

Bayesian and Maximum Likelihood Estimation of Stress–Strength Reliability for Mixed Mgamma Records with Application

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Received: 7 Jun. 2026, Revised: 21 Jul. 2026, Accepted: 25 Aug. 2026

Published online: 1 Nov. 2026

Abstract: This paper addresses the estimation of the stress–strength reliability parameter $R = P(Y < X)$ when the stress Y and the strength X are independent random variables following the one-parameter Mgamma distribution, a mixture of two gamma components recently proposed for skewed lifetime data. A closed-form representation of R in terms of the regularized incomplete beta function is obtained, and it is shown that, because the beta indices are integers, R is in fact a rational function of the two parameters whose first and second derivatives are available explicitly. Inference is developed under a mixed record scheme in which upper records are observed on the stress variable and lower records on the strength variable. Maximum likelihood estimators are obtained, and Bayesian estimates under non-informative priors are computed with the Tierney–Kadane approximation. All derivatives required by this approximation are given explicitly, including the non-zero mixed derivative of the log-posterior augmented by $\log R$. We also show that the Fisher information about the strength parameter carried by lower records converges to a finite limit as the number of records grows, which governs the finite-sample behaviour of the estimators and of Wald-type intervals; profile-likelihood intervals are proposed and attain close to nominal coverage. Monte Carlo simulations covering three reliability configurations ($R \approx 0.22, 0.50$ and 0.76) show that the Tierney–Kadane estimator has smaller mean squared error than the maximum likelihood estimator in the two configurations with $R \geq 0.5$, whereas the maximum likelihood estimator is preferable in the low-reliability configuration. An illustrative application to lung cancer survival times demonstrates the methodology.

Keywords: Bayesian inference, Mgamma distribution, Record values, Stress–strength reliability, Tierney–Kadane approximation

1 Introduction

The stress–strength model plays a central role in reliability engineering and measures the probability that a system is strong enough to withstand the stress it is subjected to, $R = P(Y < X)$. Since the seminal work of Birnbaum [9], this model has been extensively studied under various distributional assumptions and sampling schemes [7, 10, 27, 28]. The past few years have seen increased interest in flexible lifetime distributions and non-standard sampling schemes, motivated by applications in engineering, medicine and the environmental sciences.

In many reliability problems complete data are not available because of time, cost or destructive testing, and record values, which are extreme observations, arise naturally. In accelerated life testing, upper record values (which become larger and larger as the test progresses) are used to model stress, whereas lower record values (which become smaller and smaller) are used to model strength in quality control [3, 8, 12]. This asymmetry motivates a mixed record sampling scheme, in which stress is characterized through upper records and strength through lower records. Recent contributions [7, 18, 20, 21, 27] have studied record-based stress–strength models for a variety of distributions, but

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the Mgamma case has not been studied. Related recent work on multi-component and censored stress–strength reliability further illustrates the breadth of current activity in this area [16, 17, 19], including the reliability of s-out-of-k systems under inverted Z-Lindley stress–strength models with progressive censoring [1] and step-stress inference for the inverted Z-Lindley distribution under progressive Type-II censoring [4]. A parallel body of work has developed stress–strength and reliability inference for other flexible lifetime laws under incomplete-data schemes, including the discrete alpha-power Weibull distribution with record data [15], the Garhy distribution under progressive censoring compared via MCMC and Tierney–Kadane approximations [6] and under diverse record schemes [5], a one-parameter model under unified hybrid censoring [13, 32], an extended Lindley model for entropy-based loss and actuarial risk measures [22], a one-parameter decreasing-hazard lifetime model [31], new flexible distributions for radiation data [26, 41], and an odds-based Weibull model for engineering failure-time data [2].

The Mgamma distribution, recently introduced by Khan and Ruidas [24], is a one-parameter continuous distribution obtained as a finite mixture of two gamma components. It fits skewed and heavy-tailed lifetime data better than several classical one-parameter models (exponential, Lindley, Xgamma, Garhy [14], Obulezi [29], updated Lindley [30]). Its mixture representation yields tractable mathematical properties while maintaining flexibility. However, its use for stress–strength reliability with record-based sampling has not been investigated.

Bayesian inference provides a coherent way of estimating the parameters, but the required posterior integrals are generally not available in closed form. The Tierney–Kadane (TK) approximation [40] gives an accurate Laplace-type approximation of posterior moments that only involves the first and second derivatives of the log-posterior. Applications of related approximations in reliability are reported in [6, 33–38], but their use for mixed records and the Mgamma distribution is new. In this paper the TK method is used to approximate the posterior mean of the stress–strength reliability under mixed record sampling.

The paper is organized as follows. Section 2 recalls the Mgamma distribution. Section 3 derives a closed-form expression for R . Section 4 describes the mixed upper–lower record scheme and the likelihood. Section 5 presents maximum likelihood estimation, the information carried by lower records, and confidence intervals. Section 6 develops the Bayesian approach based on the TK approximation, with all required derivatives. Simulation results are presented in Section 7, Section 8 gives an illustrative medical application, and Section 9 concludes.

2 The Mgamma Distribution

A random variable X follows the Mgamma distribution [24] with parameter $\theta > 0$, denoted $X \sim \text{Mgamma}(\theta)$, if its probability density function (pdf) and cumulative distribution function (cdf) are

$$f(x; \theta) = \frac{\theta^3 x (1 + \frac{x}{2}) e^{-\theta x}}{1 + \theta}, \quad (1)$$

$$F(x; \theta) = 1 - A(x, \theta) e^{-\theta x}, \quad (2)$$

for $x > 0$, $\theta > 0$, where

$$A(x, \theta) = 1 + \frac{\theta x}{1 + \theta} \left(1 + \theta + \frac{\theta x}{2} \right). \quad (3)$$

The Mgamma distribution is a finite mixture of $\text{Gamma}(2, \theta)$ and $\text{Gamma}(3, \theta)$ with mixing probability $\theta/(1 + \theta)$ on the first component. It exhibits positive skewness and leptokurtosis. For the stress–strength model we assume independent strength $X \sim \text{Mgamma}(\theta_1)$ and stress $Y \sim \text{Mgamma}(\theta_2)$.

3 Stress–Strength Reliability: Closed-Form Expression

A system fails when the stress Y exceeds the strength X , and its reliability is $R = P(Y < X)$. In this section we derive an explicit closed-form expression for R when X and Y are independent Mgamma random variables with parameters θ_1 and θ_2 , respectively.

Theorem 1. Let $X \sim \text{Mgamma}(\theta_1)$ and $Y \sim \text{Mgamma}(\theta_2)$ be independent. Then $R = P(Y < X)$ is given by

$$R(\theta_1, \theta_2) = 1 - \frac{1}{(1 + \theta_1)(1 + \theta_2)} \left[\theta_1 \theta_2 I_\rho(2, 2) + \theta_1 I_\rho(2, 3) + \theta_2 I_\rho(3, 2) + I_\rho(3, 3) \right], \quad (4)$$

where $\rho = \theta_1/(\theta_1 + \theta_2)$, $B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$ is the beta function and $I_x(p, q) = B(p, q)^{-1} \int_0^x t^{p-1} (1-t)^{q-1} dt$ is the regularized incomplete beta function.

Proof. Step 1 (mixture representation). From the definition of the Mgamma distribution,

$$f_X(x) = \frac{\theta_1}{1 + \theta_1} g(x; 2, \theta_1) + \frac{1}{1 + \theta_1} g(x; 3, \theta_1),$$

$$F_Y(y) = \frac{\theta_2}{1 + \theta_2} G(y; 2, \theta_2) + \frac{1}{1 + \theta_2} G(y; 3, \theta_2),$$

where $g(x; p, \theta) = \theta^p x^{p-1} e^{-\theta x} / \Gamma(p)$ is the gamma pdf and $G(y; p, \theta)$ the corresponding cdf.

Step 2 (double sum). By independence,

$$R = \sum_{i=2}^3 \sum_{j=2}^3 w_i v_j P(U_i > V_j), \quad (5)$$

where $U_i \sim \text{Gamma}(i, \theta_1)$ and $V_j \sim \text{Gamma}(j, \theta_2)$ are independent, $w_2 = \theta_1/(1 + \theta_1)$, $w_3 = 1/(1 + \theta_1)$, $v_2 = \theta_2/(1 + \theta_2)$ and $v_3 = 1/(1 + \theta_2)$.

Step 3 (gamma comparison). If $U \sim \text{Gamma}(p, \theta_1)$ and $V \sim \text{Gamma}(q, \theta_2)$ are independent, then $W = \theta_1 U / (\theta_1 U + \theta_2 V) \sim \text{Beta}(p, q)$ [23], and $U > V$ if and only if $W > \rho$. Hence

$$P(U > V) = 1 - I_\rho(p, q). \quad (6)$$

Step 4 (combination). Since $\sum_i \sum_j w_i v_j = 1$,

$$\begin{aligned} R &= \sum_{i=2}^3 \sum_{j=2}^3 w_i v_j [1 - I_\rho(i, j)] \\ &= 1 - [w_2 v_2 I_\rho(2, 2) + w_2 v_3 I_\rho(2, 3) \\ &\quad + w_3 v_2 I_\rho(3, 2) + w_3 v_3 I_\rho(3, 3)]. \end{aligned}$$

Replacing the weights by their explicit expressions yields (4).

Remark 1(Polynomial form). Because the beta indices are integers, $I_\rho(p, q) = \sum_{k=p}^{p+q-1} \binom{p+q-1}{k} \rho^k (1 - \rho)^{p+q-1-k}$ is a polynomial in ρ :

$$\begin{aligned} I_\rho(2, 2) &= 3\rho^2 - 2\rho^3, \quad I_\rho(2, 3) = 6\rho^2 - 8\rho^3 + 3\rho^4, \\ I_\rho(3, 2) &= 4\rho^3 - 3\rho^4, \quad I_\rho(3, 3) = 10\rho^3 - 15\rho^4 + 6\rho^5. \end{aligned}$$

Consequently $R(\theta_1, \theta_2)$ is a rational function of (θ_1, θ_2) , it can be evaluated without special functions, and its derivatives (subsection 6.3.3) are elementary. For example, $R(0.8, 1.5) = 0.7557$, $R(1.2, 0.6) = 0.2200$ and $R(\theta, \theta) = 0.5$ for every $\theta > 0$; these values agree with direct Monte Carlo evaluation of $P(Y < X)$ based on 2×10^6 draws to within 10^{-4} .

4 Mixed Upper-Lower Record Sampling

Let $\{X_i\}_{i \geq 1}$ be a sequence of i.i.d. random variables from the strength distribution $\text{Mgamma}(\theta_1)$. Define the lower record times $L(1) = 1$ and $L(k) = \min\{j > L(k-1) : X_j < X_{L(k-1)}\}$, so that the lower record values satisfy $X_{L(1)} > X_{L(2)} > \dots$. Similarly, let $\{Y_i\}_{i \geq 1}$ be an i.i.d. sequence from the stress distribution $\text{Mgamma}(\theta_2)$, with upper record times $U(1) = 1$ and $U(k) = \min\{j > U(k-1) : Y_j > Y_{U(k-1)}\}$, so that $Y_{U(1)} < Y_{U(2)} < \dots$. Throughout the paper the observed values of the first n lower records and the first m upper records are written as

$$x_i \equiv x_{L(i)}, \quad i = 1, \dots, n; \quad y_j \equiv y_{U(j)}, \quad j = 1, \dots, m, \quad (7)$$

with $x_1 > x_2 > \dots > x_n$ and $y_1 < y_2 < \dots < y_m$.

The joint likelihood function of the observed mixed record data $\mathcal{D} = (\mathbf{x}, \mathbf{y})$, $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_m)$, is [8]

$$\begin{aligned} \mathcal{L}(\theta_1, \theta_2 | \mathcal{D}) &= \left[\prod_{i=1}^{n-1} \frac{f(x_i; \theta_1)}{F(x_i; \theta_1)} \right] f(x_n; \theta_1) \\ &\quad \times \left[\prod_{j=1}^{m-1} \frac{f(y_j; \theta_2)}{1 - F(y_j; \theta_2)} \right] f(y_m; \theta_2). \end{aligned} \quad (8)$$

The first bracket is the joint density of the first n lower records (reversed hazard f/F for all but the last record) and the second bracket is the joint density of the first m upper records (hazard $f/(1 - F)$ for all but the last record).

5 Maximum Likelihood Estimation

From (8), the log-likelihood separates as $\ell(\theta_1, \theta_2) = \ell_1(\theta_1) + \ell_2(\theta_2)$, with

$$\ell_1(\theta_1) = \sum_{i=1}^n \ln f(x_i; \theta_1) - \sum_{i=1}^{n-1} \ln F(x_i; \theta_1), \quad (9)$$

$$\ell_2(\theta_2) = \sum_{j=1}^m \ln f(y_j; \theta_2) - \sum_{j=1}^{m-1} \ln(1 - F(y_j; \theta_2)). \quad (10)$$

Hence θ_1 and θ_2 can be estimated separately by maximizing ℓ_1 and ℓ_2 .

5.1 Score equations and observed information

The maximum likelihood estimators (MLEs) $\hat{\theta}_1$ and $\hat{\theta}_2$ solve

$$\frac{d\ell_1}{d\theta_1} = \sum_{i=1}^n \psi_f(x_i; \theta_1) - \sum_{i=1}^{n-1} \psi_F(x_i; \theta_1) = 0, \quad (11)$$

$$\frac{d\ell_2}{d\theta_2} = \sum_{j=1}^m \psi_f(y_j; \theta_2) - \sum_{j=1}^{m-1} \psi_S(y_j; \theta_2) = 0, \quad (12)$$

where $\psi_f = \partial \ln f / \partial \theta$, $\psi_F = \partial \ln F / \partial \theta$ and $\psi_S = \partial \ln(1 - F) / \partial \theta$. These three component functions do not depend on the prior and are therefore common to the score equations and to the log-posterior used in Section 6. To avoid duplication, they are derived only once, in Lemma 1 of subsection 6.3.2; the MLE score equations (11)–(12) are obtained from the log-posterior derivatives in Theorem 2 by deleting the prior terms $-1/\theta_k$ and $1/\theta_k^2$. The equations have no closed-form solution and are solved numerically (subsection 7.1).

Writing $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$ and $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2)^\top$, the observed information matrix is diagonal because the log-likelihood separates:

$$\mathbf{J}(\boldsymbol{\theta}) = -\frac{\partial^2 \ell}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} = \begin{pmatrix} -\frac{d^2 \ell_1}{d\theta_1^2} & 0 \\ 0 & -\frac{d^2 \ell_2}{d\theta_2^2} \end{pmatrix}, \quad (13)$$

with the second derivatives given in Lemma 1 and, explicitly, in Appendix A. The corresponding expected (Fisher) information is

$$\mathbf{I}_{n,m}(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}}[\mathbf{J}(\boldsymbol{\theta})] = \text{diag}\{I_n^L(\theta_1), I_m^U(\theta_2)\}.$$

5.2 Information in lower and upper records

The two diagonal elements of $\mathbf{I}_{n,m}(\boldsymbol{\theta})$ behave very differently as the number of records grows, which has direct consequences for interval estimation.

Proposition 1.(i) For upper records, $I_m^U(\theta_2) \rightarrow \infty$ (linearly in m) as $m \rightarrow \infty$. (ii) For lower records, $I_n^L(\theta_1)$ is non-decreasing in n and converges to a finite limit $I_{\infty}^L(\theta_1)$ as $n \rightarrow \infty$.

Proof. (ii) A series expansion of (1)–(2) at $x = 0$ gives

$$\ln \frac{f(x; \theta)}{F(x; \theta)} = \ln \frac{2}{x} + \frac{(1 - 2\theta)x}{6} + O(x^2), \quad x \downarrow 0,$$

so the contribution of a lower record x_i to $-d^2\ell_1/d\theta_1^2$ is $O(x_i^2)$. Since $F(X_{L(k)})$ is distributed as a product of k independent uniform variables and $F(x) \sim \theta^3 x^2 / \{2(1 + \theta)\}$ near 0, $X_{L(k)}$ decreases geometrically (as $e^{-k/2}$), and the contributions are summable. The last record contributes $3/\theta_1^2 - 1/(1 + \theta_1)^2$ for every n . Hence $I_n^L(\theta_1)$ converges. (i) For large y , $\partial \ln\{f(y)/(1 - F(y))\}/\partial \theta \rightarrow 1/\theta$, so each upper record contributes a non-vanishing amount of information, approximately $1/\theta_2^2$.

Numerically, $I_n^L(0.8)$ equals 4.38 for $n = 1$, 5.55 for $n = 5$ and 5.58 for $n \geq 10$, whereas $I_m^U(0.8)$ equals 4.38, 15.5, 26.2, 45.2 and 96.9 for $m = 1, 5, 10, 20, 50$ (Monte Carlo averages over 2×10^4 record sequences). Thus, in the lower-record sample, the information about θ_1 is concentrated in the first few records: deep lower records lie close to zero, where the reversed hazard $f/F \approx 2/x$ no longer depends on θ_1 .

5.3 Asymptotic distribution and estimation of R

By Proposition 1(i) and standard regularity conditions (the Mgamma log-density is twice continuously differentiable in θ and the true value is an interior point of $(0, \infty)$), $\sqrt{m}(\hat{\theta}_2 - \theta_2)$ is asymptotically normal as $m \rightarrow \infty$. By Proposition 1(ii), the information about θ_1 converges to $I_{\infty}^L(\theta_1)$, so the sampling distribution of $\hat{\theta}_1$ stabilizes after the first few lower records. For interval estimation we use the normal approximation

$$\hat{\boldsymbol{\theta}} \overset{\text{approx.}}{\sim} \mathcal{N}_2(\boldsymbol{\theta}, \mathbf{J}(\hat{\boldsymbol{\theta}})^{-1}), \quad \mathbf{J}(\hat{\boldsymbol{\theta}})^{-1} = \text{diag}\{\hat{\sigma}_1^2, \hat{\sigma}_2^2\}, \quad (14)$$

where $\hat{\sigma}_k^2 = \{-d^2\ell_k/d\theta_k^2\}^{-1}|_{\theta_k=\hat{\theta}_k}$. Its accuracy for θ_2 increases with m , and for θ_1 it is governed by the limiting information $I_{\infty}^L(\theta_1)$.

The stress–strength reliability is estimated by plugging the MLEs into Theorem 1, $\hat{R}_{\text{MLE}} = R(\hat{\theta}_1, \hat{\theta}_2)$. By the delta method [11],

$$\widehat{\text{Var}}(\hat{R}_{\text{MLE}}) = \nabla R^{\top} \mathbf{J}(\hat{\boldsymbol{\theta}})^{-1} \nabla R = R_1^2 \hat{\sigma}_1^2 + R_2^2 \hat{\sigma}_2^2, \quad (15)$$

where $\nabla R = (R_1, R_2)^{\top}$, $R_k = \partial R / \partial \theta_k$, is given explicitly in Lemma 2 and evaluated at $\hat{\boldsymbol{\theta}}$. The Wald interval is $\hat{R}_{\text{MLE}} \pm 1.96 \widehat{\text{Var}}(\hat{R}_{\text{MLE}})^{1/2}$.

5.4 Profile-likelihood and bootstrap intervals

Because of Proposition 1(ii) and the skewness of the likelihood in θ_1 , we also consider two alternatives to the Wald interval for R (Section 7.5).

Profile-likelihood interval. The 95% profile-likelihood interval for R is

$$\left\{ r : 2[\ell(\hat{\boldsymbol{\theta}}) - \max_{\boldsymbol{\theta}: R(\boldsymbol{\theta})=r} \ell(\boldsymbol{\theta})] \leq \chi_{1;0.95}^2 = 3.841 \right\}.$$

Since ℓ separates and R is continuous, this set is the interval $[\min R, \max R]$ taken over the likelihood region $\{\boldsymbol{\theta} : \ell(\boldsymbol{\theta}) \geq \ell(\hat{\boldsymbol{\theta}}) - 1.92\}$. We evaluate it on a logarithmic grid of 500×500 values of (θ_1, θ_2) in $[0.005, 100]^2$, which is fast because ℓ_1 and ℓ_2 are computed on one-dimensional grids.

Parametric bootstrap percentile interval. Generate B mixed record samples of the same sizes from $\text{Mgamma}(\hat{\theta}_1)$ and $\text{Mgamma}(\hat{\theta}_2)$, recompute $\hat{R}^{*(b)}$, $b = 1, \dots, B$, and report the 2.5% and 97.5% quantiles. Records are generated sequentially by inversion: $F(X_{L(k+1)}) = U_{k+1} F(X_{L(k)})$ and $1 - F(Y_{U(k+1)}) = U_{k+1} \{1 - F(Y_{U(k)})\}$ with independent $U_k \sim \text{Uniform}(0, 1)$.

6 Bayesian Estimation Using the Tierney–Kadane Method

6.1 Non-informative prior

Non-informative (objective) priors are designed to have minimal influence on the posterior distribution. For a one-parameter model the Jeffreys prior is $\pi(\theta) \propto \sqrt{I(\theta)}$, where $I(\theta)$ is the expected Fisher information. For a complete Mgamma sample $I(\theta) = 3/\theta^2 - 1/(1 + \theta)^2$. Under the mixed record scheme the expected information is more complicated (Section 5.2); for simplicity and tractability we adopt independent priors

$$\pi(\theta_1, \theta_2) \propto \frac{1}{\theta_1 \theta_2}, \quad \theta_1, \theta_2 > 0, \quad (16)$$

which behave like the Jeffreys prior for small θ and correspond to a locally uniform prior on $\ln \theta_k$. This prior is improper, but the posterior is proper: as $\theta_k \rightarrow 0$ the likelihood factor of θ_k vanishes at least as fast as θ_k^3 , and as $\theta_k \rightarrow \infty$ it decays exponentially.

Prior sensitivity. To assess the influence of this choice we also consider (i) the uniform prior $\pi(\theta_k) \propto 1$ and (ii) the weakly informative prior $\theta_k \sim \text{Gamma}(0.01, 0.01)$ (shape = rate = 0.01). The numerical comparison is reported in Table 4 (Section 7.6).

6.2 The Tierney–Kadane approximation

For a smooth positive function $h(\theta)$, the posterior mean $E[h(\theta) | \mathcal{D}] = \int h e^L d\theta / \int e^L d\theta$, where

$$L(\theta) = \ell(\theta) + \ln \pi(\theta) = \ell_1(\theta_1) + \ell_2(\theta_2) - \ln \theta_1 - \ln \theta_2 \quad (17)$$

is the log-posterior up to a constant, is approximated by [40]

$$\hat{E}[h(\theta) | \mathcal{D}] = \left(\frac{\det \Sigma^*}{\det \Sigma} \right)^{1/2} \exp\{L^*(\hat{\theta}^*) - L(\tilde{\theta})\}, \quad (18)$$

where $L^*(\theta) = L(\theta) + \ln h(\theta)$, $\tilde{\theta}$ is the posterior mode (the maximizer of L , which differs from the MLE because of the prior term), $\hat{\theta}^*$ maximizes L^* , $\Sigma = [-H(\tilde{\theta})]^{-1}$, $\Sigma^* = [-H^*(\hat{\theta}^*)]^{-1}$, and H and H^* are the Hessian matrices of L and L^* . To estimate the reliability we take $h(\theta) = R(\theta_1, \theta_2)$, so that

$$L^*(\theta) = L(\theta) + \ln R(\theta_1, \theta_2). \quad (19)$$

It is essential to note that, although L separates in θ_1 and θ_2 , L^* does not: $\ln R$ couples the two parameters, so H^* is a full 2×2 matrix with a non-zero off-diagonal element (Section 6.3.4).

6.3 Derivatives required by the TK method

We need the first derivatives, the second pure derivatives and the mixed derivative of both L and L^* . Those of L follow from the component functions of Section 6.3.2; those of L^* additionally require the derivatives of R , given in Section 6.3.3.

6.3.1 Log-posterior components

From (17),

$$\frac{\partial L}{\partial \theta_k} = \frac{d\ell_k}{d\theta_k} - \frac{1}{\theta_k}, \quad \frac{\partial^2 L}{\partial \theta_k^2} = \frac{d^2 \ell_k}{d\theta_k^2} + \frac{1}{\theta_k^2}, \quad k = 1, 2, \\ \frac{\partial^2 L}{\partial \theta_1 \partial \theta_2} = 0.$$

6.3.2 Component functions

Lemma 1. For $x > 0$ and $\theta > 0$, with $A = A(x, \theta)$ as in (3) and $F = F(x; \theta)$,

$$\psi_f(x; \theta) = \frac{3}{\theta} - x - \frac{1}{1 + \theta}, \quad (20)$$

$$\psi_F(x; \theta) = \frac{F_\theta}{F}, \quad \psi_S(x; \theta) = -\frac{F_\theta}{1 - F}, \quad (21)$$

where $F_\theta = \partial F / \partial \theta$ is

$$F_\theta = \left(xA - \frac{\partial A}{\partial \theta} \right) e^{-\theta x} \\ = \frac{\theta^2 x^2 \{3 + 2\theta + (1 + \theta)x\}}{2(1 + \theta)^2} e^{-\theta x} > 0, \quad (22)$$

with

$$\frac{\partial A}{\partial \theta} = \frac{x}{(1 + \theta)^2} \left(1 + \theta + \frac{\theta x}{2} \right) + \frac{\theta x}{1 + \theta} \left(1 + \frac{x}{2} \right). \quad (23)$$

The second derivatives are

$$\frac{\partial \psi_f}{\partial \theta} = -\frac{3}{\theta^2} + \frac{1}{(1 + \theta)^2}, \quad (24)$$

$$\frac{\partial \psi_F}{\partial \theta} = \frac{F F_{\theta\theta} - F_\theta^2}{F^2}, \quad (25)$$

$$\frac{\partial \psi_S}{\partial \theta} = -\frac{(1 - F)F_{\theta\theta} + F_\theta^2}{(1 - F)^2}, \quad (26)$$

where

$$F_{\theta\theta} = F_\theta \left[\frac{2}{\theta} - \frac{2}{1 + \theta} - x + \frac{2 + x}{3 + 2\theta + (1 + \theta)x} \right]. \quad (27)$$

Proof. (20) and (24) follow by differentiating $\ln f = 3 \ln \theta + \ln x + \ln(1 + x/2) - \theta x - \ln(1 + \theta)$ term by term. Since $F = 1 - Ae^{-\theta x}$, $F_\theta = (xA - \partial A / \partial \theta) e^{-\theta x}$. Expanding $A = 1 + \theta x + \theta^2 x^2 / \{2(1 + \theta)\}$ gives $\partial A / \partial \theta = x + \theta(2 + \theta)x^2 / \{2(1 + \theta)^2\}$, and therefore $xA - \partial A / \partial \theta = \theta^2 x^2 \{3 + 2\theta + (1 + \theta)x\} / \{2(1 + \theta)^2\}$, which is (22). Taking logarithms of (22) and differentiating with respect to θ gives (27). (25)–(26) follow from the quotient rule.

Remark 2. The factorized forms (22) and (27) are free of the cancellation that affects $xA - \partial A / \partial \theta$ for very small x (deep lower records). For numerical work we also use $1 - F = Ae^{-\theta x}$, i.e. $\ln(1 - F) = \ln A - \theta x$, and $F = \frac{\theta}{1 + \theta} P(2, \theta x) + \frac{1}{1 + \theta} P(3, \theta x)$, where $P(a, t)$ is the regularized lower incomplete gamma function. All formulas of this section were checked against central finite differences (relative error below 10^{-6}).

6.3.3 Derivatives of $R(\theta_1, \theta_2)$

Write $s = \theta_1 + \theta_2$, $\rho = \theta_1/s$, $D = (1 + \theta_1)(1 + \theta_2)$ and

$$N = \theta_1 \theta_2 I_{22} + \theta_1 I_{23} + \theta_2 I_{32} + I_{33}, \quad R = 1 - \frac{N}{D},$$

where $I_{pq} = I_p(p, q)$ (Remark 1). Let $b_{pq}(\rho) = \rho^{p-1}(1-\rho)^{q-1}/B(p, q)$ be the beta density, i.e. $b_{22} = 6\rho(1-\rho)$, $b_{23} = 12\rho(1-\rho)^2$, $b_{32} = 12\rho^2(1-\rho)$, $b_{33} = 30\rho^2(1-\rho)^2$, with $b'_{pq} = b_{pq}\{(p-1)/\rho - (q-1)/(1-\rho)\}$, and define

$$M = \theta_1 \theta_2 b_{22} + \theta_1 b_{23} + \theta_2 b_{32} + b_{33}, \\ M' = \theta_1 \theta_2 b'_{22} + \theta_1 b'_{23} + \theta_2 b'_{32} + b'_{33}.$$

The derivatives of ρ are

$$\rho_1 = \frac{\theta_2}{s^2}, \quad \rho_2 = -\frac{\theta_1}{s^2}, \quad \rho_{11} = -\frac{2\theta_2}{s^3}, \\ \rho_{22} = \frac{2\theta_1}{s^3}, \quad \rho_{12} = \frac{\theta_1 - \theta_2}{s^3}.$$

Lemma 2. With the notation above,

$$N_1 = \theta_2 I_{22} + I_{23} + M\rho_1, \quad N_2 = \theta_1 I_{22} + I_{32} + M\rho_2, \\ N_{11} = 2(\theta_2 b_{22} + b_{23})\rho_1 + M'\rho_1^2 + M\rho_{11}, \\ N_{22} = 2(\theta_1 b_{22} + b_{32})\rho_2 + M'\rho_2^2 + M\rho_{22}, \\ N_{12} = I_{22} + (\theta_2 b_{22} + b_{23})\rho_2 + (\theta_1 b_{22} + b_{32})\rho_1 \\ + M'\rho_1\rho_2 + M\rho_{12}.$$

Let $Q = N/D$, $D_1 = 1 + \theta_2$, $D_2 = 1 + \theta_1$ and $Q_k = (N_k - QD_k)/D$. Then

$$R_k = \frac{\partial R}{\partial \theta_k} = -Q_k, \quad k = 1, 2, \quad (28)$$

$$R_{11} = -\frac{N_{11} - 2Q_1 D_1}{D}, \quad R_{22} = -\frac{N_{22} - 2Q_2 D_2}{D}, \quad (29)$$

$$R_{12} = -\frac{N_{12} - Q_1 D_2 - Q_2 D_1 - Q}{D}. \quad (30)$$

Proof. Since $\partial I_{pq}/\partial \theta_k = b_{pq}(\rho)\rho_k$ and $\partial b_{pq}/\partial \theta_k = b'_{pq}(\rho)\rho_k$, the expressions for N_k and N_{kl} follow from the product rule; the ρ -derivatives follow from $\rho = \theta_1/(\theta_1 + \theta_2)$. Since $D_{11} = D_{22} = 0$ and $D_{12} = 1$, applying the quotient rule twice to $Q = N/D$ gives (28)–(30).

For example, at $(\theta_1, \theta_2) = (0.8, 1.5)$ one obtains $R = 0.7557$, $(R_1, R_2) = (-0.4326, 0.2227)$ and $(R_{11}, R_{12}, R_{22}) = (0.1141, 0.2334, -0.2785)$, in agreement with finite differences. Note that $R_{12} \neq 0$.

6.3.4 Derivatives of L^* and the mixed derivative

For $L^* = L + \ln R$ in (19),

$$\frac{\partial L^*}{\partial \theta_k} = \frac{\partial L}{\partial \theta_k} + \frac{R_k}{R}, \quad k = 1, 2, \quad (31)$$

$$\frac{\partial^2 L^*}{\partial \theta_k^2} = \frac{\partial^2 L}{\partial \theta_k^2} + \frac{RR_{kk} - R_k^2}{R^2}, \quad (32)$$

$$\frac{\partial^2 L^*}{\partial \theta_1 \partial \theta_2} = \frac{RR_{12} - R_1 R_2}{R^2}. \quad (33)$$

The mixed derivative (33) is not zero in general (at (0.8, 1.5) it equals 0.4776). Hence the Hessian H of L is diagonal, whereas the Hessian H^* of L^* is a full matrix that includes the off-diagonal term (33). In the computations of this paper H^* is evaluated analytically from (32)–(33); we verified that it agrees with a finite-difference Hessian. For the posterior second moment ($h = R^2$) the terms involving R in (31)–(33) are multiplied by 2.

Theorem 2 (Derivatives for the TK method). Under the mixed record model with Mgamma distributions and prior $\pi(\theta_1, \theta_2) \propto (\theta_1 \theta_2)^{-1}$:

1. First order: $\partial L/\partial \theta_k = d\ell_k/d\theta_k - 1/\theta_k$, where $d\ell_1/d\theta_1$ and $d\ell_2/d\theta_2$ are the left-hand sides of (11) and (12), with components (20)–(22); $\partial L^*/\partial \theta_k$ is given by (31).
2. Second order (pure): $\partial^2 L/\partial \theta_k^2 = d^2 \ell_k/d\theta_k^2 + 1/\theta_k^2$, with components (24)–(27); $\partial^2 L^*/\partial \theta_k^2$ is given by (32).
3. Mixed: $\partial^2 L/\partial \theta_1 \partial \theta_2 = 0$, whereas $\partial^2 L^*/\partial \theta_1 \partial \theta_2 = (RR_{12} - R_1 R_2)/R^2 \neq 0$, with R_k, R_{kl} from Lemma 2.

Consequently $\det \Sigma = \{(\partial^2 L/\partial \theta_1^2)(\partial^2 L/\partial \theta_2^2)\}^{-1}$ at $\tilde{\theta}$, while $\det \Sigma^* = \{H_{11}^* H_{22}^* - (H_{12}^*)^2\}^{-1}$ at $\tilde{\theta}^*$.

6.4 Implementation of the TK estimator of R

Algorithm 1 Tierney–Kadane estimation of $E[R | \mathcal{D}]$

- 1: Compute the posterior mode $\tilde{\theta}$ by maximizing $L_1(\theta_1) = \ell_1 - \ln \theta_1$ and $L_2(\theta_2) = \ell_2 - \ln \theta_2$ separately.
- 2: Set $\Sigma = \text{diag}\{(-\partial^2 L/\partial \theta_1^2)^{-1}, (-\partial^2 L/\partial \theta_2^2)^{-1}\}$ at $\tilde{\theta}$ (Theorem 2).
- 3: Define $L^*(\theta) = L(\theta) + \ln R(\theta)$.
- 4: Maximize L^* to obtain $\tilde{\theta}^*$ (coordinate-wise search started at $\tilde{\theta}$, then Newton steps with gradient (31) and Hessian (32)–(33)).
- 5: Form the full Hessian $H^*(\tilde{\theta}^*)$, including H_{12}^* from (33), and $\Sigma^* = [-H^*(\tilde{\theta}^*)]^{-1}$.
- 6: Return $\hat{R}_{\text{TK}} = (\det \Sigma^*/\det \Sigma)^{1/2} \exp\{L^*(\tilde{\theta}^*) - L(\tilde{\theta})\}$.
- 7: (Optional) Repeat steps 3–6 with $h = R^2$ to obtain $\hat{E}[R^2 | \mathcal{D}]$ and the approximate posterior standard deviation $\hat{s} = (\hat{E}[R^2 | \mathcal{D}] - \hat{R}_{\text{TK}}^2)^{1/2}$.

The TK interval reported in Section 7 is $\hat{R}_{TK} \pm 1.96\hat{s}$, truncated to $[0, 1]$. It is an approximate credible interval based on a normal approximation to the posterior of R .

7 Monte Carlo Simulation Study

In this section we evaluate the finite-sample performance of the MLE and TK estimators of $R = P(Y < X)$ under mixed upper-lower record sampling.

7.1 Simulation design

Parameter scenarios. We consider three configurations of (θ_1, θ_2) :

- S1 (moderate reliability): $\theta_1 = 0.8$, $\theta_2 = 1.5$, $R = 0.7557$;
- S2 (low reliability): $\theta_1 = 1.2$, $\theta_2 = 0.6$, $R = 0.2200$;
- S3 (symmetric configuration): $\theta_1 = \theta_2 = 1.0$, $R = 0.5000$.

The values of R are computed from Theorem 1.

Sample sizes and replications. For each scenario we generate mixed records with $n = m \in \{10, 20, 30, 50\}$, which represent small to moderate record samples, and use $N = 1500$ replications.

Optimization strategy. Because the record likelihoods separate (Section 5), $\hat{\theta}_1$ and $\hat{\theta}_2$ are obtained by one-dimensional golden-section searches over $[0.02, 10]$. The cdf and its derivatives are evaluated with the numerically stable forms of Remark 2. The TK estimate is computed by Algorithm 1 with the full Hessian of L^* .

7.2 Evaluation criteria

For each estimator $\hat{\psi}$ of $\psi \in \{\theta_1, \theta_2, R\}$ we compute

$$\text{Bias}(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^N \hat{\psi}_i - \psi, \quad (34)$$

$$\text{MSE}(\hat{\psi}) = \frac{1}{N} \sum_{i=1}^N (\hat{\psi}_i - \psi)^2, \quad (35)$$

together with the coverage probability (CP) and average width of the nominal 95% intervals.

7.3 Simulation results

Table 1 gives bias, MSE and interval performance of the MLE and TK estimators of R , and Table 2 reports the average parameter estimates.

7.4 Discussion of results

Scenario S1 ($R = 0.7557$). The TK estimator has lower MSE than the MLE at all sample sizes (a reduction of about 20–23%, e.g. 0.0482 vs. 0.0605 at $n = 10$ and 0.0431 vs. 0.0559 at $n = 30$). Both estimators are biased downward ($|\text{Bias}|$ between 0.08 and 0.11), TK less so.

Scenario S2 ($R = 0.2200$). The MLE attains lower MSE than TK at every sample size (0.0228–0.0265 vs. 0.0254–0.0282). TK shows a positive bias (0.038–0.052) and the MLE a small negative bias ($|\text{Bias}| < 0.021$).

Scenario S3 ($R = 0.5$). TK has 20–26% lower MSE than the MLE at every n , and its bias (≈ -0.03) is about half that of the MLE (≈ -0.06 to -0.07).

Effect of the sample size. Bias and MSE of both estimators change very little between $n = 10$ and $n = 50$. This is explained by Proposition 1: the information about θ_1 is concentrated in the first few lower records, so the precision of $\hat{R}$ is determined mainly by the strength sample, while additional upper records improve the estimation of θ_2 (Table 2).

Choice of estimator. In the two configurations with $R \geq 0.5$ (S1 and S3) the TK estimator has the smaller MSE, whereas in the low-reliability configuration S2 the MLE is preferable. The S2 result is explained by the fact that the posterior of R is strongly right-skewed when R is close to 0, so that the posterior mean (which TK approximates) lies above the true value.

7.5 Coverage of interval estimators

The Wald intervals in Table 1 have coverage 0.76–0.85, below the nominal 0.95, and this coverage is stable in n . Two features of the record setting account for this: (i) the saturation of information about θ_1 (Proposition 1), which means that the large-sample justification of (14) does not apply to the strength parameter, and (ii) the strong right skewness of the likelihood in θ_1 (Table 2), which a symmetric interval does not reflect. The TK intervals attain higher coverage (0.88–0.94).

To evaluate alternative intervals, we ran an additional, independent simulation with the same design ($N = 1500$ replications per cell, new random numbers) in which, for every replication, we also computed the profile-likelihood interval and the parametric bootstrap percentile interval of Section 5.4 ($B = 300$ bootstrap samples). The point-estimate results of this run reproduce those of Table 1 within Monte Carlo error (the largest differences from Table 1 are 0.012 in bias and 0.0035 in MSE, and the estimator with the smaller MSE is the same in all twelve cells). The interval results are reported in Table 3. In this table all intervals are truncated to $[0, 1]$, so the reported widths of the Wald intervals are somewhat smaller than the untruncated widths of Table 1.

Three conclusions emerge from Table 3. First, the profile-likelihood interval is the only one whose coverage is close to the nominal level in every cell (0.933–0.949).

Table 1: Simulation results for the estimation of R under mixed record sampling ($N = 1500$ replications). CP (Width): coverage probability and average width of the nominal 95% interval (Wald interval for the MLE; normal approximation to the TK posterior for the Bayes estimator). Bold: smaller MSE.

Scenario	$n = m$	MLE			TK Bayesian		
		Bias	MSE	CP (Width)	Bias	MSE	CP (Width)
S1 ($R = 0.7557$)	10	−0.0993	0.0605	0.837 (0.750)	−0.0890	0.0482	0.879 (0.703)
	20	−0.1007	0.0607	0.847 (0.725)	−0.0849	0.0473	0.880 (0.683)
	30	−0.0940	0.0559	0.849 (0.720)	−0.0767	0.0431	0.887 (0.674)
	50	−0.1093	0.0609	0.851 (0.720)	−0.0890	0.0472	0.885 (0.678)
S2 ($R = 0.2200$)	10	−0.0069	0.0265	0.775 (0.616)	0.0480	0.0282	0.935 (0.741)
	20	−0.0174	0.0235	0.767 (0.582)	0.0400	0.0256	0.929 (0.726)
	30	−0.0072	0.0233	0.788 (0.597)	0.0518	0.0266	0.932 (0.737)
	50	−0.0206	0.0228	0.759 (0.561)	0.0381	0.0254	0.921 (0.716)
S3 ($R = 0.5000$)	10	−0.0627	0.0548	0.802 (0.829)	−0.0263	0.0421	0.875 (0.816)
	20	−0.0668	0.0531	0.815 (0.804)	−0.0254	0.0409	0.891 (0.804)
	30	−0.0733	0.0534	0.805 (0.791)	−0.0298	0.0412	0.893 (0.797)
	50	−0.0711	0.0497	0.828 (0.794)	−0.0260	0.0386	0.899 (0.799)

Across all cells at most 1.9% of replications had a parameter estimate near the boundary of the search interval $[0.02, 10]$, and this fraction did not increase with n .

Table 2: Average parameter estimates (standard deviations across replications in parentheses), $N = 1500$ replications. The Bayes column reports the posterior mode $\tilde{\theta}_k$ under the prior $(\theta_1 \theta_2)^{-1}$, i.e. the expansion point $\tilde{\theta}$ of the TK approximation.

Scenario	n	θ_1	$\hat{\theta}_1^{\text{MLE}}$	$\tilde{\theta}_1$ (mode)	θ_2	$\hat{\theta}_2^{\text{MLE}}$	$\tilde{\theta}_2$ (mode)
S1	10	0.8	1.276 (1.202)	0.871 (0.860)	1.5	1.599 (0.423)	1.501 (0.396)
	20	0.8	1.259 (1.159)	0.850 (0.750)	1.5	1.543 (0.297)	1.489 (0.286)
	30	0.8	1.224 (1.103)	0.836 (0.788)	1.5	1.540 (0.257)	1.501 (0.250)
	50	0.8	1.307 (1.251)	0.894 (0.913)	1.5	1.523 (0.198)	1.498 (0.195)
S2	10	1.2	1.893 (1.548)	1.261 (1.032)	0.6	0.636 (0.167)	0.598 (0.157)
	20	1.2	1.934 (1.645)	1.307 (1.201)	0.6	0.613 (0.116)	0.592 (0.112)
	30	1.2	1.832 (1.512)	1.225 (1.034)	0.6	0.613 (0.102)	0.598 (0.100)
	50	1.2	2.004 (1.735)	1.357 (1.289)	0.6	0.612 (0.081)	0.602 (0.080)
S3	10	1.0	1.589 (1.429)	1.078 (1.029)	1.0	1.057 (0.275)	0.993 (0.258)
	20	1.0	1.601 (1.412)	1.076 (0.958)	1.0	1.039 (0.196)	1.003 (0.189)
	30	1.0	1.619 (1.479)	1.107 (1.120)	1.0	1.020 (0.162)	0.994 (0.158)
	50	1.0	1.611 (1.476)	1.089 (1.039)	1.0	1.022 (0.132)	1.005 (0.130)

Note: the strength parameter $\hat{\theta}_1$ (estimated from lower records) shows a right-skewed sampling distribution and an upward mean bias that does not decrease with n . This is explained by Proposition 1: the information about θ_1 in lower records saturates after the first few records. The prior term $-\ln \theta_1$ shifts the posterior mode downwards, which reduces this bias relative to the MLE (e.g. 0.871 vs. 1.276 for S1, $n = 10$). The upper-record estimators of θ_2 do not show this effect.

Its average width is similar to that of the Wald interval in S1 and S3, and larger in S2 (0.62–0.65 versus 0.48–0.51). Second, the parametric bootstrap percentile interval attains coverage 0.82–0.86, higher than the Wald interval; being centred on the MLE, it reflects the bias of the MLE. Third, the coverage of the Wald, bootstrap and TK intervals is stable in n , in agreement with Proposition 1. We therefore recommend the profile-likelihood interval for R in this record setting.

7.6 Prior sensitivity

Table 4 reports the bias and MSE of the TK estimator of R under the three priors of Section 6.1, computed on the same simulated samples as Table 3.

The Gamma(0.01, 0.01) prior gives results practically identical to those of $\pi \propto 1/\theta$ (differences of at most 0.0005 in bias and 0.0003 in MSE), as expected because its density is close to $\theta^{-0.99}$. The uniform prior gives different results: the bias of the TK estimator is lower by 0.05–0.08 in every cell, which increases the MSE in S1 (0.067–0.069 versus 0.045–0.050) and in S3

Table 3: Coverage probability (average width) of nominal 95% intervals for R from an independent simulation run ($N = 1500$ replications per cell; bootstrap with $B = 300$). All intervals truncated to $[0, 1]$. Boldface: coverage within ± 0.02 of 0.95.

Scenario	$n = m$	Wald (MLE)	Bootstrap (MLE)	Profile likelihood	TK normal approx.
S1 ($R = 0.7557$)	10	0.829 (0.640)	0.854 (0.738)	0.941 (0.636)	0.877 (0.629)
	20	0.823 (0.616)	0.837 (0.728)	0.933 (0.606)	0.863 (0.602)
	30	0.841 (0.616)	0.830 (0.727)	0.935 (0.601)	0.882 (0.600)
	50	0.846 (0.626)	0.834 (0.728)	0.939 (0.601)	0.885 (0.606)
S2 ($R = 0.2200$)	10	0.784 (0.506)	0.860 (0.495)	0.949 (0.649)	0.938 (0.620)
	20	0.786 (0.498)	0.849 (0.473)	0.945 (0.639)	0.928 (0.619)
	30	0.773 (0.488)	0.822 (0.458)	0.936 (0.629)	0.927 (0.612)
	50	0.768 (0.479)	0.819 (0.445)	0.943 (0.624)	0.929 (0.609)
S3 ($R = 0.5000$)	10	0.809 (0.709)	0.857 (0.686)	0.945 (0.716)	0.884 (0.729)
	20	0.819 (0.713)	0.847 (0.674)	0.948 (0.708)	0.893 (0.732)
	30	0.803 (0.696)	0.829 (0.661)	0.935 (0.693)	0.872 (0.717)
	50	0.807 (0.704)	0.823 (0.652)	0.937 (0.698)	0.901 (0.730)

Table 4: Prior sensitivity of the TK estimator of R : bias and MSE under the prior $\pi(\theta) \propto 1/\theta$ used in the paper, the uniform prior and the Gamma(0.01, 0.01) prior (same samples as Table 3).

Scenario	$n = m$	$\pi \propto 1/\theta$		Uniform		Gamma(0.01, 0.01)	
		Bias	MSE	Bias	MSE	Bias	MSE
S1 ($R = 0.7557$)	10	-0.0959	0.0503	-0.1525	0.0688	-0.0957	0.0500
	20	-0.0838	0.0483	-0.1481	0.0679	-0.0836	0.0480
	30	-0.0809	0.0463	-0.1491	0.0665	-0.0807	0.0460
	50	-0.0841	0.0452	-0.1563	0.0669	-0.0838	0.0449
S2 ($R = 0.2200$)	10	0.0448	0.0268	-0.0097	0.0204	0.0451	0.0267
	20	0.0467	0.0263	-0.0151	0.0190	0.0469	0.0262
	30	0.0442	0.0264	-0.0199	0.0190	0.0445	0.0263
	50	0.0395	0.0240	-0.0270	0.0173	0.0398	0.0239
S3 ($R = 0.5000$)	10	-0.0265	0.0438	-0.0895	0.0498	-0.0261	0.0435
	20	-0.0209	0.0403	-0.0942	0.0472	-0.0206	0.0400
	30	-0.0202	0.0434	-0.0963	0.0500	-0.0199	0.0431
	50	-0.0336	0.0394	-0.1141	0.0490	-0.0332	0.0391

(0.047–0.050 versus 0.039–0.044) and decreases it in S2 (0.017–0.020 versus 0.024–0.027). This effect does not diminish as n grows. It is again a consequence of Proposition 1: since the information about θ_1 is bounded, the data never dominate the prior on θ_1 . The comparison between TK and MLE in Section 7 refers to the prior $\pi \propto 1/\theta$; under the uniform prior TK has smaller MSE than the MLE in S2 and S3 but larger in S1.

Table 5: TK approximation versus MCMC posterior mean of R , based on 100 independent record samples ($n = 30$) per scenario; $|\Delta|$ is the mean absolute difference between the TK and MCMC estimates.

Scen.	R	TK MSE	MCMC MSE	$ \Delta $	Corr.
S1	0.7557	0.0499	0.0441	0.026	0.989
S2	0.2200	0.0264	0.0308	0.030	0.978
S3	0.5000	0.0390	0.0394	0.036	0.981

7.7 Validation against Markov chain Monte Carlo

To assess the accuracy of the TK approximation directly, we compare it with a Markov chain Monte Carlo (MCMC) treatment of the same posterior. For each scenario we ran a random-walk Metropolis–Hastings sampler on $(\ln \theta_1, \ln \theta_2)$ under the joint posterior implied

by the record likelihood and the prior of Section 6.1 (3000 iterations, 1000-iteration burn-in, thinning interval 2), for 100 independent record samples of size $n = 30$ per scenario, and compared the MCMC posterior mean of R with the TK estimate on the same data.

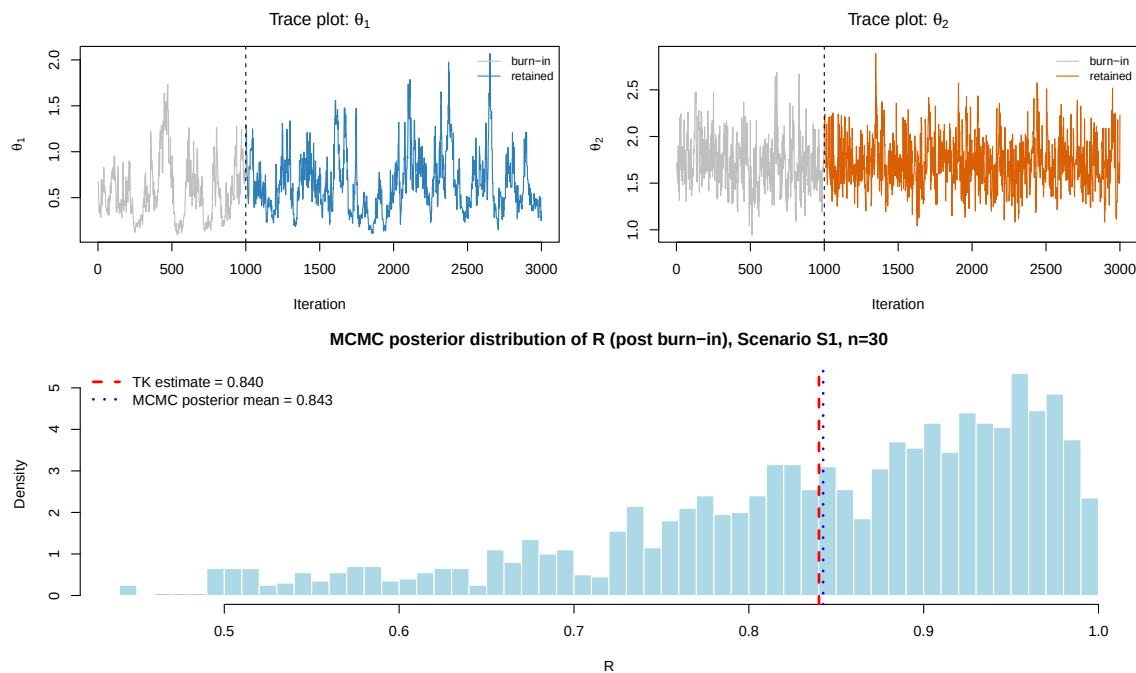


Fig. 1: MCMC diagnostics for one record sample from Scenario S1 ($\theta_1 = 0.8$, $\theta_2 = 1.5$, $R = 0.7557$, $n = m = 30$). Left: trace plots of θ_1 and θ_2 over 3000 Metropolis–Hastings iterations (the 1000-iteration burn-in is shown in grey). Right: MCMC posterior distribution of R after burn-in, with the TK estimate (dashed red line) and the MCMC posterior mean (dashed blue line).

The TK approximation and the MCMC posterior mean agree closely on a per-sample basis (Table 5, Fig. 1): their correlation exceeds 0.97 in every scenario and the mean absolute difference is 0.03–0.04. The two methods achieve comparable MSE against the true R , with neither dominating the other uniformly. In addition, for 40 samples from S1 with $n = 10$ we compared the TK estimate (computed with the full Hessian of Algorithm 1) with the exact posterior mean obtained by two-dimensional numerical integration on a 3000×3000 logarithmic grid; the correlation was 0.9997 and the largest absolute difference was 0.033.

8 Illustrative Application: NCCTG Lung Cancer Survival

Purpose of the application. This section illustrates the proposed methodology on real survival times: the fitting of the Mgamma model, the closed-form evaluation of R and the extraction of records. The stress variable is not recorded in the NCCTG data and is described through an assumed stress model.

The lung cancer survival data were obtained from the North Central Cancer Treatment Group (NCCTG) study [25] and are available in the `survival` package of R [39]. The data set contains survival times (in days) of

Table 6: First ten uncensored observations of the NCCTG lung cancer survival data.

Obs.	Time (days)	Scaled time (days/100)
1	306	3.06
2	455	4.55
3	210	2.10
4	883	8.83
5	1022	10.22
6	310	3.10
7	361	3.61
8	218	2.18
9	166	1.66
10	170	1.70

228 patients with advanced lung cancer. Of these, 63 times are right-censored. Because the record likelihood (8) is defined for exactly observed values, and censored values are not records of the survival-time sequence, the censored observations are excluded and the analysis uses the 165 uncensored times. These survival times serve as the strength variable X . Table 6 presents the first ten uncensored observations.

To bring the data to a suitable scale, survival times are divided by 100. The scaled strength variable ranges from 0.05 to 10.22, with mean 3.05 and standard deviation

Table 7: Reliability $R = P(Y < X)$ at selected quantiles of the prior of θ_2 , with $\theta_1 = \hat{\theta}_1 = 0.893$ (lung cancer data).

θ_2 prior quantile	θ_2	R
5%	0.795	0.448
25%	1.010	0.555
50% (median)	1.180	0.622
75%	1.368	0.682
95%	1.673	0.754

Table 8: Model comparison for the scaled lung cancer survival times (strength X), with the Kolmogorov–Smirnov (KS) test.

Model	$\log L$	AIC	BIC	KS p -value
Mgamma	−332.2	666.3	669.4	0.560
Exponential	−336.6	675.3	678.4	0.003
Lindley	−327.5	657.1	660.2	0.296
Xgamma	−328.9	659.7	662.9	0.133

2.31. The parameter θ_1 is estimated by maximum likelihood from these 165 scaled times (a complete-sample fit), with a nonparametric bootstrap interval ($B = 500$).

Stress variable: a prior-based treatment

The stress variable Y is not recorded in the NCCTG data set and is treated as an external burden. Rather than fixing θ_2 at a single value, we assign it a Gamma($k, k/\mu$) prior with mean $\mu = 1.2$ (an Mgamma stress distribution with mean of about 2.5 scaled units, i.e. 250 days) and shape $k = 20$, so that the central 50% prior interval of θ_2 is approximately (1.01, 1.37). The induced distribution of R is obtained by propagating this prior through Theorem 1, together with the bootstrap sampling distribution of $\hat{\theta}_1$, by Monte Carlo simulation. Because θ_2 is not estimated from data, the results for R in this section are conditional on this assumed stress model.

Averaging over the prior for θ_2 and over the bootstrap distribution of $\hat{\theta}_1$ gives a mean of R equal to 0.613 with a 95% interval of (0.408, 0.777) (Table 9). This interval reflects the uncertainty about the unobserved stress level and is therefore wider than an interval accounting only for sampling uncertainty in $\hat{\theta}_1$.

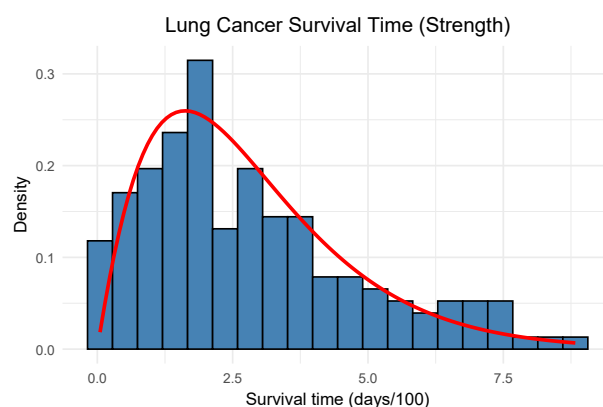
Model comparison

Table 8 reports the fits of the Mgamma, exponential, Lindley and Xgamma distributions. The Lindley distribution attains the lowest AIC and BIC, followed by the Xgamma and Mgamma models, while the Mgamma model attains the largest KS p -value (0.56). The

Table 9: Estimation results for the lung cancer data. The interval for R combines bootstrap uncertainty in $\hat{\theta}_1$ ($B = 500$) with the assumed prior for θ_2 .

Parameter	Estimate	95% interval
θ_1 (strength)	0.893	(0.824, 0.962)
θ_2 (stress, prior mean)	1.200	(0.795, 1.673) [†]
$R = P(Y < X)$	0.613	(0.408, 0.777)

[†] 5%–95% prior quantiles (assumed, not estimated).

**Fig. 2:** Histogram of the 165 uncensored NCCTG lung cancer survival times, scaled to units of 100 days (strength X), with the fitted Mgamma density ($\hat{\theta}_1 = 0.893$, red curve).

Mgamma, Lindley and Xgamma models are all retained by the KS test at the 5% level, whereas the exponential model is rejected. The Mgamma model therefore provides an adequate fit to these data according to the KS criterion, and it is used here because the application is designed to demonstrate the Mgamma-based methodology developed in this paper.

Results summary

Under the assumed stress model, the probability that the (scaled) survival time exceeds the stress is approximately 61%, with a 95% interval of (0.408, 0.777) reflecting sampling uncertainty in $\hat{\theta}_1$ and prior uncertainty in θ_2 .

Graphical illustrations

Figure 2 shows the histogram of the scaled survival times with the fitted Mgamma density, and Figure 3 the density of the assumed stress. Figure 4 overlays the two densities at the prior mean of θ_2 ; the substantial overlap explains why R is moderate rather than close to one.

Figure 5 illustrates the mixed record extraction. Since the NCCTG observations have no natural time order, we

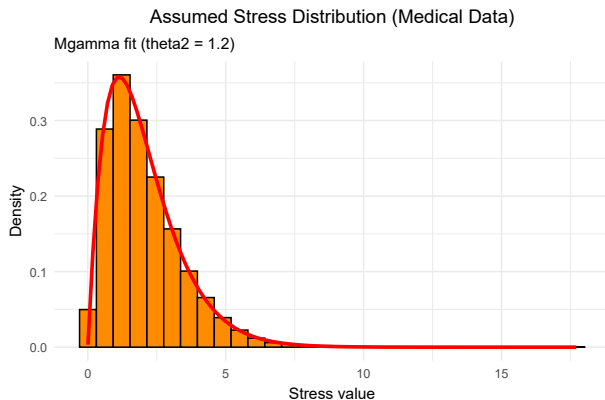


Fig. 3: Density of the assumed (not observed) stress variable $Y \sim \text{Mgamma}(\theta_2 = 1.2)$, the prior mean of θ_2 ; its mean is about 2.5 scaled units (250 days).

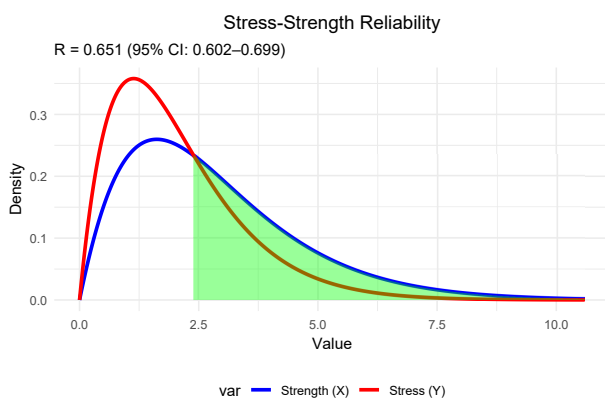


Fig. 4: Densities of the strength $X \sim \text{Mgamma}(0.893)$ (blue) and the assumed stress $Y \sim \text{Mgamma}(1.2)$ (red). At this fixed θ_2 , $R = P(Y < X) = 0.651$, with bootstrap interval $(0.602, 0.699)$ reflecting uncertainty in $\hat{\theta}_1$ only; averaging over the prior of θ_2 gives 0.613 (Table 9).

placed the 165 uncensored times in a random order (“shuffled” them, i.e. applied a random permutation with a fixed seed so that the result can be reproduced) and extracted the lower records from the resulting sequence. Since the stress is not observed, the upper records shown are those of a sequence simulated from the assumed stress model.

Figure 6 presents the bootstrap distribution of $\hat{R}$ at the prior mean of θ_2 . Accounting additionally for prior uncertainty in θ_2 widens the interval $(0.602, 0.699)$ to the one in Table 9. Figure 7 compares the empirical cdf with the fitted Mgamma cdf; the close agreement is consistent with the KS test in Table 8.

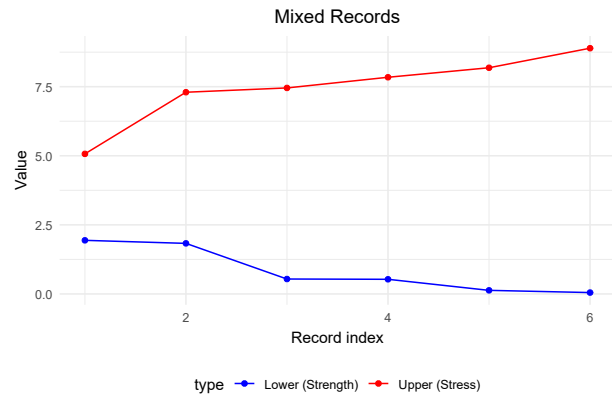


Fig. 5: Illustration of mixed record extraction. Blue: lower records of the strength sequence, obtained from the 165 uncensored scaled survival times after placing them in a random order (random permutation with a fixed seed). Red: upper records of a stress sequence generated from the assumed model $\text{Mgamma}(\theta_2 = 1.2)$. The decreasing lower and increasing upper records illustrate the sampling scheme of Section 4.

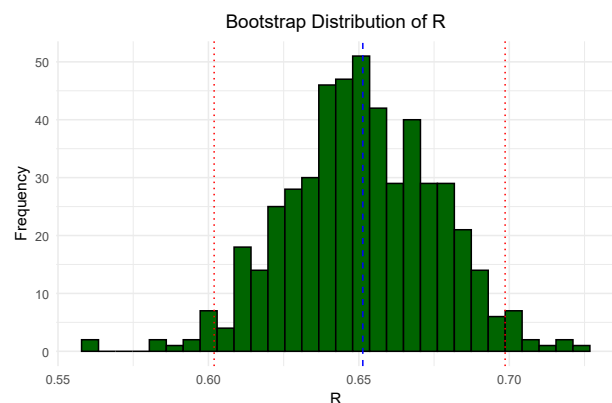


Fig. 6: Bootstrap distribution of $\hat{R}$ ($B = 500$ nonparametric resamples of the 165 uncensored times) with θ_2 fixed at its prior mean 1.2. Dashed blue line: point estimate $\hat{R} = 0.651$; dotted red lines: 95% percentile interval $(0.602, 0.699)$. This interval reflects sampling uncertainty in $\hat{\theta}_1$ only.

Discussion

Under the assumed stress model, the estimated reliability of about 0.61 means that the survival time exceeds the stress level with probability around 61%, and the 95% interval $(0.408, 0.777)$ reflects both sampling uncertainty and the uncertainty about the unobserved stress.

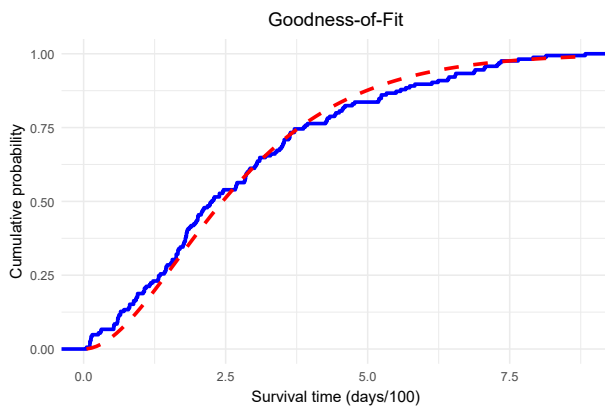


Fig. 7: Empirical cdf of the 165 scaled uncensored survival times (blue) and fitted Mgamma cdf with $\hat{\theta}_1 = 0.893$ (red dashed).

9 Conclusion

This paper developed Bayesian and frequentist estimation of the stress–strength reliability $R = P(Y < X)$ when both variables follow the Mgamma distribution under mixed upper–lower record sampling. A closed-form expression for R in terms of the regularized incomplete beta function was obtained, which reduces to a rational function of (θ_1, θ_2) . Maximum likelihood estimators were computed numerically, and a Bayesian method based on the Tierney–Kadane approximation was developed with explicit first and second derivatives of the log-posterior, of R , and of the TK-modified log-posterior $L^* = L + \ln R$, whose mixed derivative is non-zero.

The main findings are: (i) lower records carry a bounded amount of information about the strength parameter, so the precision of $\hat{R}$ is determined mainly by the first few strength records; (ii) profile-likelihood intervals attain coverage between 0.93 and 0.95 in all configurations studied; (iii) the prior sensitivity analysis shows that the prior $1/\theta$ and the $\text{Gamma}(0.01, 0.01)$ prior give practically identical results, whereas the uniform prior changes the TK estimates appreciably, because the data never dominate the prior on θ_1 ; and (iv) a comparison with MCMC confirms that the TK approximation tracks the posterior mean closely (correlation above 0.97) at a fraction of the computational cost.

In the simulations the TK estimator had smaller MSE than the MLE in the configurations with $R \approx 0.5$ and $R \approx 0.76$, whereas the MLE was better for $R \approx 0.22$. Accordingly, we recommend TK when moderate or high reliability is expected and the MLE when low reliability is expected, and the profile-likelihood interval for interval estimation. The real-data application illustrates the computations for the Mgamma model on survival times, under an assumed stress model.

Future research directions include dependent stress–strength variables, covariates, adaptive record sampling schemes, designs that combine records with additional information on the strength variable (for example inter-record times or several independent record sequences) to overcome the information bound of Proposition 1, and extension of the TK approach to other flexible lifetime distributions and to progressive censoring schemes.

Disclosure statement

The authors declare no competing interests.

Acknowledgment

The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4300206GSSR02.

Funding Statement

This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4300206GSSR02.

Data availability

The NCCTG lung cancer data are available in the R package *survival* [39]. The code used for the simulation study is available from the corresponding author upon reasonable request.

Appendix A. Observed information matrix

Because the strength and stress log-likelihoods separate, $\mathbf{J}(\boldsymbol{\theta}) = \text{diag}\{-d^2\ell_1/d\theta_1^2, -d^2\ell_2/d\theta_2^2\}$, with the lower-record sample $x_1 > \dots > x_n$ and the upper-record sample $y_1 < \dots < y_m$.

Strength block. With $F = F(x; \theta_1)$, and $F_\theta, F_{\theta\theta}$ as in (22) and (27),

$$-\frac{d^2\ell_1}{d\theta_1^2} = n \left[\frac{3}{\theta_1^2} - \frac{1}{(1 + \theta_1)^2} \right] + \sum_{i=1}^{n-1} \frac{F(x_i)F_{\theta\theta}(x_i) - F_\theta(x_i)^2}{F(x_i)^2}. \quad (\text{A.1})$$

Stress block. It is numerically preferable, and equivalent, to differentiate $\ln(1 - F(y; \theta_2)) = \ln A(y, \theta_2) - \theta_2 y$ directly:

$$\frac{\partial}{\partial \theta_2} \ln(1 - F) = \frac{1}{A} \frac{\partial A}{\partial \theta_2} - y, \quad (\text{A.2})$$

$$\frac{\partial^2}{\partial \theta_2^2} \ln(1 - F) = \frac{A \partial^2 A / \partial \theta_2^2 - (\partial A / \partial \theta_2)^2}{A^2}, \quad (\text{A.3})$$

with $\partial^2 A / \partial \theta^2 = x^2 / (1 + \theta)^3$ (evaluated at $x = y$, $\theta = \theta_2$). Then

$$-\frac{d^2 \ell_2}{d \theta_2^2} = m \left[\frac{3}{\theta_2^2} - \frac{1}{(1 + \theta_2)^2} \right] + \sum_{j=1}^{m-1} \frac{\partial^2}{\partial \theta_2^2} \ln(1 - F(y_j; \theta_2)). \quad (\text{A.4})$$

Wald intervals for θ_1 , θ_2 and, via (15), for R are obtained by inverting $\mathbf{J}(\hat{\theta})$.

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