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Solution of a Class of Advanced-Retarded Differential Equations

Héctor M. Moya-Cessa

Instituto Nacional de Astrofísica, Óptica y Electrónica, Calle Luis Enrique Erro 1, Santa María Tonantzintla, Puebla, 72840, Mexico

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Abstract: We give an operator solution to an advanced-retarded differential equation. The application of the operators involved produces a solution in terms of Bessel functions.

Keywords: Advanced-retarded differential equations, Operator techniques, Photonic circuits

1 Introduction

Recently it has been shown how photonic circuits may be constructed such that their dynamics obeys advanced-retarded (AR) differential equations [1]. Álvarez-Rodríguez *et al.* [1] showed the similarities between the AR differential equations and discrete differential equations that arise in the propagation of classical or quantum light through waveguide arrays [2,3,4,5].

AR differential equations, also known as mixed functional differential equations, are equations for which the derivative of the function explicitly depends on the same function evaluated at different values of the variable [6,7,8,9,10]. They are helpful to describe phenomena that involve feedback/feedforward interactions in their evolution [11,12,13].

2 Operator solution

We consider an AR differential equation of the following form

$$i\frac{dx(t)}{dt} = \alpha t x(t) + \lambda[x(t+\tau) + x(t-\tau)] \tag{1}$$

and, by defining the operator $D_t = \frac{d}{dt}$, with commutator $[D_t, t] = 1$, we can rewrite (1) as follows

$$i\frac{dx(t)}{dt} = [\alpha t + \lambda(e^{\tau D_t} + e^{-\tau D_t})]x(t) \tag{2}$$

that has the simple solution

$$x(t) = \frac{1}{N} e^{\beta \sinh(\tau D_t)} e^{-i\alpha \frac{t^2}{2}} x(0), \tag{3}$$

with $x(0)$ the initial condition, N a normalization constant and we have defined $\beta = \frac{2\lambda}{\alpha\tau}$. It is not difficult to show that this is a solution because

$$\frac{dx(t)}{dt} = -\frac{i}{N}\alpha e^{\beta \sinh(\tau D_t)} t e^{-i\alpha \frac{t^2}{2}} x(0), \tag{4}$$

that, by inserting a unit operator, $e^{-\beta \sinh(\tau D_t)} e^{\beta \sinh(\tau D_t)}$ after the linear term gives

$$\frac{dx(t)}{dt} = -\frac{i}{N}\alpha e^{\beta \sinh(\tau D_t)} t e^{-\beta \sinh(\tau D_t)} x(t), \tag{5}$$

and finally applying the relation $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots$, with A and B two arbitrary operators, equation (2) is recovered

$$\frac{dx(t)}{dt} = -\frac{i}{N}\alpha[t + \beta\tau\cosh(\tau D_t)]x(t). \tag{6}$$

Writing $e^{i\alpha \sinh(\tau D_t)}$ in terms of Bessel functions

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{n\tau D_t} e^{-i\alpha \frac{t^2}{2}} \tag{7}$$

we end up with the final form

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{-i\alpha \frac{(t+n\tau)^2}{2}}, \tag{8}$$

* Corresponding author e-mail: hmmc@inaoep.mx

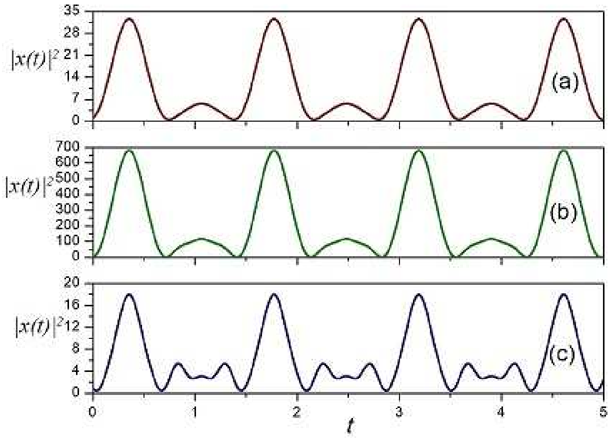

Fig. 1: $\alpha = 1.25$, $\tau = \sqrt{\pi}$ and (a) $\beta = 1.25$, (b) $\beta = 1.5$ and (c) $\beta = 2$

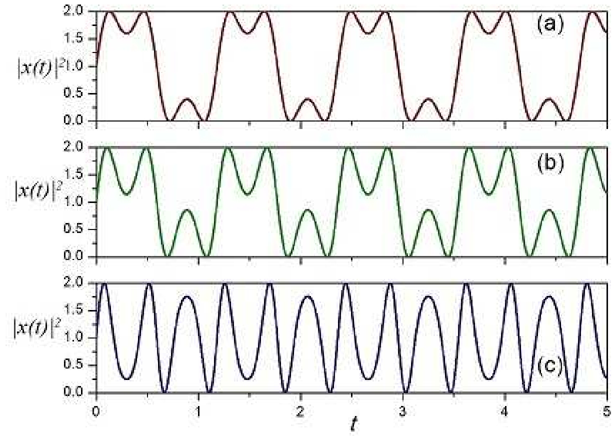

Fig. 2: $\alpha = 1.5$, $\tau = \sqrt{\pi}$ and (a) $\beta = 1.25$, (b) $\beta = 1.5$ and (c) $\beta = 2$

so that the normalization constant takes the value

$$N = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{-i\alpha \frac{n^2\tau^2}{2}}. \tag{9}$$

We plot in Figures 1-3 the absolute value squared of the amplitude $x(t)$, $|x(t)|^2$ for different values of α and β. In the figures it may be seen the periodic behaviour of the solutions, but also strong variations in the square amplitude, $|x(t)|^2$, although the parameters used are slightly modified. This is clear in Fig. 1. On the other hand, Fig. 3 (a) shows the solution given in Subsection 3.1.

3 Some special cases

For some sets of parameters the solution (8) may take some closed forms. In this Section we look at those cases:

3.1 $\alpha\tau^2 = 2\pi$

In this case we may write the solution as

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{-i\alpha \frac{(t^2+2nt\tau)}{2}} e^{-in^2\pi}, \tag{10}$$

and, because $e^{-in^2\pi} = e^{-in\pi}$ we rewrite the above-mentioned equation as follows

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{-i\alpha \frac{t^2}{2}} e^{-in(t\alpha\tau - \pi)}, \tag{11}$$

that may be added using the generating function of Bessel functions [14, 15, 16, 17]

$$x(t) = \frac{x(0)}{N} e^{-i\alpha \frac{t^2}{2}} e^{i\beta \sin(t\alpha\tau)}. \tag{12}$$

3.2 $\alpha\tau^2 = \pi$

The solution in this case takes the form

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{-i\alpha \frac{(t^2+2nt\tau)}{2}} e^{-in^2\frac{\pi}{2}}. \tag{13}$$

We see that the relevant term is $e^{-i\frac{\pi}{2}n^2} = (-i)^{n^2}$. For n odd gives $-i$ while for even gives 1, such that we can split (13) into even and odd series

$$x(t) = \frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_{2n}(\beta) e^{-i\alpha 2nt\tau} - i\frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_{2n+1}(\beta) e^{-i\alpha(2n+1)t\tau}. \tag{14}$$

We rewrite the above equation in the following form

$$x(t) = \frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{-i\alpha nt\tau}[1+(-1)^n] - i\frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{-i\alpha nt\tau}[1-(-1)^n], \tag{15}$$

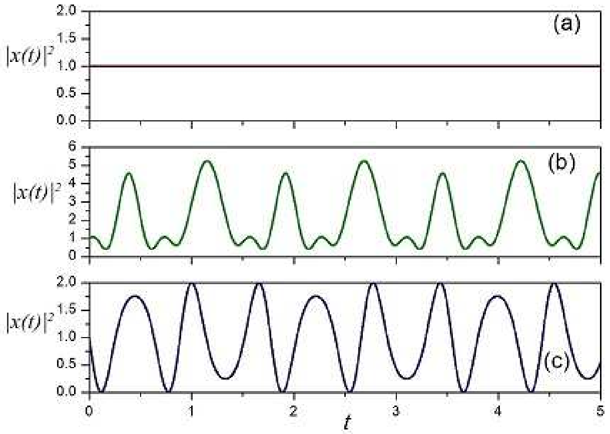

Fig. 3: $\alpha = 2$, $\beta = 2$ and (a) $\tau = \sqrt{\pi}$, (b) $\tau = \sqrt{\pi/3}$ and (c) $\tau = \sqrt{\pi/4}$

that, using the generating functions of Bessel functions [14, 15, 16, 17] gives

$$x(t) = \frac{x(0)e^{-i\alpha\frac{t^2}{2}}}{N} \left([1-i]e^{-i\beta\sin(t\alpha\tau)} + [1+i]e^{i\beta\sin(t\alpha\tau)}\right). \quad (16)$$

4 Conclusion

In the present paper was shown that we can use operator techniques to solve equation of the form given in equation (1). The solution produces infinite series of Bessel functions that for particular values of parameters may give a closed form. The series solution we have provided may be a hint to find solutions to other types of AR differential equations (different time dependent parameters).

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Conflict of Interest

The authors declare that they have no conflict of interest.

A

In this appendix we show, by using properties of the Bessel functions, that indeed, equation (8) is a solution to equation (1). Given the equation

$$x(t) = \frac{x(0)}{N} \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha\frac{(t+n\tau)^2}{2}}, \quad (17)$$

its derivative gives

$$\frac{dx(t)}{dt} = \frac{-i\alpha x(0)}{N} \sum_{n=-\infty}^{\infty} (t+n\tau)J_n(\beta)\, e^{-i\alpha\frac{(t+n\tau)^2}{2}}, \quad (18)$$

that by applying the identity $\frac{2n}{x}J_n(x) = J_{n+1}(x) + J_{n-1}(x)$ may be rewritten as

$$\frac{dx(t)}{dt} = -i\alpha t x(t) - \frac{i\alpha\tau\beta x(0)}{2N} \sum_{n=-\infty}^{\infty} [J_{n+1}(\beta) + J_{n-1}(\beta)]e^{-i\alpha\frac{(t+n\tau)^2}{2}}. \quad (19)$$

Changing the indices of the sums gives

$$\frac{dx(t)}{dt} = -i\alpha t x(t) - i\lambda\frac{x(0)}{N}\left[\sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha\frac{[t+(n-1)\tau]^2}{2}} - \sum_{n=-\infty}^{\infty} J_n(\beta)\, e^{-i\alpha\frac{[t+(n+1)\tau]^2}{2}}\right], \quad (20)$$

that gives equation (1).

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Héctor Moya-Cessa obtained his PhD at Imperial College in 1993 and since then he is a researcher/lecturer at Instituto Nacional de Astrofísica, Óptica y Electrónica in Puebla, Mexico where he works on Quantum Optics. He has published over 180 papers in international peer reviewed journals. He is fellow of the Alexander von Humboldt Foundation.