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Statistical Inferences Under Inverse Weibull Distribution Based on Generalized Type-II Progressive Hybrid Censoring Scheme

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Abstract: In this paper, point and interval estimation problems of the parameters of inverse Weibull (IW) distribution have been investigated based on generalized Type-II progressive hybrid censoring scheme(Generalized $Type-II\ PHCS$) using Bayesian and non-Bayesian approaches. The obtained results have been applied to a real data set as an illustrative example.

Keywords: Inverse Weibull distribution, Type-I progressive hybrid censoring scheme, Type-II progressive hybrid censoring scheme, Likelihood inference, Bayes inference, *MCMC* algorithm

1 Introduction

Suppose that n identical units, from certain distribution with PDF, $f(x;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector of parameters and RF, $R(x;\boldsymbol{\theta})$, are placed on a lifetime test. At the time of the i^{th} failure, R_i surviving units are randomly withdrawn from the experiment, $1 \leq i \leq r$. Thus, if r failures are observed, $R_1+R_2+...+R_r$ units are progressively censored, so $n=r+R_1+R_2+...+R_r$ and $X^{\boldsymbol{M}}_{1:r:n} < X^{\boldsymbol{M}}_{2:r:n} < ... < X^{\boldsymbol{M}}_{r:r:n}$ describe the progressively censored failure times, where $\boldsymbol{M}=(R_1,R_2,...,R_r)$ and $\sum_{i=1}^{r} R_i = n-r$.

The previous progressively type-II censored data can be written in the following form:
$\boldsymbol{x}=(x^{\boldsymbol{M}}_{1:r:n},x^{\boldsymbol{M}}_{2:r:n},...,x^{\boldsymbol{M}}_{r:r:n})$ which can be written for simplicity as $\boldsymbol{x}=(x_1,x_2,...,x_r)$. For more details, see [4] and [5].

Two new censoring schemes related to the previous censoring scheme are introduced.The first is called Type-I progressive hybrid censoring scheme $(Type-I\ PHCS)$ and the other is called Type-II progressive hybrid censoring scheme $(Type-II\ PHCS)$. The two models are studied in [6].

Lee et al. in [8] combined $Type-I\ PHCS$ and $Type-II\ PHCS$ to give a new censoring scheme called generalized Type-II progressive hybrid censoring scheme which can be described as follows: For fixed $r \in (1,2,...,n)$ and time points $T_1,T_2 \in (0,\infty)$ with $T1 < T2$. If the r^{th} failure occurs before the time point T_1, terminate the experiment at T_1. If the r^{th} failure occurs between T_1 and T_2, terminate the experiment at $X_{r:n}$. Finally, if the r^{th} failure occurs after T_2, terminate the experiment at T_2. Under this censoring scheme, one of the following three forms can be observed:

1. $0 < X_{r:n} < T_1 < T_2$ in which case we terminate at T_1,
2. $0 < T_1 < T_2 < X_{r:n}$ in which case we terminate at T_2,
3. $0 < T_1 < X_{r:n} < T_2$ in which case we terminate at $X_{r:n}$.

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Let d_i denote the number of failures until time T_i, $i = 1,2$. Then, the likelihood function of this generalized $Type-II$ $PHCS$ is as follows:

$$L(\boldsymbol{\theta}|data) \propto \begin{cases} \left[\prod_{i=1}^{r-1} f(x_i;\boldsymbol{\theta})(R(x_i;\boldsymbol{\theta}))^{R_i}\right]\left[\prod_{i=r}^{d_1} f(x_i;\boldsymbol{\theta})\right]\left[f(T_1;\boldsymbol{\theta})(R(T_1;\boldsymbol{\theta}))^{R^*_{d_1}}\right], \\ R^*_{d_1} = n - d_1 - \sum_{i=1}^{r-1} R_i, R_r = 0 \; for \; d_1 \geq r \; and d_1 = r, r+1, ..., n, \\ \left[\prod_{i=1}^{d_2} f(x_i;\boldsymbol{\theta})(R(x_i;\boldsymbol{\theta}))^{R_i}\right]\left[f(T_2;\boldsymbol{\theta})(R(T_2;\boldsymbol{\theta}))^{R^*_{d_2}}\right], \\ R^*_{d_2} = \sum_{i=d_2+1}^{r} R_i, d_2 = 1,2,...,r-1, \\ \prod_{i=1}^{r} f(x_i;\boldsymbol{\theta})(R(x_i;\boldsymbol{\theta}))^{R_i}, d_1 = 0,1,...,r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{1}$$

Special Cases:
From the proposed model, other well-known models can be obtained as special cases such as:

1. Type-I PHCS when $T_1 \to 0$.
2. Type-II PHCS when $T_2 \to \infty$.
3. Hybrid Type-I censoring scheme when $T_1 \to 0$, $R_i = 0, i = 1,2,...,r-1, R_r = n-r$.
4. Hybrid Type-II censoring scheme when $T_2 \to \infty$, $R_i = 0, i = 1,2,...,r-1, R_r = n-r$.

The prediction problem based on the previous censoring scheme under Burr-XII Distribution is explored in [3].

A random variable X has IW distribution with the parameters α and β ($IW(\alpha,\beta)$) if its probability density function (PDF) is given by

$$f(x;\alpha,\beta) = \alpha\,\beta\,x^{-\alpha-1}\,e^{-\beta\,x^{-\alpha}},\; x \geq 0,\; (\alpha > 0,\; \beta > 0). \tag{2}$$

The reliability function (RF) of this distribution can be written as

$$R(x;\alpha,\beta) = 1 - e^{-\beta\,x^{-\alpha}},\; x \geq 0,\; (\alpha > 0,\; \beta > 0). \tag{3}$$

For more details about IW distribution, some of its generalizations and related distributions with applications, see [9].

2 Point estimation

In this section, the estimates of the parameters α and β of the IW distribution have been obtained under the generalized $Type-II$ $PHCS$ using the maximum likelihood (ML) and Bayes (B) methods.

2.1 Maximum likelihood estimation

Let $\boldsymbol{x} = (x_1, x_2, ..., x_r)$ be a progressive type-II censored sample of failure times, distributed as IW distribution, of n items putted in life-time experiment with censored scheme $\boldsymbol{M} = (R_1, R_2, ..., R_r)$.

Under the generalized $Type-II$ $PHCS$, the likelihood function (LF) of the parameters α and β given the vector of observations $\boldsymbol{x}$ can be obtained by substituting from (2) and (3) in (1) to be of the form

$$L(\alpha,\beta|\boldsymbol{x}) \propto \begin{cases} \alpha^{d_1+1}\,\beta^{d_1+1}\left(1-e^{-\beta\,T_1^{-\alpha}}\right)^{R^*_{d_1}}\left[T_1\prod_{i=1}^{d_1} x_i\right]^{-\alpha-1}\prod_{i=1}^{r-1}\left(1-e^{-\beta\,x_i^{-\alpha}}\right)^{R_i}\times \\ e^{-\beta\,(T_1^{-\alpha}+\sum_{i=1}^{d_1} x_i^{-\alpha})}, R^*_{d_1} = n - d_1 - \sum_{i=1}^{r-1} R_i, R_r = 0 \; for \; d_1 \geq r \; and \\ d_1 = r, r+1, ..., n, \\ \alpha^{d_2+1}\,\beta^{d_2+1}\left(1-e^{-\beta\,T_2^{-\alpha}}\right)^{R^*_{d_2}}\left[T_2\prod_{i=1}^{d_2} x_i\right]^{-\alpha-1}\prod_{i=1}^{d_2}\left(1-e^{-\beta\,x_i^{-\alpha}}\right)^{R_i}\times \\ e^{-\beta(T_2^{-\alpha}+\sum_{i=1}^{d_2} x_i^{-\alpha})}, R^*_{d_2} = \sum_{i=d_2+1}^{r} R_i, d_2 = 1,2,...,r-1, \\ \alpha^r\,\beta^r\left[\prod_{i=1}^{r} x_i\right]^{-\alpha-1}\prod_{i=1}^{r}\left(1-e^{-\beta\,x_i^{-\alpha}}\right)^{R_i} e^{-\beta\,\sum_{i=1}^{r} x_i^{-\alpha}}, \\ d_1 = 0,1,...,r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{4}$$

Differentiating $LogL(\alpha,\beta|\boldsymbol{x})$ with respect to α and β then setting them to zero, a system of two nonlinear equations has been obtained and can be solved simultaneously using some iteration schemes, such as Newton-Raphson, to obtain the maximum likelihood estimates ($MLE's$) of α and β, denoted by $\hat{\alpha}$ and $\hat{\beta}$, respectively.

2.2 Bayes estimation

Let $u(\boldsymbol{\theta})$ be a general function of the vector of parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_m)$. Under the squared error loss (SEL) function and linear exponential *LINEX* loss function, the Bayes estimates of $u(\boldsymbol{\theta})$ are given, respectively, by

$$\hat{u}_S(\boldsymbol{\theta}) = E(u(\boldsymbol{\theta})|\,\boldsymbol{x}) = \int ... \int u(\boldsymbol{\theta})\,\pi^*(\boldsymbol{\theta}|\boldsymbol{x})\,d\theta_1 ... d\theta_m. \tag{5}$$

$$\hat{u}_L(\boldsymbol{\theta}) = \frac{-1}{a}\ln[E(e^{-a\,u(\boldsymbol{\theta})}|\boldsymbol{x})] = \frac{-1}{a}\ln\left[\int ... \int e^{-a\,u(\boldsymbol{\theta})}\,\pi^*(\boldsymbol{\theta}|\boldsymbol{x})\,d\theta_1 ... d\theta_m\right], \tag{6}$$

where $\pi^*(\boldsymbol{\theta}|\,\boldsymbol{x}) \propto \pi(\boldsymbol{\theta})L(\boldsymbol{\theta}|\,\boldsymbol{x})$ is the posterior *PDF* of the vector of parameters $\boldsymbol{\theta}$ given the vector of observations $\boldsymbol{x}$, $\pi(\boldsymbol{\theta})$ is a prior *PDF* of $\boldsymbol{\theta}$ and $L(\boldsymbol{\theta}|\boldsymbol{x})$ is the likelihood function of $\boldsymbol{\theta}$ given $\boldsymbol{x}$. The integrals are taken over the m-dimensional space R^m. To compute the integrals, Markov Chain Monte Carlo $(MCMC)$, method has been used to generate a random sample $[\boldsymbol{\theta}^i = (\theta_1^i, ..., \theta_m^i),\ i = 1, 2, ..., K]$ from the posterior *PDF* $\pi^*(\boldsymbol{\theta}|\,\boldsymbol{x})$ and then (5) and (6) have been written, respectively in the forms,

$$\hat{u}_S(\boldsymbol{\theta}) = \frac{\sum_{i=1}^{K} u(\boldsymbol{\theta}^i)}{K} \tag{7}$$

and

$$\hat{u}_L(\boldsymbol{\theta}) = (-1/a)\ln\left[\frac{1}{K}\sum_{i=1}^{K} e^{-a\,u(\boldsymbol{\theta}^i)}\right]. \tag{8}$$

To generate from the posterior *PDF*, $\pi^*(\boldsymbol{\theta}|\,\boldsymbol{x})$, Gibbs sampler and Metropolis-Hastings techniques have been used.
In this subsection the Bayes estimates $(BE's)$ of α and β have been obtained.
To estimate α and β, a function $u(\alpha, \beta)$ has been defined as

$$u(\alpha, \beta) = \alpha^{\delta_1}\,\beta^{\delta_2}. \tag{9}$$

The *BE* of $u(\alpha, \beta)$ has been obtained in two cases:

1. when $\delta_1 = 1, \delta_2 = 0$, which is equivalent to estimating α,
2. when $\delta_2 = 1, \delta_1 = 0$, which is equivalent to estimating β.

Using the bivariate prior suggested in [1]and [2] which is of the form

$$\pi(\alpha, \beta) \propto \alpha^{c_1+c_3-1}\,\beta^{c_3-1}e^{-\alpha\,(\beta+c_2)}, \alpha > 0,\ \beta > 0,\ (c_1 > 0,\ c_2 > 0,\ c_3 > 0), \tag{10}$$

where c_1, c_2 and c_3 are the prior parameters (also known as hyperparameters) and the *LF* (4). Then, the posterior *PDF* can be written in the form

$$\pi^*(\alpha, \beta|\boldsymbol{x}) = \begin{cases} A_1\,\alpha^{d_1+c_1+c_3}\,\beta^{d_1+c_3}\left(1 - e^{-\beta\,T_1^{-\alpha}}\right)^{R^*_{d_1}}\left[T_1\prod_{i=1}^{d_1} x_i\right]^{-\alpha-1}\prod_{i=1}^{r-1}\left(1 - e^{-\beta\,x_i^{-\alpha}}\right)^{R_i} \times \\ e^{-[c_2\,\alpha+\beta\,(T_1^{-\alpha}+\sum_{i=1}^{d_1} x_i^{-\alpha}+\alpha)]}, R^*_{d_1} = n - d_1 - \sum_{i=1}^{r-1} R_i, R_r = 0\ for\ d_1 \geq r\ and \\ d_1 = r, r+1, ..., n, \\ A_2\,\alpha^{d_2+c_1+c_3}\,\beta^{d_2+c_3}\left(1 - e^{-\beta\,T_2^{-\alpha}}\right)^{R^*_{d_2}}\left[T_2\prod_{i=1}^{d_2} x_i\right]^{-\alpha-1}\prod_{i=1}^{d_2}\left(1 - e^{-\beta\,x_i^{-\alpha}}\right)^{R_i} \times \\ e^{-[c_2\,\alpha+\beta(T_2^{-\alpha}+\sum_{i=1}^{d_2} x_i^{-\alpha}+\alpha)]}, R^*_{d_2} = \sum_{i=d_2+1}^{r} R_i, d_2 = 1, 2, ..., r-1, \\ A_3\,\alpha^{r+c_1+c_3-1}\,\beta^{r+c_3-1}\left[\prod_{i=1}^{r} x_i\right]^{-\alpha-1}\prod_{i=1}^{r}\left(1 - e^{-\beta\,x_i^{-\alpha}}\right)^{R_i} e^{-[c_2\,\alpha+\beta(\sum_{i=1}^{r} x_i^{-\alpha}+\alpha)]}, \\ d_1 = 0, 1, ..., r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{11}$$

where A_i are normalizing constants, $i = 1, 2, 3$.
By generating $(\alpha^{(1)}, \beta^{(1)}), (\alpha^{(2)}, \beta^{(2)}), ..., (\alpha^{(K)}, \beta^{(K)})$ from the posterior *PDF* (11) and using the function $u(\alpha, \beta)$ (9) in (7) and (8), the $BE's$ of the considered parameters have been obtained under *SEL* function and *LINEX* loss function, respectively, using *MCMC* algorithm.

3 Interval estimation

In this section, the approximate confidence interval (CI), bootstrap-p CI, Bayesian CI (credibility interval) and highest posterior density interval (HPD) have been studied for the two parameters α and β.

3.1 Approximate Confidence Intervals

Using the sample $\boldsymbol{x} = (x_1, x_2, ..., x_r)$ which represents a progressive type-II censored failure times from IW distribution with censored scheme $\boldsymbol{M} = (R_1, R_2, ..., R_r)$ and under the generalized $Type-II\ PHCS$, the $MLE's$ of the parameters α and β will be $\hat{\alpha}$ and $\hat{\beta}$, respectively.

With large censoring value r, $(\hat{\alpha},\hat{\beta}) \sim N((\alpha,\beta), I_0^{-1}(\hat{\alpha},\hat{\beta}))$, where $I_0(\hat{\alpha},\hat{\beta})$ is observed information matrix given by

$$I_0(\hat{\alpha},\hat{\beta}) = \begin{bmatrix} -\frac{\partial^2 Log(L(\alpha,\beta|\boldsymbol{x}))}{\partial\alpha^2} & -\frac{\partial^2 Log(L(\alpha,\beta|\boldsymbol{x}))}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 Log(L(\alpha,\beta|\boldsymbol{x}))}{\partial\alpha\partial\beta} & -\frac{\partial^2 Log(L(\alpha,\beta|\boldsymbol{x}))}{\partial\beta^2} \end{bmatrix}_{(\hat{\alpha},\hat{\beta})}. \tag{12}$$

The Approximate confidence intervals for α and β can be obtained, respectively, by

$$\hat{\alpha} \mp z_{\frac{\tau}{2}}\sqrt{v_{11}} \quad \texttt{and} \quad \hat{\gamma} \mp z_{\frac{\tau}{2}}\sqrt{v_{22}}, \tag{13}$$

where v_{11} and v_{22} are the elements on the main diagonal of the covariance matrix $I_0^{-1}(\hat{\alpha},\hat{\beta})$ and $z_{\frac{\tau}{2}}$ is the standard normal variate.

3.2 Bootstrap confidence intervals

In this subsection, confidence intervals based on the parametric percentile bootstrap method $(Bootstrap-p)$ have been obtained based on the idea of Efron [7]. The algorithms for estimating the confidence intervals of the parameters using $Bootstrap-p$ method are illustrated as the following:

1. From the original data $\boldsymbol{x} = (x_1, x_2, ..., x_r)$ compute the $MLE's$ of the parameters α and β, $\hat{\alpha}$ and $\hat{\beta}$, respectively.
2. Using $\hat{\alpha}$ and $\hat{\beta}$, a bootstrap sample of upper ordered values $\boldsymbol{x}^*$ is generated .
3. As in Step 1, based on $\boldsymbol{x}^*$, compute the bootstrap sample estimates of α and β say $\hat{\alpha}^*$ and $\hat{\beta}^*$.
4. Repeat Steps 2 and 3 N times representing N bootstrap $MLE's$ of α and β based on N bootstrap samples.
5. Arrange all $\hat{\alpha}^*$'s and $\hat{\beta}^*$'s in an ascending order to obtain the bootstrap samples $(\hat{\alpha}^{*1}, \hat{\alpha}^{*2}, ..., \hat{\alpha}^{*N})$ and $(\hat{\beta}^{*1}, \hat{\beta}^{*2}, ..., \hat{\beta}^{*N})$.
6. A two-sided $(1-\tau)\times 100\%$ $bootstrap-p$ confidence interval of α, say $[\alpha_L^*, \alpha_U^*]$ is then given by $[\hat{\alpha}^{*N(\tau/2)}, \hat{\alpha}^{*N(1-\tau/2)}]$.
7. Also, a two-sided $(1-\tau)\times 100\%$ $bootstrap-p$ confidence interval of β, say $[\beta_L^*, \beta_U^*]$ is given by $[\hat{\beta}^{*N(\tau/2)}, \hat{\beta}^{*N(1-\tau/2)}]$.

3.3 Credibility intervals

For a specified value of τ, we define the $(1-\tau)\times 100\%$ CI (L_α, U_α) for α and $(1-\tau)100\%$ CI (L_β, U_β) for β, respectively by

$$\begin{aligned} &\int_{L_\alpha}^{\infty} \pi_1^*(\alpha \mid \boldsymbol{x}) d\alpha = 1 - \tfrac{\tau}{2}, \quad \int_{U_\alpha}^{\infty} \pi_1^*(\alpha \mid \boldsymbol{x}) d\alpha = \tfrac{\tau}{2}, \\ &\int_{L_\beta}^{\infty} \pi_2^*(\beta \mid \boldsymbol{x}) d\beta = 1 - \tfrac{\tau}{2}, \quad \int_{U_\beta}^{\infty} \pi_2^*(\beta \mid \boldsymbol{x}) d\beta = \tfrac{\tau}{2}, \end{aligned} \tag{14}$$

where $\pi_1^*(\alpha \mid \boldsymbol{x})$ and $\pi_2^*(\beta \mid \boldsymbol{x})$ are the marginal $PDF's$ of α and β, respectively. In many cases it will be very difficult to obtain the marginal PDF from the posterior PDF. Hence, Gibbs sampler and Metropolis Hastings algorithms have been used to generate $(\alpha^1,\beta^1), (\alpha^2,\beta^2), ..., (\alpha^K,\beta^K)$ from $\pi^*(\alpha,\beta \mid \boldsymbol{x})$. Using these generated values of α and β, the marginal posteriors $PDF's$ of α and β given $\boldsymbol{x}$ have been given by

$$\pi_1^*(\alpha \mid \boldsymbol{x}) = \frac{1}{K}\sum_{i=1}^{K} \pi^*(\alpha,\beta^i \mid \boldsymbol{x}), \quad \pi_2^*(\beta \mid \boldsymbol{x}) = \frac{1}{K}\sum_{i=1}^{K} \pi^*(\beta,\alpha^i \mid \boldsymbol{x}). \tag{15}$$

Substituting from (15) in (14), simple formulas have been obtained to compute the credibility intervals for α and β in the following form

$$\begin{array}{ll} \frac{1}{K}\sum_{i=1}^{K}\int_{L_\alpha}^{\infty}\pi^*(\alpha,\beta^i\mid\boldsymbol{x})d\alpha = 1-\frac{\tau}{2}, & \frac{1}{K}\sum_{i=1}^{K}\int_{U_\alpha}^{\infty}\pi^*(\alpha,\beta^i\mid\boldsymbol{x})d\alpha = \frac{\tau}{2}, \\ \frac{1}{K}\sum_{i=1}^{K}\int_{L_\beta}^{\infty}\pi^*(\beta,\alpha^i\mid\boldsymbol{x})d\beta = 1-\frac{\tau}{2}, & \frac{1}{K}\sum_{i=1}^{K}\int_{U_\beta}^{\infty}\pi^*(\beta,\alpha^i\mid\boldsymbol{x})d\beta = \frac{\tau}{2}. \end{array} \tag{16}$$

Another method can be used to compute the $(1-\tau)\times 100\%$ *CI* (L_α, U_α) for α and $(1-\tau)100\%$ *CI* (L_β, U_β) for β. This method can be described as follows:

1.From the posterior *PDF*, generate $(\alpha^{(1)},\beta^{(1)}),(\alpha^{(2)},\beta^{(2)}),...,(\alpha^{(K)},\beta^{(K)})$.
2.Arrange all α's and β's in an ascending order to obtain the following samples $(\alpha_{(1)},\alpha_{(2)},...,\alpha_{(K)})$ and $(\beta_{(1)},\beta_{(2)},...,\beta_{(K)})$
3.A two-sided $(1-\tau)\times 100\%$ *CI* for α, say $[\alpha_L^*,\alpha_U^*]$ is then given by $[\alpha_{(K\tau/2)},\alpha_{(K(1-\tau/2))}]$.
4.Also, a two-sided $(1-\tau)\times 100\%$ *CI* for β, say $[\beta_L^*,\beta_U^*]$ is then given by $[\beta_{(K\tau/2)},\beta_{(K(1-\tau/2))}]$.

3.4 Highest posterior density intervals

A $(1-\tau)\times 100\%$ *HPD* interval for α is obtained by solving the two following nonlinear equations

$$\frac{1}{K}\sum_{i=1}^{K}\int_{L_\alpha}^{U_\alpha}\pi^*(\alpha,\beta^i\mid\boldsymbol{x})d\alpha = 1-\tau, \quad \sum_{i=1}^{K}\pi^*(L_\alpha,\beta^i\mid\boldsymbol{x}) = \sum_{i=1}^{K}\pi^*(U_\alpha,\beta^i\mid\boldsymbol{x}). \tag{17}$$

Similarly, the $(1-\tau)\times 100\%$ *HPD* interval for β is obtained by solving the two following nonlinear equations

$$\frac{1}{K}\sum_{i=1}^{K}\int_{L_\beta}^{U_\beta}\pi^*(\beta,\alpha^i\mid\boldsymbol{x})d\beta = 1-\tau, \quad \sum_{i=1}^{K}\pi^*(L_\beta,\alpha^i,\mid\boldsymbol{x}) = \sum_{i=1}^{K}\pi^*(U_\beta,\alpha^i\mid\boldsymbol{x}). \tag{18}$$

4 Results

4.1 Simulated results

In the following, the $MLE's$ and $BE's$ have been compared based on a Monte Carlo simulation as follows:

1. For a given set of prior parameters c_1, c_2 and c_3, the population parameters α and β have been generated from the joint prior (10).
2. Making use of α and β obtained in step 1, a progressive type-II censored sample with censoring scheme $\boldsymbol{M} = (R_1,R_2,...,R_m)$ from *IW* distribution has been generated.
3. For different values of T_1 and T_2, the $MLE's$ of α and β have been computed as explained in section (2.1).
4. For the same values of T_1 and T_2, the $BE's$ of α and β based on *SEL* function and *LINEX* loss function using *MCMC* method have been given, respectively, as explained in section (2.2).
5. The above steps (2-4) have been repeated N times.
6. If $\hat{\theta}_j$ is an estimate of θ, based on sample j, $j = 1,2,...,N$, the average estimate $\bar{\hat{\theta}}$ and the mean squared error (MSE) over the N samples have been given, respectively, by $\bar{\hat{\theta}} = \frac{1}{N}\sum_{j=1}^{N}\hat{\theta}_j$ and $MSE(\hat{\theta}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\theta}_j-\theta)^2$.
7. Using step 6, the quantities $MSE(\hat{\alpha})$ and $MSE(\hat{\beta})$ have been computed.
8. The approximate, *Bootstrap*, Bayes(credible) and *HPD* $CI's$ have been computed for different values of T_1 and T_2.
9. The lengths and the coverage probabilities $CP's$ of the previous $CI's$ have been computed.
10. The results have been summarized in Tables 1 and 2.

Table 1:- Averages and $MSE's$ of the ML and B estimates over 10000 samples (under SEL and $LINEX$ loss functions) $(a = 0,1,2)$, $(\alpha = 3.2981, \beta = 1.4921)$,$(c_1 = 1.6, c_2 = 1.5, c_3 = 1.75)$ based on Generalized $Type-II$ $PHCS$.

(T_1,T_2)				$(0.3,0.5)$		$(0.3,1.5)$	
$(n,r,\boldsymbol{M})$	$Method$			$\bar{\hat{\alpha}}, MSE(\hat{\alpha})$	$\bar{\hat{\beta}}, MSE(\hat{\beta})$	$\bar{\hat{\alpha}}, MSE(\hat{\alpha})$	$\bar{\hat{\beta}}, MSE(\hat{\beta})$
$(30,10,\boldsymbol{M}_1)$	ML			(3.45,0.6103)	(1.58,0.2002)	(3.59,0.3221)	(1.56,0.1102)
	B	SEL		(3.56,0.4219)	(1.58,0.0423)	(3.33,0.3120)	(1.60,0.0561)
		$LINEX$	$a = 0.00001$	(3.56,0.4219)	(1.58,0.0423)	(3.33,0.3120)	(1.60,0.0561)
			$a = 1.0$	(3.35,0.4113)	(1.56,0.0403)	(3.35,0.2151)	(1.61,0.0421)
			$a = 2.0$	(3.22,0.2091)	(1.57,0.0182)	(3.54,0.2012)	(1.54,0.0442)
$(30,20,\boldsymbol{M}_2))$	ML			(3.59,0.6112)	(1.60,0.1713)	(3.25,0.3001)	(1.59,0.0912)
	B	SEL		(3.27,0.3902)	(1.55,0.0392)	(3.55,0.2801)	(1.59,0.0513)
		$LINEX$	$a = 0.00001$	(3.27,0.3902)	(1.55,0.0392)	(3.55,0.2801)	(1.59,0.0513)
			$a = 1.0$	(3.27,0.3781)	(1.57,0.0321)	(3.61,0.2113)	(1.56,0.0182)
			$a = 2.0$	(3.17,0.2013)	(1.55,0.0132)	(3.54,0.2010)	(1.56,0.0121)
$(30,30,\boldsymbol{M}_3)$	ML			(3.53,0.3692)	(1.56,0.1210)	(3.55,0.2891)	(1.58,0.0816)
	B	SEL		(3.42,0.3441)	(1.58,0.0389)	(3.25,0.2212)	(1.58,0.0352)
		$LINEX$	$a = 0.00001$	(3.42,0.3441)	(1.58,0.0389)	(3.25,0.2212)	(1.58,0.0352)
			$a = 1.0$	(3.24,0.2012)	(1.59,0.0194)	(3.27,0.2021)	(1.57,0.0154)
			$a = 2.0$	(3.30,0.2004)	(1.59,0.0104)	(3.38,0.1615)	(1.60,0.0102)
(T_1,T_2)				$(1.3,2.5)$		$(0.1,3.5)$	
$(30,10,\boldsymbol{M}_1)$	ML			(3.56,0.3204)	(1.57,0.1106)	(3.32,0.2071)	(1.56,0.0237)
	B	SEL		(3.59,0.3014)	(1.56,0.0317)	(3.59,0.1716)	(1.58,0.0215)
		$LINEX$	$a = 0.00001$	(3.59,0.3014)	(1.56,0.0317)	(3.59,0.1716)	(1.58,0.0215)
			$a = 1.0$	(3.57,0.1821)	(1.58,0.0152)	(3.37,0.1391)	(1.55,0.0142)
			$a = 2.0$	(3.45,0.1621)	(1.59,0.0101)	(3.48,0.1044)	(1.59,0.0101)
$(30,20,\boldsymbol{M}_2)$	ML			(3.40,0.2713)	(1.59,0.0182)	(3.40,0.1714)	(1.56,0.0218)
	B	SEL		(3.14,0.2515)	(1.57,0.0122)	(3.38,0.1602)	(1.62,0.0157)
		$LINEX$	$a = 0.00001$	(3.14,0.2515)	(1.57,0.0122)	(3.38,0.1602)	(1.62,0.0157)
			$a = 1.0$	(3.34,0.1515)	(1.58,0.0114)	(3.31,0.1201)	(1.57,0.0101)
			$a = 2.0$	(2.99,0.1481)	(1.58,0.0113)	(3.31,0.1001)	(1.57,0.0061)
$(30,30,\boldsymbol{M}_3)$	ML			(3.32,0.2614)	(1.61,0.0143)	(3.31,0.1416)	(1.57,0.0213)
	B	SEL		(3.28,0.1839)	(1.58,0.0116)	(3.29,0.1318)	(1.58,0.0126)
		$LINEX$	$a = 0.00001$	(3.28,0.1839)	(1.58,0.0116)	(3.29,0.1318)	(1.58,0.0126)
			$a = 1.0$	(3.21,0.1339)	(1.57,0.0108)	(3.31,0.1001)	(1.57,0.0101)
			$a = 2.0$	(3.28,0.1096)	(1.57,0.0091)	(3.24,0.0911)	(1.58,0.0031)

where, $\boldsymbol{M}_1 = (1,4,0,2,3,4,0,3,1,2)$, $\boldsymbol{M}_2 = (1,0,0,2,0,2,0,0,0,1,0,0,1,0,0,2,0,0,1,0)$ and $\boldsymbol{M}_3 = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$.
From tables 1 and 2, observe the following:

1. For fixed T_1 and T_2, the $MSE's$ of the $BE's$ based on SEL function and $LINEX$ loss function are less than that obtained for the $MLE's$ which means that the $BE's$ are better than the $MLE's$.
2. For fixed T_1 and T_2, the $MSE's$ of the $BE's$ based on $LINEX$ loss function decrease by increasing a.
3. For fixed T_1 and T_2, the $MSE's$ of the $BE's$ based on $LINEX$ loss function are the same as that obtained based on SEL function when $a \to 0$.
4. For fixed T_1 and T_2, the lengths of the $CI's$ decrease by increasing r.
5. For fixed T_1 and T_2, the length of the approximate $CI >$ that computed for the $bootstrap-p$ $CI>$ that computed for Bayes(credible) $CI >$ that computed for the HPD interval.

Table 2:- $CI's$ of the parameters α and β.

(T_1,T_2)		$(0.3,0.5)$		$(0.3,1.5)$	
$(n,r,\boldsymbol{M})$	*Method*	(L_α,U_α) *Length* *CP*	(L_β,U_β) *Length* *CP*	(L_α,U_α) *Length* *CP*	(L_β,U_β) *Length* *CP*
$(30,10,\boldsymbol{M}_1)$	*Approximate CI*	(0.5512,4.5530) 4.0018 93.66	(0.9015,2.6927) 1.7912 94.19	(1.9319,4.0426) 2.1107 94.81.4	(0.8192,1.9693) 1.1501 93.55
	Bootstrap − p CI	(0.4427,3.7617) 3.3190 95.95	(0.9014,2.1815) 1.2801 95.89	(1.8012,3.8031) 2.0019 96.97	(1.0176,2.1185) 1.1009 95.98
	Credible CI	(2.9752,4.1499) 1.1747 97.98	(1.1770,2.3594) 1.1824 96.29	(1.9621,3.6346) 1.6725 97.16	(1.3739,2.3885) 1.0146 98.15
	HPD CI	(2.0102,3.2211) 1.2109 95.71	(1.2109,2.2438) 1.0329 95.87	(2.8121,3.6735) 0.8614 96.74	(1.2105,2.0281) 0.8176 96.54
$(30,20,\boldsymbol{M}_2)$	*Approximate CI*	(0.7017,3.4215) 2.7198 96.93	(0.6102,1.6284) 1.0182 96.03	(1.3192,3.3306) 2.0114 96.91	(1.0182,2.1001) 1.0819 96.93
	Bootstrap − p CI	(1.2901,3.4003) 2.1102 96.78	(1.1185,2.3294) 1.2109 96.89	(1.3192,3.2107) 1.8915 95.99	(1.3192,2.3379) 1.0187 97.69
	Credible CI	(2.9077,4.2484) 1.3407 96.28	(1.0116,2.2064) 1.1948 95.33	(2.6464,3.5976) 0.9512 98.76	(1.4149,2.3628) 0.9479 96.32
	HPD CI	(2.2180,3.2592) 1.0412 96.21	(1.3187,2.2113) 0.8926 95.17	(2.5021,3.5131) 1.0110 97.36	(1.6125,2.2840) 0.6715 95.56
$(30,30,\boldsymbol{M}_3)$	*Approximate CI*	(1.2519,3.5426) 2.2907 95.41	(0.9107,1.8914) 0.9807 95.98	(1.3017,3.2820) 1.9803 95.04	(0.7908,1.8325) 1.0417 95.94
	Bootstrap − p CI	(2.4331,3.3304) 0.8973 97.03	(0.9838,1.8940) 0.9102 96.92	(2.5184,3.4915) 0.9731 96.06	(1.0539,1.9743) 0.9204 96.86
	Credible CI	(2.8620,3.5376) 0.7756 96.8	(1.1998,2.0500) 0.8502 98.98	(2.8630,3.5714) 0.7084 98.76	(1.4153,2.2765) 0.8612 97.01
	HPD CI	(2.7012,3.3228) 0.6216 96.2	(1.4183,2.1299) 0.7116 97.33	(2.8298,3.3541) 0.5243 96.83	(1.4414,2.0752) 0.6338 96.21

4.2 Data Analysis

In this section, a real data set of 100 observations from *IW* distribution have been introduced. These real data are from [2] For $T_1 = 0.2$, $T_2 = 1.5$, $r = 30$ and censoring scheme $\boldsymbol{M} = (3,0,2,0,0,2,3,2,0,6,2,1,4,4,1,0,5,3,2,4,0,1,5,0,6,1,3,2,6,2)$, a generated $Type-II\ PHCS$ from the real data can be obtained. The results of the point and interval estimation have been shown in Tables 3 and 4.

Table 3:- Estimates of the parameters α and β using *ML* and *B* methods (under *SEL* and *LINEX* loss functions) $(a = 0,1,2)$ based on Generalized $Type-II\ PHCS$ from real data.

(T_1,T_2)				$(0.2,1.5)$	
(n,r)	*Method*			$\hat{\alpha}$	$\hat{\beta}$
$(100,30)$	*ML*			2.6188	0.06132
	B	*SEL*		2.7716	0.05819
		LINEX	$a = 0.00001$	2.7716	0.05819
			$a = 1.0$	2.8019	0.06904
			$a = 2.0$	2.7916	0.06718

Table 4:- $CI's$ of the parameters α and β based on Generalized $Type-II\ PHCS$ from real data.

(T_1,T_2)		$(0.2,1.5)$	
(n,r)	$Method$	(L_α,U_α) $Length$	(L_β,U_β) $Length$
$(100,30)$	$Approximate CI$	(1.0155, 3.1042) 2.0887	(0.0112, 0.2016) 0.1904
	$Bootstrap-p\ CI$	(1.1031,2.9048) 1.8017	(0.0141,0.1759) 0.1618
	$Credible CI$	(1.5504,3.1521) 1.6017	(0.0206,0.1750) 0.1544
	$HPD\ CI$	(1.8093, 3.2784) 1.4691	(0.0273,0.1325) 0.1052

5 Conclusion

In this paper, the point estimates of the parameters α and β of the IW distribution have been obtained using the ML and B methods based on generalized $Type-II\ PHCS$ and for different values of T_1 and T_2. Also, the approximate, bootstrap-p, Bayes(credible) and $HPD\ CI's$ have been obtained. The lengths and the $CP's$ of all these $CI's$ have been computed. Finally, the estimates of all parameters and all $CI's$ have been computed based on the studied censoring scheme from a real data set.

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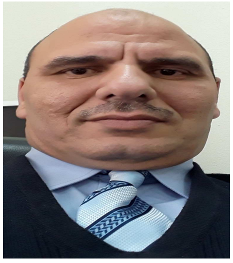
References

[1] S.F. Ateya, Estimation under modified Weibull distribution based on right censored generalized order statistics, *J. Appl. Stat.*, 40, 2720-2734 (2013).
[2] S.F. Ateya, Estimation under Inverse Weibull Distribution based on Balakrishnan's Unied Hybrid Censored Scheme, *Commun. Stat. Simul. Comput.*, 46, 3645 - 3666 (2017).
[3] S.F. Ateya, H.S. Mohammed, Prediction Under Burr-XII Distribution Based on Generalized Type-II Progressive Hybrid Censoring Scheme, *JOEMS*, 26, 491- 508 (2018).
[4] N. Balakrishnan, R. Aggrawala, *Progessive censoring, theory, methods and applications*, Birkhauser, Boston (2000).
[5] N. Balakrishnan, E. Cramer. *The Art of Progressive Censoring. Applications to Reliability and Quality*, New York, Birkhäuser (2014).
[6] A. Childs, B. Chandrasekar, N. Balakrishnan, Exact likelihood inference for an exponential parameter under progressive hybrid censoring, In: *Statistical Models and Methods for Biomedical and Technical Systems* (Eds., F. Vonta, M. Nikulin, N. Limnios and C. Huber-Carol), pp. 319 - 330, Birkhauser, Boston, MA (2008).
[7] B. Efron, *The Jackknife, the Bootstrap and Other Re- sampling Plans*, CBMS-NSF Regional Conference Serie- sin, Applied Mathematics, SIAM, Philadelphia, 38(1982).
[8] K. Lee, H. Sun, Y. Cho, J, Exact likelihood inference of the exponential parameter under generalized Type-II progressive hybrid censoring, *Korean Statist. Soc.*, 45, 123-136 (2016).
[9] B.O. Oluyede, T. Yang, Generalizations of the Inverse Weibull and Related Distributions with Applications, *Elec. J. of Appl. Statist. Analy.*, 7, 94-116 (2014).



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