Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/150408

Some New Inequalities for LR-Log-h-Convex Interval-Valued Functions by Means of Pseudo Order Relation

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Received: 21 Feb. 2021, Revised: 22 Mar. 2021, Accepted: 24 Mar. 2021.
Published online: 1 May 2021.

Abstract: In this paper, we introduce the concept of LR-log-h-convex interval-valued functions. Under the new concept, we present new versions of Hermite-Hadamard inequalities (i.e. LR-interval Hermite-Hadamard type inequalities) by means of pseudo order relation. This order relation is defined on interval space. Some Jensen type inequalities are also derived for LR-log-h-convex interval-valued functions. Moreover, we have shown that our results include a wide class of new and known inequalities for LR-log-h-convex interval-valued functions and their variant forms as special cases. Useful examples that verify the applicability of the theory developed in this study are presented. The concepts and techniques of this paper will be a starting point for further research in this area.

Keywords: Interval-valued functions, Riemann integrals, LR-log-$\hbar$-convex interval-valued functions, Hermite-Hadamard inequality, Hermite-Hadamard-Fejer inequality, Jensen inequality.

Mathematics Subject Classification: 26A33, 26A51, 26D10

1 Introduction

The concept of convexity is well-acknowledged in optimization concepts and plays a vital role in operation research, economics, decision making, and management sciences. Recently, many extensions and generalizations of convex functions have been established. For more useful details, see [1-5] and the references therein. In classical approach, a real valued function $\mathcal{F}: K \to \mathbb{R}$ is called convex if

$$\mathcal{F}(\tau x + (1-\tau)y) \leq \tau\mathcal{F}(x) + (1-\tau)\mathcal{F}(y), \tag{1}$$

for all $x, y \in K, \tau \in [0,1]$.

The concept of convexity with integral problem is an interesting area for research. The integral inequalities are a useful technique for developing the qualitative and quantitative properties of convexity and nonconvexity. Because of diverse applications of these inequalities in different fields, there has been continuous growth of interest in such an area of research. Therefore, many inequalities have been introduced as applications of convex functions and generalized convex function, see [6-11] and the references therein. Among those, the following integral inequality is familiar in the literature as the Hermite-Hadamard inequality (in shortly, HH-inequality) [12, 13]:

$$\mathcal{F}\left(\frac{u+\vartheta}{2}\right) \leq \frac{1}{\vartheta-u}\int_u^\vartheta \mathcal{F}(x)dx \leq \frac{\mathcal{F}(u)+\mathcal{F}(\vartheta)}{2}, \tag{2}$$

where $\mathcal{F}: K \to \mathbb{R}$ is a convex function on the interval $K = [u, \vartheta]$ with $u < \vartheta$. In 2007, Noor [14] presented the following HH-inequality for preinvex function:

$$\mathcal{F}\left(\frac{2u+\partial(\vartheta,u)}{2}\right) \leq \frac{1}{\partial(\vartheta,u)}\int_u^{u+\partial(\vartheta,u)} \mathcal{F}(x)dx \leq [\mathcal{F}(u) + \mathcal{F}(\vartheta)]\int_0^1 \tau d\tau, \tag{3}$$

where $\mathcal{F}: K \to \mathbb{R}$ is a preinvex function on the invex set $K = [u, u+\partial(\vartheta,u)]$ with $u < u+\partial(\vartheta,u)$.

Furthermore, the concept of interval analysis was proposed and investigated by Moore [15]. It is a discipline in which an uncertain variable is represented by an interval of real numbers.

Recently, several classical discrete and integral inequalities have been generalized not only to the environment of the IVF and fuzzy-IVFs by Costa [16], Costa and Roman-Flores [17], Flores-Franulic et al. [18], Roman-Flores et al.

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[19, 20], and Chalco-Cano et al. [21], but also to more general set valued maps by Nikodem et al. [22], and Matkowski and Nikodem [23]. In particular, Zhang et al. [24] derived the new version of Jensen's inequalities for set-valued and fuzzy set-valued functions by means of a pseudo order relation and proved that these Jensen's inequalities generalized a form of Costa Jensen's inequalities [16]. Inspired by the above literature, Zhao et al. [25] introduced $\hbar$-convex interval-valued functions (IVFs, in short) in 2018 and proved that the HH-inequality for convex IVFs, as a follows:

Let $\mathcal{F}: [u, \vartheta] \subset \mathbb{R} \to \mathbb{R}_I^+$ be a convex IVF given by $\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)]$ for all $x \in [u, \vartheta]$, where $\mathcal{F}_*(x)$ is a convex function and $\mathcal{F}^*(x)$ is a concave function. If $\mathcal{F}$ is Riemann integrable, then

$$\mathcal{F}\left(\frac{u+\vartheta}{2}\right) \supseteq \frac{1}{\vartheta-u}\ (IR)\int_u^\vartheta \mathcal{F}(x)dx \supseteq \frac{\mathcal{F}(u)+\mathcal{F}(\vartheta)}{2}. \quad (4)$$

It is a familiar fact that log-convex functions have serious importance in convex theory because using these functions, we can derive more accurate inequalities compared to convex functions. Recently, some authors have discussed different classes and related inequalities of log-convex and log-nonconvex functions. Inspired by the above literature, Guo et al. [26] introduced log-$\hbar$-convex-IVF in 2018 and proved the HH-inequality for log-$\hbar$-convex IVFs, as follows:

Let $\mathcal{F}: [u, \vartheta] \subset \mathbb{R} \to \mathbb{R}_I^+$ be a log-$\hbar$-convex-IVF given by $\mathcal{F}(x) = [\mathcal{F}_*(x), \mathcal{F}^*(x)]$ for all $x \in [u, \vartheta]$, where $\mathcal{F}_*(x)$ is a log-$\hbar$-convex function and $\mathcal{F}^*(x)$ is a log-$\hbar$-concave function. If $\mathcal{F}$ is Riemann integrable, then

$$\mathcal{F}\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \supseteq exp\left[\frac{1}{\vartheta-u}\int_u^\vartheta ln\mathcal{F}(x)dx\right] \supseteq [\mathcal{F}(u)\mathcal{F}(\vartheta)]^{\int_0^1 \hbar(\tau)d\tau}. \quad (5)$$

Inspired by Costa and Roman-Flores [10], and Zhang et al. [38], we present LR-interval Jensen inequality, HH-inequality and HH-Fejer inequality for LR-log-$\hbar$-interval IVFs by means of pseudo order relation.

We urge the readers for further analysis of literature on the applications and properties of generalized convex functions, Jensen and HH-integral inequalities, see [27-48] and the references therein.

This paper is organized, as follows: Section 2 presents preliminary and results in interval space and for Riemann integrals. Section 3 also introduces the new classes of log-$\hbar$-convex functions and investigates its properties. Section 4 obtains LR-interval Hermite-Hadamard inequalities via LR-log-$\hbar$-convex IVSs. In addition, some interesting examples are also presented to verify our results. Section 5 is dedicated to conclusions and future works.

2 Preliminaries

In this section, we recall some basic preliminary notions, definitions and results. Based on these results, some new basic definitions and results are also discussed.

We begin by recalling basic notations and definitions. We define interval,

$[r_*, r^*] = \{x \in \mathbb{R}: r_* \le x \le r^* \text{ and } r_*, r^* \in \mathbb{R}\}$, where $r_* \le r^*$.

We write $\text{len}[r_*, r^*] = r^* - r_*$, If $\text{len}\ [r_*, r^*] = 0$, then $[r_*, r^*]$ is called degenerate. In this paper, all intervals will be non-degenerate intervals. The collection of all closed and bounded intervals of $\mathbb{R}$ is defined as $\mathbb{R}_I = \{[r_*, r^*]: r_*, r^* \in \mathbb{R} \text{ and } r_* \le r^*\}$. If $r_* \ge 0$, then $[r_*, r^*]$ is called positive interval. The set of all positive interval is denoted by $\mathbb{R}_I^+$ and defined as $\mathbb{R}_I^+ = \{[r_*, r^*]: [r_*, r^*] \in \mathbb{R}_I \text{ and } r_* \ge 0\}$.

Now we discuss some properties of intervals under the arithmetic operations addition, multiplication and scalar multiplication. If $[r_*, r^*]$, $[s_*, s^*] \in \mathbb{R}_I$ and $\rho \in \mathbb{R}$, then arithmetic operations are defined by

$[r_*, r^*] + [s_*, s^*] = [r_* + s_*,\ r^* + s^*]$, $[r_*, r^*] \times [s_*, s^*] = [min\{r_*s_*, r^*s_*, r_*s^*, r^*s^*\}, max\{r_*s_*, r^*s_*, r_*s^*, r^*s^*\}]$,

$$\rho.[r_*, r^*] = \begin{cases} [\rho r_*, \rho r^*] & \text{if } \rho > 0, \\ \{0\} & \text{if } \rho = 0, \\ [\rho r^*, \rho r_*] & \text{if } \rho < 0. \end{cases}$$

For $[r_*, r^*]$, $[s_*, s^*] \in \mathbb{R}_I$, the inclusion " $\subseteq$ " is defined by

$[r_*, r^*] \subseteq [s_*, s^*]$, if and only if $s_* \le r_*, r^* \le s^*$.

Remark 2.1. [24] (i) The relation " $\le_p$ " defined on $\mathbb{R}_I$ by

$[r_*, r^*] \le_p [s_*, s^*]$ if and only if $r_* \le s_*,\ r^* \le s^*$,

for all $[r_*, r^*]$, $[s_*, s^*] \in \mathbb{R}_I$, it is a pseudo order relation. For given $[r_*, r^*]$, $[s_*, s^*] \in \mathbb{R}_I$, we say that $[r_*, r^*] \le_p [s_*, s^*]$ if and only if $r_* \le s_*,\ r^* \le s^*$ or $r_* \le s_*,\ r^* < s^*$. The relation $[r_*, r^*] \le_p [s_*, s^*]$ is coincident to $[r_*, r^*] \le [s_*, s^*]$ on $\mathbb{R}_I$.

(ii) It can be easily seen that " $\le_p$ " looks like "left and right" on the real line $\mathbb{R}$, so we call " $\le_p$ " is "left and right" (or "LR" order, in short).

The concept of Riemann integral for IVF first introduced by Moore [15] is defined, as follows:

Theorem 2.2. [15] If $f: [u, \vartheta] \subset \mathbb{R} \to \mathbb{R}_I$ is an IVF on such that $f(x) = [f_*, f^*]$. Then, f is Riemann integrable over $[u, \vartheta]$ if and only if f_* and f^* are Riemann integrable over $[u, \vartheta]$ such that

$(IR)\int_u^\vartheta f(x)dx = \left[(R)\int_u^\vartheta f_*(u)dx\,,(R)\int_u^\vartheta f^*(u)dx\right].$

The collection of all Riemann integrable real valued functions and Riemann integrable IVF is denoted by $\mathcal{R}_{[u,\vartheta]}$ and $\mathcal{IR}_{[u,\vartheta]}$, respectively.

Definition 2.3. [36] A function $f:[u,\vartheta]\to\mathbb{R}^+$ is said to be log-convex function if

$$f(\lambda x+(1-\lambda)y)\le f(x)^\lambda f(y)^{1-\lambda},\quad \forall\, x,y\in[u,\vartheta],\lambda\in[0,1], \tag{6}$$

where $f(x)\ge 0$. If (6) is reversed, then f is called log-concave.

Definition 2.4. [34] A function $f:[u,\vartheta]\to\mathbb{R}^+$ is said to be log-P-convex function if

$$f(\lambda x+(1-\lambda)y)\le f(x)f(y),\quad \forall\, x,y\in[u,\vartheta],\lambda\in[0,1]. \tag{7}$$

If (7) is reversed, then f is called log-P-concave.

Definition 2.5. [39] A function $f:K\to\mathbb{R}^+$ is said to be log-s-convex function in the second sense if

$$f(\lambda x+(1-\lambda)y)\le f(x)^{\lambda^s}f(y)^{(1-\lambda)^s},\ \forall\, x,y\in[u,\vartheta],\lambda\in[0,1], \tag{8}$$

where $s\in(0,1)$. If (8) is reversed, then f is called log-s-concave.

Definition 2.6. [34] A function $f:[u,\vartheta]\to\mathbb{R}^+$ is said to be log-$\hbar$-convex function if for all $x,y\in[u,\vartheta],\lambda\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)y)\le f(x)^{\hbar(\lambda)}f(y)^{\hbar(1-\lambda)}, \tag{9}$$

where $\hbar:L\to\mathbb{R}^+$. If (9) is reversed, then f is called log-$\hbar$-concave.

If inequality (9) is reversed, then f is said to be log-$\hbar$-concave on $[u,\vartheta]$. f is log-$\hbar$-affine if and only if it is both log-$\hbar$-convex and log-$\hbar$-concave. The set of all log-$\hbar$-convex (log-$\hbar$-concave, log-$\hbar$-affine) functions is denoted by

$SX([u,\vartheta],\mathbb{R}^+,log-\hbar)$ $\big(SV([u,\vartheta],\mathbb{R}^+,log-\hbar),$ $SA([u,\vartheta],\mathbb{R}^+,log-\hbar)\big)$.

A function $\hbar:L\to\mathbb{R}^+$ is called super multiplicative if for all $x,y\in L$, we have

$$\hbar(xy)\ge\hbar(x)\hbar(y). \tag{10}$$

If (10) is reversed then, $\hbar$ is known as sub multiplicative. If the equality holds in (10) then, $\hbar$ is called multiplicative

Definition 2.7. Let $\hbar:L\to\mathbb{R}^+$ such that $\hbar\not\equiv 0$. Then, an IVF $f:[u,\vartheta]\to\mathbb{R}_I^+$ is said to be LR-log-$\hbar$-convex if for all $x,y\in[u,\vartheta],\lambda\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)y)\le_p f(x)^{\hbar(\lambda)}f(y)^{\hbar(1-\lambda)}. \tag{11}$$

If inequality (11) is reversed, then f is said to be LR-log-$\hbar$-concave on $[u,\vartheta]$. f is LR-log-$\hbar$-affine if and only if it is both LR-log-$\hbar$-convex and LR-log-$\hbar$-concave. The set of all LR-log-$\hbar$-convex (LR-log-$\hbar$-concave, LR-log-$\hbar$-affine) functions is denoted by

$LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$ $\big(LRSV([u,\vartheta],\mathbb{R}_I^+,log-\hbar),$ $LRSA([u,\vartheta],\mathbb{R}_I^+,log-\hbar)\big)$.

Remark 2.8. If $\hbar(\lambda)=\lambda^s$ with $s\in(0,1)$, then LR-log-$\hbar$-convex-IVF becomes LR-log-s-convex-IVF in the second sense, i.e.

$$f(\lambda x+(1-\lambda)y)\le_p f(x)^{\lambda^s}f(y)^{(1-\lambda)^s},\forall\, x,y\in K,\lambda\in[0,1]. \tag{12}$$

If $\hbar(\lambda)=\lambda$, then LR-log-$\hbar$-convex-IVF becomes LR-log-convex-IVF [30], i.e.

$$f(\lambda x+(1-\lambda)y)\le_p f(x)^{\lambda}f(y)^{1-\lambda},\forall\, x,y\in K,\lambda\in[0,1]. \tag{13}$$

If $\hbar(\lambda)\equiv 1$, then LR-log-$\hbar$-convex-IVF becomes LR-log-P-convex-IVF. That is,

$$f(\lambda x+(1-\lambda)y)\le_p f(x)f(y),\forall\, x,y\in K,\lambda\in[0,1]. \tag{14}$$

Theorem 2.9. Let $f:[u,\vartheta]\to\mathbb{R}_I^+$ be an IVF defined by $f(x)=[f_*(x),f^*(x)]$, for all $x\in[u,\vartheta]$. Then, $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$ if and only if $f_*,f^*\in SX([u,\vartheta],\mathbb{R}^+,log-\hbar)$.

Proof. Assume that $f_*,f^*\in SX([u,\vartheta],\mathbb{R}^+,log-\hbar)$. Then, for all $x,y\in[u,\vartheta],\lambda\in[0,1]$, we have

$$f_*(\lambda x+(1-\lambda)y)\le[f_*(x)]^{\hbar(\lambda)}[f_*(y)]^{\hbar(1-\lambda)},$$

and

$$f^*(\lambda x+(1-\lambda)y)\le[f^*(x)]^{\hbar(\lambda)}[f^*(y)]^{\hbar(1-\lambda)}.$$

From Definition 2.6 and order relation $\le_p$, we have

$$[f_*(\lambda x+(1-\lambda)y),f^*(\lambda x+(1-\lambda)y)]\le_p\left[[f_*(x)]^{\hbar(\lambda)}[f_*(y)]^{\hbar(1-\lambda)},\ [f^*(x)]^{\hbar(\lambda)}[f^*(y)]^{\hbar(1-\lambda)}\right]$$

$$=\left[[f_*(x)]^{\hbar(\lambda)},\ [f^*(x)]^{\hbar(\lambda)}\right]\left[[f_*(y)]^{\hbar(1-\lambda)},\ [f^*(y)]^{\hbar(1-\lambda)}\right],$$

i.e.

$$f(\lambda x+(1-\lambda)y)\le_p[f(x)]^{\hbar(\lambda)}[f(y)]^{\hbar(1-\lambda)},\forall\, x,y\in[u,\vartheta],\lambda\in[0,1].$$

Hence, $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$.

Conversely, let $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$. Then, for all $x,y\in[u,\vartheta]$ and $\lambda\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)y) \leq_p [f(x)]^{\hbar(\lambda)}[f(y)]^{\hbar(1-\lambda)},$$

that is

$[f_*(\lambda x+(1-\lambda)y), f^*(\lambda x+(1-\lambda)y)] \leq_p \left[[f_*(x)]^{\hbar(\lambda)}[f_*(y)]^{\hbar(1-\lambda)},\ [f^*(x)]^{\hbar(\lambda)}[f^*(y)]^{\hbar(1-\lambda)}\right].$

It follows that

$$f_*(\lambda x+(1-\lambda)y) \leq [f_*(x)]^{\hbar(\lambda)}[f_*(y)]^{\hbar(1-\lambda)},$$

and

$$f^*(\lambda x+(1-\lambda)y) \leq [f^*(x)]^{\hbar(\lambda)}[f^*(y)]^{\hbar(1-\lambda)}.$$

Hence, the result follows.

Example 2.10. We consider $\hbar(\lambda) \equiv m\,(m \geq 1)$, for $\lambda \in [0,1]$ and the IVF $f:[1,4] \to \mathbb{R}_I^+$ defined by, $f(x) = \left[e^x,\ e^{x^2}\right]$. Since end point functions $f_*(x)$ and $f^*(x)$ are log-$\hbar$-convex functions, then by Theorem 2.9, $f(x)$ is LR-log-$\hbar$-convex-IVF.

Remark 2.11. If $f_*(u) = f^*(\vartheta)$, then LR-log-$\hbar$-convex-IVF becomes log-$\hbar$-convex function, see [34].

If $f_*(u) = f^*(\vartheta)$ with $\hbar(\lambda) = \lambda^s$ with $s \in (0,1)$, then LR-log-$\hbar$-convex-IVF becomes log-s-convex function in the second sense, see [39].

If $f_*(u) = f^*(\vartheta)$ with $\hbar(\lambda) = \lambda$, then LR-log-$\hbar$-convex-IVF becomes log-convex function, see [36].

If $f_*(u) = f^*(\vartheta)$ with $\hbar(\lambda) \equiv 1$, then LR-log-$\hbar$-convex-IVF reduces to the log-P-convex function, see [34].

Theorem 2.12. Let $f:[u,\vartheta] \to \mathbb{R}_I^+$ be an IVF defined by $f(x) = [f_*(x), f^*(x)]$, for all $x \in [u,\vartheta]$. Then, $f \in LRSV([u,\vartheta], \mathbb{R}_I^+, log-\hbar)$ if and only if $f_*, f^* \in SV([u,\vartheta], \mathbb{R}^+, log-\hbar)$.

Proof. The proof is similar to that of Theorem 2.9.

Example 2.13. We consider the IVF $f:[u,\vartheta] = [0,4] \to \mathbb{R}_I^+$ defined by

$$f(x) = [-5x, -x],$$

where $\hbar(\lambda) = \lambda$, since end point functions $f_*(x) = -5x$, $f^*(x) = -x$ are log-$\hbar$-concave functions, then by Theorem 2.12, $f(x)$ is LR-log-$\hbar$-concave-IVF.

3 Main Results

In this section, we establish interval HH-inequality, interval HH-Fejer inequality and interval Jensen inequality, and their variant forms for LR-log-$\hbar$-convex-IVF. We also provide some nontrivial examples.

Theorem 3.1. (Interval HH-inequality for LR-log-$\hbar$-convex-IVF). Let $f:[u,\vartheta] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x), f^*(x)]$ for all $x \in [u,\vartheta]$ and $f \in \mathcal{IR}_{([u,\vartheta])}$. If $f \in LRSX([u,\vartheta], \mathbb{R}_I^+, log-\hbar)$, then

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^\vartheta lnf(x)dx\right]$$

$$\leq_p [f(u)f(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda}. \qquad (15)$$

If $f \in LRSV([u,\vartheta], \mathbb{R}_I^+, log-\hbar)$, then

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \geq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^\vartheta lnf(x)dx\right] \geq_p [f(u)\,f(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda}.$$

Proof. Let $f \in LRSX([u,\vartheta], \mathbb{R}_I^+, log-\hbar)$. Then, by hypothesis, we have

$$f\left(\frac{u+\vartheta}{2}\right) \leq_p [f(\lambda u+(1-\lambda)\vartheta)]^{\hbar\left(\frac{1}{2}\right)}\left[f\big((1-\lambda)u+\lambda\vartheta\big)\right]^{\hbar\left(\frac{1}{2}\right)}.$$

Then, we have

$$f_*\left(\frac{u+\vartheta}{2}\right) \leq [f_*(\lambda u+(1-\lambda)\vartheta)]^{\hbar\left(\frac{1}{2}\right)}\left[f_*\big((1-\lambda)u+\lambda\vartheta\big)\right]^{\hbar\left(\frac{1}{2}\right)},$$
$$f^*\left(\frac{u+\vartheta}{2}\right) \leq [f^*(\lambda u+(1-\lambda)\vartheta)]^{\hbar\left(\frac{1}{2}\right)}\left[f^*\big((1-\lambda)u+\lambda\vartheta\big)\right]^{\hbar\left(\frac{1}{2}\right)}. \qquad (16)$$

Taking logarithms on both sides of (16), we obtain

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\,lnf_*\left(\frac{u+\vartheta}{2}\right) \leq lnf_*(\lambda u+(1-\lambda)\vartheta) + lnf_*\big((1-\lambda)u+\lambda\vartheta\big),$$
$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\,lnf^*\left(\frac{u+\vartheta}{2}\right) \leq lnf^*(\lambda u+(1-\lambda)\vartheta) + lnf^*\big((1-\lambda)u+\lambda\vartheta\big).$$

Then,

$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\int_0^1 lnf_*\left(\frac{u+\vartheta}{2}\right)d\lambda \leq \int_0^1 lnf_*(\lambda u+(1-\lambda)\vartheta)d\lambda + \int_0^1 lnf_*\big((1-\lambda)u+\lambda\vartheta\big)d\lambda,$$
$$\frac{1}{\hbar\left(\frac{1}{2}\right)}\int_0^1 lnf^*\left(\frac{u+\vartheta}{2}\right)d\lambda \leq \int_0^1 lnf^*(\lambda u+(1-\lambda)\vartheta)d\lambda + \int_0^1 lnf^*\big((1-\lambda)u+\lambda\vartheta\big)d\lambda.$$

It follows that

$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}lnf_*\left(\frac{u+\vartheta}{2}\right) \leq \frac{1}{\vartheta-u}\int_u^\vartheta lnf_*(x)dx,$$
$$\frac{1}{2\hbar\left(\frac{1}{2}\right)}\,lnf^*\left(\frac{u+\vartheta}{2}\right) \leq \frac{1}{\vartheta-u}\int_u^\vartheta lnf^*(x)dx,$$

which implies that

$$f_*\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq exp\left(\frac{1}{\vartheta-u}\int_u^\vartheta lnf_*(x)dx\right),$$
$$f^*\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq exp\left(\frac{1}{\vartheta-u}\int_u^\vartheta lnf^*(x)dx\right).$$

That is,

$$\left[f_*\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}}, f^*\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}}\right] \leq_p \left[\begin{array}{l} exp\left(\frac{1}{\vartheta-u}\int_u^{\vartheta} lnf_*(x)dx\right), \\ exp\left(\frac{1}{\vartheta-u}\int_u^{\vartheta} lnf^*(x)dx\right)\end{array}\right].$$

Thus,

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^{\vartheta} lnf(x)dx\right]. \tag{17}$$

Similarly, we have

$$exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^{\vartheta} lnf(x)dx\right] \leq_p [f(u)f(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda}. \tag{18}$$

Combining (17) and (18), we have

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^{\vartheta} lnf(x)dx\right]$$

$$\leq_p [f(u)f(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda},$$

the required result.

Remark 3.2. If $\hbar(\lambda)=\lambda^s$ with $s\in(0,1)$, then Theorem 3.1 reduces to the result for LR-log-s-convex-IVF in the second sense:

$$f\left(\frac{u+\vartheta}{2}\right)^{2^{s-1}} \leq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^{\vartheta} lnf(x)dx\right] \leq_p [f(u)f(\vartheta)]^{\frac{1}{s+1}}.$$

If $\hbar(\lambda)=\lambda$, then Theorem 3.1 reduces to the result for LR-log-convex-IVF:

$$f\left(\frac{u+\vartheta}{2}\right) \leq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^{\vartheta} lnf(x)dx\right] \leq_p \sqrt{f(u)\,f(\vartheta)}.$$

If $\hbar(\lambda)\equiv 1$, then Theorem 3.1 reduces to the result for LR-log-P-convex-IVF;

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2}} \leq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^{\vartheta} lnf(x)dx\right] \leq_p f(u)\,f(\vartheta).$$

If $f_*(u)=f^*(\vartheta)$, then Theorem 3.1 reduces to the result for log-$\hbar$-convex function, see [34]:

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq exp\left[\frac{1}{\vartheta-u}\,(R)\int_u^{\vartheta} lnf(x)dx\right] \leq [f(u)f(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda}.$$

If $f_*(u)=f^*(\vartheta)$ and $\hbar(\lambda)=\lambda^s$, then Theorem 3.1 reduces to the result for log-s-convex function, see [34]:

$$f\left(\frac{u+\vartheta}{2}\right)^{2^{s-1}} \leq exp\left[\frac{1}{\vartheta-u}\,(R)\int_u^{\vartheta} lnf(x)dx\right] \leq [f(u)f(\vartheta)]^{\frac{1}{s+1}}.$$

If $f_*(u)=f^*(\vartheta)$ and $\hbar(\lambda)=\lambda$, then Theorem 3.1 reduces to the result for log-convex function, see [31]:

$$f\left(\frac{u+\vartheta}{2}\right) \leq exp\left[\frac{1}{\vartheta-u}\,(R)\int_u^{\vartheta} lnf(x)dx\right] \leq \sqrt{f(u)f(\vartheta)}.$$

If $f_*(u)=f^*(\vartheta)$ and $\hbar(\lambda)\equiv 1$, then Theorem 3.1 reduces to the result for log-P-convex function, see [34]:

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2}} \leq exp\left[\frac{1}{\vartheta-u}\,(R)\int_u^{\vartheta} lnf(x)dx\right] \leq f(u)f(\vartheta).$$

Example 3.3. We consider $\hbar(\lambda)=\lambda$, for $\lambda\in[0,1]$, and the IVF $f:[u,\vartheta]=[1,4]\to\mathbb{R}_I^+$ defined by, $f(x)=\left[e^{2x},\ e^{x^2}\right]$, then $f(x)$ is LR-log-$\hbar$-convex-IVF. Since $f_*(x)=e^{x}$ and $f^*(x)=e^{x^2}$ then we have

$$f_*\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} = \left[e^{\frac{5}{2}}\right]^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} = e^5,$$

$$exp\left(\frac{1}{\vartheta-u}\int_u^{\vartheta} lnf_*(x)dx\right) = exp\left(\frac{1}{3}\int_1^4 ln(e^x)dx\right) = e^5,$$

$$[f_*(u)f_*(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda} = [e^2.e^8]^{\frac{1}{2}} = e^5,$$

i.e.

$$e^5 \leq e^5 \leq e^5.$$

Similarly, it can be easily shown that

$$f^*\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq exp\left[\frac{1}{\vartheta-u}\int_u^{\vartheta} lnf^*(x)dx\right] \leq [f^*(u)+f^*(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda}.$$

Now,

$$f^*\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} = \left[e^{\left(\frac{5}{2}\right)^2}\right]^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} = e^{\frac{25}{4}},$$

$$exp\left(\frac{1}{\vartheta-u}\int_u^{\vartheta} lnf^*(x)dx\right) = exp\left(\frac{1}{3}\int_1^4 ln\left(e^{x^2}\right)dx\right) = e^7,$$

$$[f^*(u)f^*(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda} = [e.e^{16}]^{\frac{1}{2}} = e^{\frac{17}{2}}.$$

From which, it follows that

$$e^{\frac{25}{4}} \leq e^7 \leq e^{\frac{17}{2}}.$$

That is,

$$\left[e^5, e^{\frac{25}{4}}\right] \leq_p [e^5,\ e^7] \leq_p \left[e^5, e^{\frac{17}{2}}\right].$$

Hence,

$$f\left(\frac{u+\vartheta}{2}\right)^{\frac{1}{2\hbar\left(\frac{1}{2}\right)}} \leq_p exp\left[\frac{1}{\vartheta-u}\,(IR)\int_u^{\vartheta} f(x)dx\right]$$

$$\leq_p [f(u)f(\vartheta)]^{\int_0^1 \hbar(\lambda)d\lambda}.$$

Theorem 3.4. (Second HH-Fejer inequality for LR-log-$\hbar$-convex-IVF) Let $f:[u,\vartheta]\to\mathbb{R}_I^+$ be an IVF with $u<\vartheta$, such that $f(x)=[f_*(x),f^*(x)]$ for all $x\in[u,\vartheta]$ and $f\in\mathcal{IR}_{([u,\vartheta])}$. If $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$, then $\mathcal{W}:[u,\vartheta]\to\mathbb{R},\mathcal{W}(x)\geq 0$, symmetric with respect to $\frac{u+\vartheta}{2}$, then

$$\frac{1}{\vartheta-u}\,(IR)\int_u^\vartheta[lnf(x)]\mathcal{W}(x)dx \leq_p ln[f(u)f(\vartheta)]\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)d\lambda. \quad (19)$$

If $f\in LRSV([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$, then inequality (19) is reversed.

Proof. Let $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$. Then, we have

$$[lnf_*(\lambda u+(1-\lambda)\vartheta)]\mathcal{W}(\lambda u+(1-\lambda)\vartheta) \leq \big(\hbar(\lambda)\,lnf_*(u)+\hbar(1-\lambda)\,lnf_*(\vartheta)\big)\mathcal{W}\begin{pmatrix}\lambda u\\+(1-\lambda)\vartheta\end{pmatrix},$$

$$[lnf^*(\lambda u+(1-\lambda)\vartheta)]\mathcal{W}(\lambda u+(1-\lambda)\vartheta) \leq \big(\hbar(\lambda)\,lnf^*(u)+\hbar(1-\lambda)\,lnf^*(\vartheta)\big)\mathcal{W}\begin{pmatrix}\lambda u\\+(1-\lambda)\vartheta\end{pmatrix}, \quad (20)$$

and

$$\big[lnf_*\big((1-\lambda)u+\lambda\vartheta\big)\big]\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big) \leq \big(\hbar(1-\lambda)\,lnf_*(u)+\hbar(\lambda)\,lnf_*(\vartheta)\big)\mathcal{W}\begin{pmatrix}(1-\lambda)u\\+\lambda\vartheta\end{pmatrix}$$

$$\big[lnf^*\big((1-\lambda)u+\lambda\vartheta\big)\big]\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big) \leq \big(\hbar(1-\lambda)\,lnf^*(u)+\hbar(\lambda)\,lnf^*(\vartheta)\big)\mathcal{W}\begin{pmatrix}(1-\lambda)u\\+\lambda\vartheta\end{pmatrix}. \quad (21)$$

Adding (19) and (20), and integrating over $[0,1]$, we get

$$\int_0^1[lnf_*(\lambda u+(1-\lambda)\vartheta)]\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\,d\lambda + \int_0^1 lnf_*\big((1-\lambda)u+\lambda\vartheta\big)\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda$$
$$\leq\int_0^1\begin{bmatrix}lnf_*(u)\begin{Bmatrix}\hbar(\lambda)\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\\+\hbar(1-\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\end{Bmatrix}\\+lnf_*(\vartheta)\begin{Bmatrix}\hbar(1-\lambda)\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\\+\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\end{Bmatrix}\end{bmatrix}d\lambda,$$

$$\int_0^1\big[lnf^*\big((1-\lambda)u+\lambda\vartheta\big)\big]\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda + \int_0^1 lnf^*(\lambda u+(1-\lambda)\vartheta)\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\,d\lambda$$
$$\leq\int_0^1\begin{bmatrix}lnf^*(u)\begin{Bmatrix}\hbar(\lambda)\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\\+\hbar(1-\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\end{Bmatrix}\\+lnf^*(\vartheta)\begin{Bmatrix}\hbar(1-\lambda)\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\\+\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\end{Bmatrix}\end{bmatrix}d\lambda.$$

$$=2lnf_*(u)\int_0^1\hbar(\lambda)\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\,d\lambda +2lnf_*(\vartheta)\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda,$$
$$=2lnf^*(u)\int_0^1\hbar(\lambda)\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\,d\lambda +2lnf^*(\vartheta)\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda.$$

Since $\mathcal{W}$ is symmetric, then

$$\begin{aligned}&=2ln[f_*(u)f_*(\vartheta)]\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda,\\&=2ln[f^*(u)f^*(\vartheta)]\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda.\end{aligned} \quad (22)$$

Since

$$\begin{aligned}&\int_0^1[lnf_*(\lambda u+(1-\lambda)\vartheta)]\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\,d\lambda\\&\quad=\int_0^1\big[lnf_*\big((1-\lambda)u+\lambda\vartheta\big)\big]\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda\\&\quad=\frac{1}{\vartheta-u}\int_u^\vartheta[lnf_*(x)]\mathcal{W}(x)dx,\\&\int_0^1\big[lnf^*\big((1-\lambda)u+\lambda\vartheta\big)\big]\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda\\&\quad=\int_0^1[lnf^*(\lambda u+(1-\lambda)\vartheta)]\mathcal{W}(\lambda u+(1-\lambda)\vartheta)\,d\lambda\\&\quad=\frac{1}{\vartheta-u}\int_u^\vartheta[lnf^*(x)]\mathcal{W}(x)dx.\end{aligned} \quad (23)$$

From (22) and (23), we have

$$\frac{1}{\vartheta-u}\int_u^\vartheta[lnf_*(x)]\mathcal{W}(x)dx \leq ln[f_*(u)f_*(\vartheta)]\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda,$$
$$\frac{1}{\vartheta-u}\int_u^\vartheta[lnf^*(x)]\mathcal{W}(x)dx \leq ln[f^*(u)f^*(\vartheta)]\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)\,d\lambda.$$

That is,

$$\left[\frac{1}{\vartheta-u}\int_u^\vartheta[lnf_*(x)]\mathcal{W}(x)dx\,,\frac{1}{\vartheta-u}\int_u^\vartheta[lnf^*(x)]\mathcal{W}(x)dx\right]$$
$$\leq_p\big[ln[f_*(u)f_*(\vartheta)],\ ln[f^*(u)f^*(\vartheta)]\big]\int_0^1\hbar(\lambda)\,\mathcal{W}\begin{pmatrix}(1-\lambda)u\\+\lambda\vartheta\end{pmatrix}d\lambda.$$

Hence,

$$\frac{1}{\vartheta-u}\,(IR)\int_u^\vartheta[lnf(x)]\mathcal{W}(x)dx \leq_p ln[f(u)f(\vartheta)]\int_0^1\hbar(\lambda)\,\mathcal{W}\big((1-\lambda)u+\lambda\vartheta\big)d\lambda.$$

This concludes the proof.

Theorem 3.5. (First HH-Fejer inequality for LR-log-$\hbar$-convex-IVF) Let $f:[u,\vartheta]\to\mathbb{R}_I^+$ be an IVF with $u<\vartheta$, such that $f(x)=[f_*(x),f^*(x)]$ for all $x\in[u,\vartheta]$ and $f\in\mathcal{IR}_{([u,\vartheta])}$. If $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$ and $\mathcal{W}:[u,\vartheta]\to\mathbb{R},\mathcal{W}(x)\geq 0$, symmetric with respect to $\frac{u+\vartheta}{2}$, and $\int_u^\vartheta\mathcal{W}(x)dx>0$, then

$$lnf\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{2\hbar\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx} (IR)\int_u^\vartheta [lnf(x)]\mathcal{W}(x)dx. \quad (24)$$

If $f \in LRSV([u,\vartheta], \mathbb{R}_I^+, log-\hbar)$, then inequality (24) is reversed.

Proof. Since $f \in LRSX([u,\vartheta], \mathbb{R}_I^+, log-\hbar)$, then for $u,\vartheta \in [u,\vartheta]$, $\lambda \in [0,1]$, we have

$$\begin{aligned} &\frac{1}{\hbar\left(\frac{1}{2}\right)} lnf_*\left(\frac{u+\vartheta}{2}\right) \\ &\quad \leq lnf_*(\lambda u + (1-\lambda)\vartheta) + lnf_*((1-\lambda)u + \lambda\vartheta), \\ &\frac{1}{\hbar\left(\frac{1}{2}\right)} lnf^*\left(\frac{u+\vartheta}{2}\right) \\ &\quad \leq lnf^*(\lambda u + (1-\lambda)\vartheta) + lnf^*((1-\lambda)u + \lambda\vartheta), \end{aligned} \quad (25)$$

Multiplying (25) by $\mathcal{W}((1-\lambda)u+\lambda\vartheta) = \mathcal{W}(\lambda u + (1-\lambda)\vartheta)$ and integrating it by λ over $[0,1]$, we obtain

$$\begin{aligned} &\frac{1}{\hbar\left(\frac{1}{2}\right)}\left[lnf_*\left(\frac{u+\vartheta}{2}\right)\right]\int_0^1 \mathcal{W}((1-\lambda)u+\lambda\vartheta)d\lambda \\ &\quad \leq \int_0^1 [lnf_*(\lambda u + (1-\lambda)\vartheta)]\mathcal{W}(\lambda u + (1-\lambda)\vartheta)d\lambda \\ &\quad + \int_0^1 [lnf_*((1-\lambda)u+\lambda\vartheta)]\mathcal{W}\begin{pmatrix}(1-\lambda)u \\ +\lambda\vartheta\end{pmatrix}d\lambda, \\ &\frac{1}{\hbar\left(\frac{1}{2}\right)}\left[lnf^*\left(\frac{u+\vartheta}{2}\right)\right]\int_0^1 \mathcal{W}((1-\lambda)u+\lambda\vartheta)d\lambda \\ &\quad \leq \int_0^1 [lnf^*(\lambda u + (1-\lambda)\vartheta)]\mathcal{W}(\lambda u + (1-\lambda)\vartheta)d\lambda \\ &\quad + \int_0^1 [lnf^*((1-\lambda)u+\lambda\vartheta)]\mathcal{W}\begin{pmatrix}(1-\lambda)u \\ +\lambda\vartheta\end{pmatrix}d\lambda, \end{aligned} \quad (26)$$

Since

$$\begin{aligned} &\int_0^1 [lnf_*(\lambda u + (1-\lambda)\vartheta)]\mathcal{W}(\lambda u + (1-\lambda)\vartheta)\,d\lambda \\ &\quad = \int_0^1 [lnf_*((1-\lambda)u+\lambda\vartheta)]\mathcal{W}((1-\lambda)u+\lambda\vartheta)\,d\lambda \\ &\quad\quad = \frac{1}{\vartheta-u}\int_u^\vartheta [lnf_*(x)]\mathcal{W}(x)dx, \\ &\int_0^1 [lnf^*(\lambda u + (1-\lambda)\vartheta)]\mathcal{W}(\lambda u + (1-\lambda)\vartheta)\,d\lambda \\ &\quad = \int_0^1 [lnf^*((1-\lambda)u+\lambda\vartheta)]\mathcal{W}((1-\lambda)u+\lambda\vartheta)\,d\lambda \\ &\quad\quad = \frac{1}{\vartheta-u}\int_u^\vartheta [lnf^*(x)]\mathcal{W}(x)dx. \end{aligned} \quad (27)$$

Then, from (27), we have

$$lnf_*\left(\frac{u+\vartheta}{2}\right) \leq \frac{2\hbar\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx}\int_u^\vartheta [lnf_*(x)]\mathcal{W}(x)dx,$$

$$lnf^*\left(\frac{u+\vartheta}{2}\right) \leq \frac{2\hbar\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx}\int_u^\vartheta [lnf^*(x)]\mathcal{W}(x)dx.$$

From which, we have

$$\begin{aligned} &\left[lnf_*\left(\frac{u+\vartheta}{2}\right),\ lnf^*\left(\frac{u+\vartheta}{2}\right)\right] \\ &\quad \leq_p \frac{2\hbar\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx}\begin{bmatrix}\int_u^\vartheta [lnf_*(x)]\mathcal{W}(x)dx, \\ \int_u^\vartheta [lnf^*(x)]\mathcal{W}(x)dx\end{bmatrix}. \end{aligned}$$

That is,

$$lnf\left(\frac{u+\vartheta}{2}\right) \leq_p \frac{2\hbar\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx} (IR)\int_u^\vartheta [lnf(x)]\mathcal{W}(x)dx.$$

Then, we complete the proof.

Remark 3.6. If $\hbar(\lambda) = \lambda^s$ with $s \in (0,1)$, then inequalities in Theorem 3.4 and Theorem 3.5 reduce for LR-log-s-convex-IVFs in the second sense.

If $\hbar(\lambda) = \lambda$, then inequalities in Theorem 3.4 and Theorem 3.5 reduce for LR-log-convex-IVFs.

If $f_*(u) = f^*(u)$, then Theorem 3.4 and Theorem 3.5 reduce to classical first and second HH-Fejer inequality for log-$\hbar$-convex function, see [26].

If $f_*(u) = f^*(u)$ and $\hbar(\lambda) = \lambda^s$, then Theorem 3.4 and Theorem 3.5 reduce to classical second HH-Fejer inequality for log-s-convex function, see [38].

Example 3.7. We consider $\hbar(\lambda) = \lambda$, for $\lambda \in [0,1]$and the IVF $f: [u,\vartheta] = [1,8] \to \mathbb{R}_I^+$ defined by,

$$f(x) = \left[e^{2x}, e^{x^2}\right].$$

Since end point functions $f_*(x) = e^{2x}$ and $f^*(x) = e^{x^2}$ are log-$\hbar$-convex functions, then by Theorem 2.9, $f(x)$ is LR-log-$\hbar$-convex-IVF. If

$$\mathcal{W}(x) = \begin{cases} x-1, & \sigma \in \left[1, \frac{9}{2}\right] \\ 8-x, & \sigma \in \left(\frac{9}{2}, 8\right], \end{cases}$$

then we have

$$\begin{aligned} &\frac{1}{\vartheta-u}\int_u^\vartheta [lnf_*(x)]\mathcal{W}(x)dx = \frac{1}{7}\int_1^8 [lnf_*(x)]\mathcal{W}(x)dx \\ &\quad = \frac{1}{7}\int_1^{\frac{9}{2}} [lnf_*(x)]\mathcal{W}(x)dx + \frac{1}{7}\int_{\frac{9}{2}}^8 lnf_*(x)\mathcal{W}(x)dx, \\ &\frac{1}{\vartheta-u}\int_u^\vartheta [lnf^*(x)]\mathcal{W}(x)dx = \frac{1}{7}\int_1^8 [lnf^*(x)]\mathcal{W}(x)dx \\ &\quad = \frac{1}{7}\int_1^{\frac{9}{2}} [lnf^*(x)]\mathcal{W}(x)dx + \frac{1}{7}\int_{\frac{9}{2}}^8 [lnf^*(x)]\mathcal{W}(x)dx, \\ &= \frac{2}{7}\int_1^{\frac{9}{2}} [(x)](x-1)dx + \frac{2}{7}\int_{\frac{9}{2}}^8 (x)(8-x)dx = \frac{63}{4}, \\ &= \frac{1}{7}\int_1^{\frac{9}{2}} (x^2)(x-1)dx + \frac{1}{7}\int_{\frac{9}{2}}^8 (x^2)(8-x)dx \approx 39, \end{aligned} \quad (28)$$

and

$$ln[f_*(u)f_*(\vartheta)]\int_0^1 \hbar(\lambda)\,\mathcal{W}(u+\lambda\partial(\vartheta,u))\,d\lambda$$

$$ln[f^*(u)f^*(\vartheta)]\int_0^1 \hbar(\lambda)\,\mathcal{W}(u+\lambda\partial(\vartheta,u))\,d\lambda$$

$$= (66)\left[\int_0^{\frac{1}{2}} 7\lambda^2 dx + \int_{\frac{1}{2}}^1 \lambda(7-7\lambda)d\lambda\right] = \frac{63}{4}.$$
$$= (65)\left[\int_0^{\frac{1}{2}} 7\lambda^2 dx + \int_{\frac{1}{2}}^1 \lambda(7-7\lambda)d\lambda\right] = \frac{455}{8}. \tag{29}$$

Then, from (29), we have

$$\left[\frac{63}{4},\ 39\right] \leq_p \left[\frac{63}{4},\ \frac{455}{8}\right].$$

Hence, Theorem 3.4 is verified.

For Theorem 3.5, we have

$$lnf_*\left(\frac{u+\vartheta}{2}\right) = 9\,,$$
$$lnf^*\left(\frac{u+\vartheta}{2}\right) = \frac{81}{4}\,, \tag{30}$$

$$\int_u^\vartheta \mathcal{W}(x)dx = \int_1^{\frac{9}{2}}(x-1)dx + \int_{\frac{9}{2}}^8 (8-x)dx = \frac{49}{4},$$

$$\frac{2\hbar\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx}\int_u^\vartheta [lnf_*(x)]\mathcal{W}(x)dx = 9,$$
$$\frac{2\hbar\left(\frac{1}{2}\right)}{\int_u^\vartheta \mathcal{W}(x)dx}\int_u^\vartheta [lnf^*(x)]\mathcal{W}(x)dx \approx \frac{156}{7}. \tag{31}$$

From (30) and (31), we have

$$\left[9,\frac{81}{4}\right] \leq_p \left[9,\frac{156}{7}\right].$$

Hence, Theorem 3.5 is verified.

Theorem 3.8. (Jensen inequality for LR-log-$\hbar$-convex-IVF) Let $w_j \in \mathbb{R}^+$, $u_j \in [u,\vartheta]$, $(j = 1,2,3,\dots\dots\dots\dots k, k \geq 2)$ and let $f:[u,\vartheta] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x),\ f^*(x)]$ for all $x \in [u,\vartheta]$. If $f \in LRSX([u,\vartheta],\mathbb{R}_I^+, log-\hbar)$ and $\hbar$ is a nonnegative supermultiplicative function then

$$f\left(\frac{1}{W_k}\sum_{j=1}^k w_j\, u_j\right) \leq_p \prod_{j=1}^k [f(u_j)]^{\hbar\left(\frac{w_j}{W_k}\right)}, \tag{32}$$

where $W_k = \sum_{j=1}^k w_j$. I $\hbar$ is a nonnegative submultiplicative function f $f \in LRSV([u,\vartheta],\mathbb{R}_I^+, log-\hbar)$, Then Eq. (32) is reversed.

Proof. If $k = 2$, Then Eq. (32) is true. Consider Eq. (11) is true for $k = n-1$, then

$$f\left(\frac{1}{W_{n-1}}\sum_{j=1}^{n-1} w_j\, u_j\right) \leq_p \prod_{j=1}^{n-1}[f(u_j)]^{\hbar\left(\frac{w_j}{W_{n-1}}\right)},$$

Now, let us prove that Eq. (32) holds for $k = n$, such that

$$f\left(\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j\right) = f\left(\frac{1}{W_{n-2}}\sum_{j=1}^{n-2} w_j\, u_j + \frac{w_{n-1}+w_n}{W_n}\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1} + \frac{w_n}{w_{n-1}+w_n}u_n\right)\right).$$

Therefore, we have

$$f_*\left(\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j\right)$$
$$f^*\left(\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j\right)$$
$$\leq f_*\left(\frac{1}{W_n}\sum_{j=1}^{n-2} w_j\, u_j + \frac{w_{n-1}+w_n}{W_n}\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1} + \frac{w_n}{w_{n-1}+w_n}u_n\right)\right),$$
$$\leq f^*\left(\frac{1}{W_n}\sum_{j=1}^{n-2} w_j\, u_j + \frac{w_{n-1}+w_n}{W_n}\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1} + \frac{w_n}{w_{n-1}+w_n}u_n\right)\right),$$

$$\leq \left[\prod_{j=1}^{n-2}[f_*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)} f_*\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1} + \frac{w_n}{w_{n-1}+w_n}u_n\right)\right]^{\hbar\left(\frac{w_{n-1}+w_n}{W_n}\right)},$$
$$\leq \prod_{j=1}^{n-2}[f^*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}\left[f^*\left(\frac{w_{n-1}}{w_{n-1}+w_n}u_{n-1} + \frac{w_n}{w_{n-1}+w_n}u_n\right)\right]^{\hbar\left(\frac{w_{n-1}+w_n}{W_n}\right)},$$

$$\leq \prod_{j=1}^{n-2}[f_*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}\left[[f_*(u_{n-1})]^{\hbar\left(\frac{w_{n-1}}{w_{n-1}+w_n}\right)}[f_*(u_n)]^{\hbar\left(\frac{w_n}{w_{n-1}+w_n}\right)}\right]^{\hbar\left(\frac{w_{n-1}+w_n}{W_n}\right)},$$
$$\leq \prod_{j=1}^{n-2}[f^*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}\left[[f^*(u_{n-1})]^{\hbar\left(\frac{w_{n-1}}{w_{n-1}+w_n}\right)}[f^*(u_n)]^{\hbar\left(\frac{w_n}{w_{n-1}+w_n}\right)}\right]^{\hbar\left(\frac{w_{n-1}+w_n}{W_n}\right)},$$

$$\leq \prod_{j=1}^{n-2}[f_*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}[f_*(u_{n-1})]^{\hbar\left(\frac{w_{n-1}}{W_n}\right)}[f_*(u_n)]^{\hbar\left(\frac{w_n}{W_n}\right)},$$
$$\leq \prod_{j=1}^{n-2}[f^*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}[f^*(u_{n-1})]^{\hbar\left(\frac{w_{n-1}}{W_n}\right)}[f^*(u_n)]^{\hbar\left(\frac{w_n}{W_n}\right)},$$

$$= \prod_{j=1}^n [f_*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)},$$
$$= \prod_{j=1}^n [f^*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}.$$

From which, we have

$$\left[f_*\left(\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j\right), f^*\left(\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j\right)\right] \leq_p \left[\prod_{j=1}^n [f_*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}, \prod_{j=1}^n [f^*(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)}\right].$$

That is,

$$f\left(\frac{1}{W_n}\sum_{j=1}^n w_j\, u_j\right) \leq_p \prod_{j=1}^n [f(u_j)]^{\hbar\left(\frac{w_j}{W_n}\right)},$$

and the result follows.

If $w_1 = w_2 = w_3 = \dots\dots\dots\dots\dots = w_k = 1$, then Theorem 3.10 reduces to the following result:

Corollary 3.9. Let $u_j \in [u,\vartheta]$, $(j = 1,2,3,\dots\dots\dots\dots k, k \geq 2)$ and let $f:[u,\vartheta] \to \mathbb{R}_I^+$ be an IVF such that $f(x) = [f_*(x),\ f^*(x)]$ for all $x \in [u,\vartheta]$. If $f \in$

$LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$ and $\hbar$ is a nonnegative supermultiplicative function, then

$$f\left(\frac{1}{k}\sum_{j=1}^{k}u_j\right)\leq_p\prod_{j=1}^{k}[f(u_j)]^{\hbar\left(\frac{1}{k}\right)}. \tag{33}$$

If $f\in LRSV([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$, then Eq. (33) is reversed.

To obtain a refinement of Jensen inequality for LR-log-$\hbar$-convex-IVFs, we prove the following the result:

Theorem 3.10. Let $\hbar: L\to\mathbb{R}^+$ be a nonnegative supermultiplicative function and $f:[u,\vartheta]\to\mathbb{R}_I^+$ be an IVF such that $f(x)=[f_*(x),\ f^*(x)]$ for all $x\in[u,\vartheta]$. If $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$, then for $u_1,u_2,u_3\in[u,\vartheta]$, $u_1<u_2<u_3$ such that $u_3-u_1, u_3-u_2, u_2-u_1\in L$, we have

$$f(u_2)\leq_p f(u_1)^{\hbar(u_3-u_2)}f(u_3)^{\hbar(u_2-u_1)}. \tag{34}$$

If $f\in LRSV([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$, then Eq. (34) is reversed.

Proof. Let $u_1,u_2,u_3\in[u,\vartheta]$ and $\hbar(u_3-u_2)>0$. Then, by hypothesis, we have

$$\hbar\left(\frac{u_3-u_2}{u_3-u_1}\right)=\frac{\hbar(u_3-u_2)}{\hbar(u_3-u_1)} \text{ and } \hbar\left(\frac{u_2-u_1}{u_3-u_1}\right)=\frac{\hbar(u_2-u_1)}{\hbar(u_3-u_1)}.$$

Consider $\lambda=\frac{u_3-u_2}{u_3-u_1}$, then $u_2=\lambda u_1+(1-\lambda)u_3$. Since $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$, then by hypothesis, we have

$$f_*(u_2)\leq[f_*(u_1)]^{\hbar\left(\frac{u_3-u_2}{u_3-u_1}\right)}[f_*(u_3)]^{\hbar\left(\frac{u_2-u_1}{u_3-u_1}\right)},$$
$$f^*(u_2)\leq[f^*(u_1)]^{\hbar\left(\frac{u_3-u_2}{u_3-u_1}\right)}[f^*(u_3)]^{\hbar\left(\frac{u_2-u_1}{u_3-u_1}\right)}, \tag{35}$$

$$=[f_*(u_1)]^{\frac{\hbar(u_3-u_2)}{\hbar(u_3-u_1)}}[f_*(u_3)]^{\frac{\hbar(u_2-u_1)}{\hbar(u_3-u_1)}},$$
$$=[f^*(u_1)]^{\frac{\hbar(u_3-u_2)}{\hbar(u_3-u_1)}}[f^*(u_3)]^{\frac{\hbar(u_2-u_1)}{\hbar(u_3-u_1)}}. \tag{36}$$

From (36), we have

$$f_*(u_2)\leq[f_*(u_1)]^{\hbar(u_3-u_2)}[f_*(u_3)]^{\hbar(u_2-u_1)},$$
$$f^*(u_2)\leq[f^*(u_1)]^{\hbar(u_3-u_2)}[f^*(u_3)]^{\hbar(u_2-u_1)}.$$

That is

$$[f_*(u_2),\ f^*(u_2)]\leq_p\left[\begin{array}{l}[f_*(u_1)]^{\hbar(u_3-u_2)}[f_*(u_3)]^{\hbar(u_2-u_1)}\\ ,[f^*(u_1)]^{\hbar(u_3-u_2)}[f^*(u_3)]^{\hbar(u_2-u_1)}\end{array}\right].$$

Hence,

$$f(u_2)\leq_p f(u_1)^{\hbar(u_3-u_2)}f(u_3)^{\hbar(u_2-u_1)}.$$

Now, we obtain a refinement of Jensen inequality for LR-log-$\hbar$-convex-IVF which is given in the following results.

Theorem 3.11. Let $w_j\in\mathbb{R}^+$, $u_j\in[u,\vartheta]$, $(j=1,2,3,\dots\dots\dots k, k\geq2)$, $\hbar$ be an nonnegative supermultiplicative function, and let $f:[u,\vartheta]\to\mathbb{R}_I^+$ be an IVF such that $f(x)=[f_*(x),\ f^*(x)]$ for all $x\in[u,\vartheta]$. If $f\in LRSX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$ and $(L,U)\subseteq[u,\vartheta]$, then

$$\prod_{j=1}^{k}[f(u_j)]^{\hbar\left(\frac{w_j}{W_k}\right)}\leq_p$$
$$\prod_{j=1}^{k}\left([f(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\right), \tag{37}$$

where $W_k=\sum_{j=1}^{k}w_j$. If $f\in LRSV([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$, then Eq. (37) is reversed.

Proof. Consider $=u_1, u_j=u_2$, $(j=1,2,3,\dots\dots\dots\dots k)$, $U=u_3$. Then, by hypothesis and Eq. (35), we have

$$f_*(u_j)\leq[f_*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)}[f_*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)},$$
$$f^*(u_j)\leq[f^*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)}[f^*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)}.$$

The above inequality can be written as

$$f_*(u_j)^{\hbar\left(\frac{w_j}{W_k}\right)}\leq[f_*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f_*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)},$$
$$f^*(u_j)^{\hbar\left(\frac{w_j}{W_k}\right)}\leq[f^*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f^*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}. \tag{38}$$

Taking multiplication of all inequalities (38) for $j=1,2,3,\dots\dots\dots\dots k$, we have

$$\prod_{j=1}^{k}f_*(u_j)^{\hbar\left(\frac{w_j}{W_k}\right)}\leq\prod_{j=1}^{k}\left([f_*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f_*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\right),$$
$$\prod_{j=1}^{k}f^*(u_j)^{\hbar\left(\frac{w_j}{W_k}\right)}\leq\prod_{j=1}^{k}\left([f^*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f^*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\right).$$

That is,

$$\prod_{j=1}^{k}f(u_j)^{\hbar\left(\frac{w_j}{W_k}\right)}=\left[\prod_{j=1}^{k}f_*(u_j)^{\hbar\left(\frac{w_j}{W_k}\right)},\ \prod_{j=1}^{k}f^*(u_j)^{\hbar\left(\frac{w_j}{W_k}\right)}\right]$$

$$\leq_p\left[\prod_{j=1}^{k}\left([f_*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f_*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\right),\right.$$
$$\left.\prod_{j=1}^{k}\left([f^*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f^*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\right)\right],$$

$$\leq_p\prod_{j=1}^{k}\left(\left[[f_*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)},\right.\right.$$

$$[f^*(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\Big]\Big).\prod_{j=1}^{k}\Big(\Big[[f_*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)},$$

$$[f^*(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\Big]\Big),$$

$$=\prod_{j=1}^{k}[f(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}.\prod_{j=1}^{k}[f(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}.$$

Thus,

$$\prod_{j=1}^{k}[f(u_j)]^{\hbar\left(\frac{w_j}{W_k}\right)}\leq_p\prod_{j=1}^{k}\left([f(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\right).$$

This completes the proof.

Now, we consider some special cases of Theorem 3.8 and 3.11.

If $f_*(x)=f_*(x)$, then Theorem 3.8 and 3.11 reduce to :

Corollary 3.12. [26] (Jensen inequality for log-$\hbar$-convex function) Let $w_j\in\mathbb{R}^+$, $u_j\in[u,\vartheta]$, $(j=1,2,3,\ldots\ldots\ldots k, k\geq 2)$ and let $f:[u,\vartheta]\to\mathbb{R}^+$ be a non-negative real-valued function. If $f\in SX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$ and $\hbar$ is a nonnegative supermultiplicative function, then

$$f\left(\frac{1}{W_k}\sum_{j=1}^{k}w_j\,u_j\right)\leq\prod_{j=1}^{k}[f(u_j)]^{\hbar\left(\frac{w_j}{W_k}\right)}, \quad (39)$$

where $W_k=\sum_{j=1}^{k}w_j$. If $f\in SV([u,\vartheta],\mathbb{R}^+,log-\hbar)$, then Eq. (39) is reversed.

Corollary 3.13. [26] Let $w_j\in\mathbb{R}^+$, $u_j\in[u,\vartheta]$, $(j=1,2,3,\ldots\ldots\ldots k,k\geq 2)$, $\hbar$ be a nonnegative supermultiplicative function and let $f:[u,\vartheta]\to\mathbb{R}^+$ be an non-negative real-valued function. If $f\in SX([u,\vartheta],\mathbb{R}_I^+,log-\hbar)$ and $(L,U)\subseteq[u,\vartheta]$, then

$$\prod_{j=1}^{k}[f(u_j)]^{\hbar\left(\frac{w_j}{W_k}\right)}\leq$$
$$\prod_{j=1}^{k}\left([f(L)]^{\hbar\left(\frac{U-u_j}{U-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}[f(U)]^{\hbar\left(\frac{u_j-L}{M-L}\right)\hbar\left(\frac{w_j}{W_k}\right)}\right), \quad (40)$$

where $W_k=\sum_{j=1}^{k}w_j$. If $f\in SV([u,\vartheta],\mathbb{R}^+,log-\hbar)$, then Eq. (40) is reversed.
Note that, if $\hbar$ is a nonnegative multiplicative function , then results 3.8-3.13 reduce to new ones.

5 Conclusion and Future Plan

We have proposed a new class of log-convex-IVFs, which is called LR-log-$\hbar$-convex-IVFs. We have derived several new types of HH- and Jensen inequalities for this class. The examples helped show that our results include a wide class of new and known inequalities for LR-log-$\hbar$-convex-IVFs and their variant forms as special cases. In the future, we will attemt to explore this concept for fuzzy-interval-valued functions. We hope that the concepts and techniques of this paper will be starting point for further research in this area.

Acknowledgments: The authors would like to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan, for providing excellent research and academic environments.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or non-for-profit sectors.

Data Availability

No data were used to support this study.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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