0,\ 3)$ and go to step 1 in the next iteration.
5	1	$x^*=(0,3)$, $z_1=0.4545$,$z_2=2.00$,z_3=0.4
	2	$f_{Tw}=0$. $\bar{x}=(0,\ 3)$ is efficient.
	3	$\bar{x}=(0,3)$ is not desirable in the DM's view.
	4	$K_1=\{1\},K_2=\{3\},K_3=\{2\}$, $\tilde{x}=(0,3)$, $z_1=0.4545$,$z_2=2.00$,z_3=0.4
	5	$f_{Inw}=0$. go to step 3 in the next iteration.
6	3	$x^*=(0,3)$ is desirable in the DM's view.

linear program as:

$$\begin{aligned} f^*_{Inw} = max\ \ & t, \\ s.t\ \ & t \leq d_2^- + d_2^+, \\ & x_1 + x_2 + 2 + d_1^+ = 7.25\theta_1, \\ & x_1 + 2x_2 + 5 - d_1^- = 13.25\theta_1, \\ & x_1 + x_2 + 1 - d_2^+ = 6.25\theta_2, \\ & 5x_1 + x_2 - 1 + d_2^- = 13.25\theta_2, \\ & x_1 + x_2 - 1 = 4.25\theta_3, \\ & 3x_1 + 2x_2 - 1 = 11.75\theta_3, \\ & 3x_1 + 2x_2 \geq 6, \\ & x_1 \leq 3, \\ & x_2 \leq 3, \\ & x_1, x_2 \geq 0, \\ & d_k^- \geq 0,\ d_k^+ \geq 0, \quad k = 1,2, \\ & \theta_k \geq 0, \quad k = 1,2,3. \end{aligned} \tag{15}$$

A feasible solution of problem (12) which is derived by solving the above model and shown in the fourth row of Table 2 is $(\tilde{x}_1, \tilde{x}_2) = (0.3829, 2.4255)$.

[Step 5.] Return step

Because $f^*_{Inw} \neq 0$, we put $\bar{x} = \tilde{x}$ and go to step 1 to test the efficiency of new $\bar{x}$. The process can be continued successfully to reach a satisfactory solution from the DM's point of view. A sample of running the algorithm in some iterations is reported in Table 2.

Pramanik and Dey [12] proposed a distance function to compare efficient solutions in MOLFPP. In their method, membership function $\mu_k(.)$ is constructed for each linear fractional objective function $z_k(x)$. This membership function is transformed to a linear function $\tilde{\mu}_k(.)$ by using Taylor series. In continuation, two fuzzy goal programming problems are suggested to solve MOLFPP. At last, the obtained efficient solutions, by solving to models, are compared based on the scores from the distance function $(\sum_{k=1}^{K}(1-\mu_k(.))^2)^{1/2}$. Efficient solution with smaller score has priority than the other, for DM.

Here, we compare the obtained efficient solutions in Table 2 by using the distance function presented by Pramanik and Dey [12]. The example in this section is also solved by the proposed method of Pramanik and Dey [12]. Following Table 3 provides efficient solutions along with their scores.

The last column of Table 3 shows that efficient solution $(0,3)$, which is the latest obtained efficient solution by the proposed algorithm in this paper, has the smallest score among efficient solutions in Table 2. We only obtained efficient solution $(1.000728, 3)$ by solving two proposed models of Pramanik and Dey [12]. The score of this solution based on the distance function has the highest score among all of mentioned efficient solutions.

As a result, we can use the distance function proposed by Pramanik and Dey [12] as a tool to help the DM for selecting an efficient solution to concluding condition for the suggested algorithm in this paper.

Table 3: Efficient solutions and their scores.

Method	Efficient solution	Distance function
Iteration 1	(2.25,3)	1.0197
Iteration 2	(0.3564,3)	1.0281
Iteration 5	(0,3)	0.9997
Model I	(1.000728,3)	1.056
Model II	(1.000728,3)	1.056

7 Conclusions

In this article, a linearization procedure is applied to present an interactive method for solving an MOLFPP which includes a simple calculation process and is also easy to understand. By interacting with the DM, the final solution intended to meet the judgments of the DM. In the future, we try to implement the described procedure on the multiple objective quadratic fractional programming problems and also work out the multiple objective fractional programming problems with imprecise data.

References

[1] V.J. Bowman, On the relationship of the Tchebycheff norm and the efficient frontier of multiple-criteria objective, Lect. Notes. Econ. Math. **135**, 76-85 (1976).

[2] A. Charnes and W.W. Cooper, Goal programming and multiple objective optimization, Eur. J. Oper. Res. **1**, 39-54 (1977).

[3] E.U. Choo and D.R. Atkins, Connectedness in multiple linear fractional programming, Manag. Sci. **29** , 250-255 (1983).

[4] J.P. Costa, An interactive method for multiple objective linear fractional programming problems, OR. Spectrum. **27** , 633-652 (2005).

[5] M. Duran Toksarı, Taylor series approach to fuzzy multiobjective linear fractional programming, Inform. Sciences. **178** (4) , 1189-1204 (2008).

[6] J.B.G. Frenk, S. Schaible, Fractional Programming, in: N. Hadjisavvas, S. Komlosi, S. Schaible (Eds.), Handbook of Generalized Convexity and Generalized Monotonicity, Springer, Boston, 2005.

[7] F. Hosseinzadeh Lotfi, A.A. Noora, G.R. Jahanshahloo, M. Khodabakhshi and A. Payan, A linear programming approach to test efficiency in multi-objective linear fractional programming problems, Appl. Math. Model. **34** , 4179-4183 (2010).

[8] G.R. Jahanshahloo, A. Memariani, F. Hosseinzadeh Lotfi and H. Rezai, A note on some of DEA models and finding efficiency and complete ranking using common set of weights, Appl. Math. Comput. **166** , 265-281 (2005).

[9] J.S.H. Kornbluth and R.E. Steuer, Multiple objective linear fractional programming, Manag. Sci. **27** , 1024-1039 (1981).
[10] J.S.H. Kornbluth and R.E. Steuer, Goal programming with linear fractional criteria, Eur. J. Oper. Res. **8** , 58-65 (1981).
[11] T. Peric and Z. Balic, Financial structure operation by using a goal programming approach, Croat. Operat. Res. Rev. **3** , 150-162 (2012).
[12] S. Pramanik and P.P. Dey, Multi-objective linear fractional programming problem based on fuzzy goal programming, Int. J. Math. Archive. **2**, 1875-1881 (2011).
[13] V. Ravi and P.J. Reddy, Fuzzy linear fractional goal programming applied to refinery operations planning, Fuzzy. Set. Syst. **96** , 173-182 (1998).
[14] M. Sakawa and H. Yano, An interactive fuzzy satisficing method for multiobjective linear fractional programming problems, Fuzzy. Set. Syst. **28** , 129-144 (1988).
[15] I.M. Stancu-Minasian, Fractional Programming: Theory, Methods and Application, Springer, Dordrecht, 1988.
[16] B. Stanojevic and M. Stanojevic, On the efficiency test in multi-objective linear fractional programming problems by Lotfi et al. 2010, Appl. Math. Model. **37** , 7086-7093 (2013).

Tahere Hossein-Abadi is graduate student on operational research, Islamic Azad University, Zahedan Branch, Zahedan, Iran, was born in Zahedan on 1991. His major area of interest is data envelopment analysis, multi-objective programming problem.

Ali Payan is Assistant Professor, Islamic Azad University, Zahedan Branch, Zahedan, Iran, was born in Zahedan on 1981. He has received Ph.D. from Islamic Azad University, Science and Research Branch on 2010. His Ph.D thesis is about relation between multi-objective linear fractional programming problem and common set of weights method in data envelopment analysis. His major area of interest is operational research, in particular, data envelopment analysis and multiple criteria programs. He teaches advanced operational research and non-linear programming problem courses in the department of mathematics of Zahedan branch.