Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/060105

Decomposition Formulas for Srivastava's Hypergeometric Functions of Three Complex Variables I

Mosaed M. Makky

Departement of Mathematics, Faculty of Science, South Valley University, Qena, Egypt.

Received: 23 Nov. 2017, Revised: 21 Dec. 2017, Accepted: 27 Dec. 2017
Published online: 1 Jan. 2018

Abstract: In the spirit of Hasanov, Srivastava, and Turaev (2006), Ernst introduce new inverse operators together with a more general operator and find a summation formula for the last one. Based these operators and the earlier known q-analogues of the Burchnall-Chaundy operators. In section 1 some definitions and notations are introduce the inverse q-analogue operators for an arbitrary number of three complex variables, also, we derive some computations formulas for H_B Srivastava's hypergeometric functions. Finally, in section 2 we investigate some derivations of decomposition formulas for Srivastava's qH_B- hypergeometric functions of three complex variables are given.

Keywords: Decomposition formulas, Srivastava's hypergeometric functions, Gauss hypergeometric function, complex variables.

REtracted

1 Introduction

The major development of the theory of hypergeometric function was carried out by Gauss. This work is also noted as being the real beginning of in mathematics. Some important results concerning the hypergeometric functions have been developed earlier by Euler and others, but it was Gauss who made the first systematic study of the series that defines this function.
Hypergeometric series are very important in mathematics. Almost all of the elementary functions of mathematics are either hypergeometric, ratios of hypergeometric functions or limiting cases of a hypergeometric series.
The study of hypergeometric series was essentially started by Gauss when he considered the infinite series

$$1+\frac{ab}{1!c}z+\frac{a(a+1)b(b+1)}{2!c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}z^3+....$$

as a function of a, b, c and z, where it is assumed that c cannot be zero or a negative integer, so that no zero factor appears in the denominators of the terms of the series.
By the above hypergeometric series, is meant the power series

$$\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{n!\,(c)_n}\,z^n$$

where z is the complex variable, a, b and c are parameters which can take arbitrary real or complex values (provided that $(c\neq\ 0,-1,-2,....)$, and the symbol $(a)_n$ denotes the Pochhammer symbol or the shifted factorial defined as

$$(a)_n=\begin{cases}1, & n=0\\ a(a+1)(a+2).....(a+n-1), & n=1,2,3,...\end{cases}$$

The sum of this series is called the hypergeometric function and denoted by (c.f. [1]) ${}_2F_1(a,b;c;z)$. One of the most important properties of the hypergeometric function is that terms of the series do not change if the numerator parameters a and b are permuted, we obtain symmetry property

$${}_2F_1(a,b;c;z)={}_2F_1(b,a;c;z).$$

The concept of decomposition formulas for multiple hypergeometric functions is well known from the articles by Hasanov [2] and Bin-Saad [3].This paper follows the first one by using q-analogues of 15 symbolic operator formulas.
Also, we give the basic definitions and introduce a new inverse q-analogue of the Burchnall-Chaundy [4,5]

* Corresponding author e-mail: mosaed_makky11@yahoo.com, mosaed_makky@sci.svu.edu.eg

operators for an arbitrary number of variables, together with an explicit summation formula for one of these; the proof uses a q-Lauricella function summation formula. Also, we express the three triple Srivastava's q-hypergeometric functions in terms of these operators and q-hypergeometric and q-Appell's functions.
We start by defining the umbral notation [6], a mixture of Heine and [7].
Definition: The power function is defined by $q^n = e^{a\log(q)}$. We always assume that $0 < q < 1$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < Im(\log\, q) \leq \pi + \delta$, This defines a simply connected space in the complex plane.
The variables $a, b, c, \ldots.. \in C$ denote certain parameters. The variables i, j, k, l, m, n will denote natural numbers except for certain cases where it will be clear from the context that i will denote the imaginary unit. The q-shifted factorial given as follows:

$$\langle a;q\rangle_n = \prod_{m=0}^{n-1}\left(1-q^{a+m}\right) \quad ; \quad (a,q)_n = \prod_{m=0}^{n-1}\left(1-aq^m\right). \tag{1}$$

Since products of q-shifted factorials occur so often, to simplify them, we will frequently use the more compact notation:

$$\langle a_1, a_2, a_3, \ldots, a_m; q\rangle_n = \prod_{j=0}^{m}\langle a_j; q\rangle_n \tag{2}$$

take

$$\overline{\langle a;q\rangle}_n = \prod_{m=0}^{n-1}\left(1+q^{a+m}\right) \tag{3}$$

Suppose that (m,l) = 1; that is, m and l relatively prime. The operator

$$\frac{\widetilde{m}}{l} : \frac{C}{Z} \to \frac{C}{Z} \tag{4}$$

is defined by

$$a \to a + \frac{2\pi\, im}{l\, \log q}. \tag{5}$$

Furthermore

$$(a;q)_\infty = \prod_{m=0}^{\infty}\left(1-aq^m\right) \;\; ; \quad 0 < |q| < 1 \tag{6}$$

$$(a;q)_\alpha = \frac{(a,q)_\infty}{(aq^\alpha;q)_\infty} \;\; ; \quad \alpha \neq q^{-m-\alpha} \;\; , \;\; m = 0, 1, \ldots$$

Suppose that a q-Appell's hypergeometric function in the form (c.f.[4,8,9])

$$F_1(\alpha;\beta,\beta';\gamma|q;z_1,z_2) = \sum_{n_1,n_2=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_2}\langle\beta,q\rangle_{n_1}\langle\beta',q\rangle_{n_2}}{\langle 1,q\rangle_{n_1}\langle 1,q\rangle_{n_2}\langle\gamma,q\rangle_{n_1+n_2}}(z_1)^{n_1}(z_2)^{n_2} \tag{7}$$

where
$\max(|z_1|,\, |z_2|) < 1$

$$F_2(\alpha;\beta,\beta';\gamma,\gamma'|q;z_1,z_2) = \sum_{n_1,n_2=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_2}\langle\beta,q\rangle_{n_1}\langle\beta',q\rangle_{n_2}}{\langle 1,q\rangle_{n_1}\langle 1,q\rangle_{n_2}\langle\gamma,q\rangle_{n_1}\langle\gamma',q\rangle_{n_2}}(z_1)^{n_1}(z_2)^{n_2} \tag{8}$$

where
$\max(|z_1|,\, |z_2|) < 1$

$$F_3(\alpha,\alpha';\beta,\beta';\gamma|q;z_1,z_2) = \sum_{n_1,n_2=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1}\langle\alpha',q\rangle_{n_2}\langle\beta,q\rangle_{n_1}\langle\beta',q\rangle_{n_2}}{\langle 1,q\rangle_{n_1}\langle 1,q\rangle_{n_2}\langle\gamma,q\rangle_{n_1+n_2}}(z_1)^{n_1}(z_2)^{n_2} \tag{9}$$

where
$\max(|z_1|,\, |z_2|) < 1$

$$F_4(\alpha;\beta;\gamma,\gamma'|q;z_1,z_2) = \sum_{n_1,n_2=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_2}\langle\beta,q\rangle_{n_1+n_2}}{\langle 1,q\rangle_{n_1}\langle 1,q\rangle_{n_2}\langle\gamma,q\rangle_{n_1}\langle\gamma',q\rangle_{n_2}}(z_1)^{n_1}(z_2)^{n_2} \tag{10}$$

where
$\left|\sqrt{z_1}\right| + \left|\sqrt{z_2}\right| < 1.$

The q-analogues of the Srivastava triple hypergeometric functions are (c.f. [10])

$$H_B(\alpha;\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3) \tag{11}$$

$$= \sum_{n_1,n_2,n_3=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_3}\langle\beta,q\rangle_{n_1+n_2}\langle\beta',q\rangle_{n_2+n_3}}{\langle 1,\gamma;q\rangle_{n_1}\langle 1,\gamma';q\rangle_{n_2}\langle 1,\gamma'';q\rangle_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

Now, we can write the following inverse pair of symbolic operators (see [11,12]) as follows:

$$\theta_{q,j} = z_j D_{q,z_j}.$$

Then

$$\nabla_{z_1,z_2;q}(h) = \Gamma_q\begin{bmatrix} h, h+\theta_{q,1}+\theta_{q,2} \\ h+\theta_{q,1}, h+\theta_{q,2}\end{bmatrix};$$

$$\Delta_{z_1,z_2;q}(h) = \Gamma_q\begin{bmatrix} h+\theta_{q,1}, h+\theta_{q,2} \\ h+\theta_{q,1}+\theta_{q,2}, h\end{bmatrix}$$

We limit our considerations to the case $h \neq -n$.

On the ring of polynomials $C[x]$, we define the functions $\varepsilon_1 : C[x] \to C[x]$by

$$[\varepsilon_i]\, f(x_i) \equiv f(qx_i).$$

Then one has

$$\nabla_{z_1,z_2;q}(h)\langle h,q\rangle_{n_1}\langle h,q\rangle_{n_2}(z_1)^{n_1}(z_2)^{n_2} = \langle h,q\rangle_{n_1+n_2}(z_1)^{n_1}(z)^{n_2} \tag{12}$$

$$\nabla_{z_1,z_2;q}(h)=\sum_{r=0}^{\infty}\frac{\langle -\theta_{q,1},-\theta_{q,2};q\rangle_r}{\langle 1,h;q\rangle_r}(q)^{rh}\in_1^r\in_2^r. \tag{13}$$

Similarly, we obtain

$$\Delta_{z_1,z_2;q}(h)=\sum_{r=0}^{\infty}\frac{\langle -\theta_{q,1},-\theta_{q,2};q\rangle_r}{\langle 1,1-h-\theta_{q,1}-\theta_{q,2};q\rangle_r}(q)^{r}, \tag{14}$$

$$\Delta_{z_1,z_2;q}(h)=\sum_{r=0}^{\infty}\frac{(-1)^r\langle -\theta_{q,1},-\theta_{q,2};q\rangle_r\langle h;q\rangle_{2r}}{\langle 1,h+r-1,h+\theta_{q,1},h+\theta_{q,2};q\rangle_r}q^{\binom{r}{2}+rh}\in_1^r\in_2^r. \tag{15}$$

$$\nabla_{z_1,z_2;q}(h)\Delta_{z_1,z_2;q}(k)=\sum_{r=0}^{\infty}\frac{\langle -\theta_{q,1},-\theta_{q,2};q,k-h;q\rangle_{2r}}{\langle 1,h,k+r-1,k+\theta_{q,1},k+\theta_{q,2};q\rangle_r}q^{rh}\in_1^r\in_2^r. \tag{16}$$

Now, introduce the inverse pair of operators:

$$\tilde{\Delta}_{z_1,z_2,z_3;q}(h)\frac{1}{\langle h;q\rangle_{n_1}\langle h;q\rangle_{n_2+n_3}}z_1^{n_1}z_2^{n_2}z_3^{n_3}=\frac{1}{\langle h;q\rangle_{n_1+n_2+n_3}}z_1^{n_1}z_2^{n_2}z_3^{n_3} \tag{17}$$

$$\tilde{\nabla}_{z_1,z_2,z_3;q}(h)\frac{1}{\langle h;q\rangle_{n_1+n_2+n_3}}z_1^{n_1}z_2^{n_2}z_3^{n_3}=\frac{1}{\langle h;q\rangle_{n_1}\langle h;q\rangle_{n_2+n_3}}z_1^{n_1}z_2^{n_2}z_3^{n_3}. \tag{18}$$

Consider the q-analogues for the function $H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3)$ as given in the relation (11) by using the operators (12) to (18) as follows:

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3) \tag{19}$$

$$=\nabla_{z_1,z_3;q}(\alpha)\,\nabla_{z_1,z_2;q}(\beta)\,\nabla_{z_2,z_3;q}(\gamma')\;.\,{}_2\Phi_1(\alpha,\beta;\gamma|q;-z_1)\,\Phi_1(\beta',\beta,\alpha;\gamma'|q;-z_2,-z_3)$$

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3) \tag{20}$$

$$=\nabla_{z_1,z_3;q}(\alpha)\,\nabla_{z_1,z_2;q}(\beta)\,{}_2\Phi_1(\alpha,\beta;\gamma|q;-z_1)\;.\,\Phi_2(\beta',\beta,\alpha;\gamma',\gamma''|q;-z_2,-z_3)$$

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3) \tag{21}$$

$$=\nabla_{z_1,z_3;q}(\alpha)\,\nabla_{z_1,z_2;q}(\beta)\,\nabla_{z_2,z_3;q}(\beta')\,\nabla_{z_2,z_3;q}(\gamma)\;.\,{}_2\Phi_1(\alpha,\beta;\gamma|q;-z_1)\,\Phi_3(\beta',\alpha,\beta,\beta';\gamma|q;-z_2,-z_3)$$

$$H_B(\alpha,\alpha,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3) \tag{22}$$

$$=\nabla_{z_1,z_3;q}(\alpha)\,\nabla_{z_1,z_2;q}(\alpha)\,\Delta_{z_2,z_3;q}(\alpha)\,{}_2\Phi_1(\alpha,\alpha;\gamma|q;-z_1)\;.\,\Phi_4(\alpha,\beta';\gamma',\gamma''|q;-z_2,-z_3)$$

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3) \tag{23}$$

$$=\nabla_{z_1,z_3;q}(\alpha)\,\nabla_{z_1,z_2;q}(\beta)\,\nabla_{z_2,z_3;q}(\beta')\,{}_2\Phi_1(\alpha,\beta;\gamma|q;-z_1)\;.\,{}_2\Phi_1(\beta,\beta';\gamma'|q;-z_2)\,{}_2\Phi_1(\alpha,\beta';\gamma''|q;-z_3)$$

2 Decompositions for Srivastava's qH_B -hypergeometric functions

In this section, we will present a study of the decompositions of qH_B- hypergeometric functions using the operators given in relationships (19) to (23) for the H_B- hypergeometric. This study is considered for hypergeometric functions when the variables are complex which as given in (11).

Now, we prove the following theorems:

Theorem (1): Suppose that the Srivastava qH_B-hypergeometric functions

$$H_B(\alpha;\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3)$$

$$=\Sigma_{n_1,n_2,n_3=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_3}\langle\beta,q\rangle_{n_1+n_2}\langle\beta',q\rangle_{n_2+n_3}}{\langle 1,\gamma;q\rangle_{n_1}\langle 1,\gamma';q\rangle_{n_2}\langle 1,\gamma';q\rangle_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}.$$

Then by using the operator given in (19) the following relation is true.

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3) \tag{24}$$

$$=\Sigma_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_1+n_2}\langle\beta';q\rangle_{n_1+n_2+2n_3}q^{n_2(\beta+n_2-1)}}{\langle 1,1,1,\beta;q\rangle_{n_2}\langle 1,1,1,\alpha;q\rangle_{n_1}\langle 1,1,1,\gamma;q\rangle_{n_3}}$$

$$.\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_2+n_3}(-z_3)^{n_1+n_3}q^{n_1(\alpha+n_1-1)+n_3(\beta'+n_3-1)}\langle 1,\alpha;q\rangle_{n_1+n_3}\langle 1,\beta;q\rangle_{n_2+n_3}}{\langle\gamma;q\rangle_{n_1+n_2}\langle\gamma';q\rangle_{n_1+n_2+2n_3}}$$

$$.{}_4\Phi_3\left[\begin{matrix}1,\alpha+n_1+n_2,\beta+n_1+n_2,1+n_1+n_2\\ 1+n_2,\gamma+n_1+n_2,1+n_1\end{matrix}\middle|q;-z_1\right]$$

$$.\Phi_1\left[\begin{matrix}\beta'+n_1+n_2+2n_3:1,\beta+n_2+n_3,1+n_2+n_3;1,\alpha+n_1+n_3,1+n_1+n_3\\ \gamma'+n_1+n_2+2n_3:1+n_3,1+n_2;1+n_3,1+n_1\end{matrix}\middle|q;-z_2,-z_3\right].$$

Proof:
By the relation (19), the computation goes as follows:

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3)$$

$$=\Sigma_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{\langle -n_4,-n_6;q\rangle_{n_1}q^{n_1(\alpha+n_4+n_6)}\langle -n_4,-n_5;q\rangle_{n_2}q^{n_2(\beta+n_4+n_5)}\langle -n_5,-n_6;q\rangle_{n_3}}{\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}\langle 1,\gamma';q\rangle_{n_3}}$$

$$.\,Z^M\frac{q^{n_3(\gamma'+n_5+n_6)}\langle\alpha,\beta;q\rangle_{n_4}\langle\beta';q\rangle_{n_5+n_6}\langle\beta;q\rangle_{n_5}\langle\alpha;q\rangle_{n_6}}{\langle 1,\gamma;q\rangle_{n_4}\langle\gamma';q\rangle_{n_5+n_6}\langle 1;q\rangle_{n_5}\langle 1;q\rangle_{n_6}}$$

$$=\Sigma_{n_1,n_2,n_3,n_4,n_5=0}^{\infty}\frac{q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)+n_3(\gamma'+n_3-1)}}{\langle 1;q\rangle_{n_4-n_1}\langle 1,\alpha;q\rangle_{n_1}\langle 1;q\rangle_{n_4-n_2}\langle 1;q\rangle_{n_6-n_1}\langle 1;q\rangle_{n_6-n_3}\langle 1,\beta;q\rangle_{n_2}\langle 1,\gamma';q\rangle_{n_3}}$$

$$.\,Z^M\frac{\langle 1,\alpha,\beta;q\rangle_{n_4}\langle\beta';q\rangle_{n_5+n_6}\langle 1,\beta;q\rangle_{n_5}\langle 1,\alpha;q\rangle_{n_6}}{\langle 1;q\rangle_{n_5-n_2}\langle 1;q\rangle_{n_5-n_3}\langle\gamma;q\rangle_{n_4}\langle\gamma';q\rangle_{n_5+n_6}}$$

$$=\Sigma_{n_1,n_2,n_3,n_4,n_5=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_1+n_2+n_4}\langle 1,\alpha;q\rangle_{n_1+n_3+n_6}\langle 1,\beta;q\rangle_{n_2+n_3+n_5}}{\langle 1;q\rangle_{n_4+n_2}\langle 1;q\rangle_{n_4+n_1}\langle 1;q\rangle_{n_3+n_6}\langle 1;q\rangle_{n_1+n_6}}$$

$$.\frac{\langle\beta';q\rangle_{n_1+n_2+2n_3+n_5+n_6}}{\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}\langle 1,\gamma';q\rangle_{n_3}\langle 1;q\rangle_{n_2+n_5}}$$

$$.\frac{(-z_1)^{n_1+n_2+n_4}(-z_2)^{n_2+n_5+n_3}(-z_3)^{n_1+n_3+n_6}q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)+n_3(\gamma'+n_3-1)}}{\langle 1;q\rangle_{n_3+n_5}\langle\gamma';q\rangle_{n_1+n_2+2n_3+n_5+n_6}\langle\gamma;q\rangle_{n_1+n_2+n_4}}=RHS$$

i.e. the relation (24) is true, since $Z^M=(-z_1)^{n_4}(-z_2)^{n_5}(-z_3)^{n_6}$.

Theorem (2):
Let the Srivastava qH_B- hypergeometric functions

$$H_B(\alpha;\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3)$$

$$=\Sigma_{n_1,n_2,n_3=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_3}\langle\beta,q\rangle_{n_1+n_2}\langle\beta',q\rangle_{n_2+n_3}}{\langle 1,\gamma,q\rangle_{n_1}\langle 1,\gamma';q\rangle_{n_2}\langle 1,\gamma'';q\rangle_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

Then by using the operator given in (20) the following relation is true.

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3) \tag{25}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5=0}^{\infty}\frac{\langle 1,\alpha,\beta,\beta';q\rangle_{n_1+n_2}}{\langle 1,1,\gamma'';q\rangle_{n_1}\langle\gamma;q\rangle_{n_1+n_2}\langle 1,1,\gamma';q\rangle_{n_2}}$$
$$.(-z_1)^{n_1+n_2}(-z_2)^{n_2}(-z_3)^{n_1}q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)}$$

$$._4\Phi_3\left[\begin{matrix}\alpha+n_1+n_2,\beta+n_1+n_2,1+n_1+n_2,1\\ 1+n_1,1+n_2,\gamma+n_1+n_2\end{matrix}\middle|q;-z_1\right]$$
$$.\Phi_2\left[\begin{matrix}\beta'+n_1+n_2:1,\beta+n_2,\infty;\alpha+n_1,\infty\\ \infty:\gamma'+n_2;\gamma''+n_1\end{matrix}\middle|q;-z_2,-z_3\right]$$

Proof:
By the relation (20), the computation goes as follows:

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3)$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5=0}^{\infty}\frac{\langle -n_3,-n_5;q\rangle_{n_1}q^{n_1(\alpha+n_3+n_5)}\langle -n_3,-n_4;q\rangle_{n_2}q^{n_2(\beta+n_3+n_4)}}{\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}}$$

$$.Z^M\frac{\langle\alpha,\beta;q\rangle_{n_3}\langle\beta';q\rangle_{n_4+n_5}\langle\beta;q\rangle_{n_4}\langle\alpha;q\rangle_{n_5}}{\langle 1,\gamma;q\rangle_{n_3}\langle 1,\gamma';q\rangle_{n_4}\langle 1,\gamma'';q\rangle_{n_5}}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_3}\langle\beta';q\rangle_{n_4+n_5}\langle\beta;q\rangle_{n_4}\langle\alpha;q\rangle_{n_5}}{\langle 1;q\rangle_{n_3-n_1}\langle 1,\alpha;q\rangle_{n_1}\langle 1;q\rangle_{n_3-n_2}\langle 1;q\rangle_{n_5-n_1}\langle 1,\beta;q\rangle_{n_2}}$$

$$.Z^M\frac{q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)}}{\langle 1;q\rangle_{n_4-n_2}\langle\gamma;q\rangle_{n_3}\langle\gamma';q\rangle_{n_4}\langle\gamma'';q\rangle_{n_5}}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_3+n_1+n_2}\langle\alpha;q\rangle_{n_1+n_5}\langle\beta;q\rangle_{n_2+n_4}\langle\beta';q\rangle_{n_1+n_2+n_4+n_5}}{\langle 1;q\rangle_{n_3+n_1}\langle 1;q\rangle_{n_3+n_2}\langle 1;q\rangle_{n_5}\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}\langle 1;q\rangle_{n_4}}$$

$$.\frac{(-z_1)^{n_3+n_1+n_2}(-z_2)^{n_2+n_4}(-z_3)^{n_1+n_5}q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)}}{\langle\gamma;q\rangle_{n_3+n_1+n_2}\langle\gamma';q\rangle_{n_4+n_2}\langle\gamma'';q\rangle_{n_5+n_1}}=RHS$$

i.e. the relation (25) is true.
Theorem (3):
Let the Srivastava qH_B- hypergeometric functions

$$H_B(\alpha;\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3)$$
$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_3}\langle\beta,q\rangle_{n_1+n_2}\langle\beta',q\rangle_{n_2+n_3}}{\langle 1,\gamma;q\rangle_{n_1}\langle 1,\gamma';q\rangle_{n_2}\langle 1,\gamma';q\rangle_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

Then by using the operator given in (21) the following relation is true:

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3) \tag{26}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6,n_7=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_1+n_2}q^{n_2(\beta+n_2-1)+n_4(\gamma'+n_4-1)}q^{n_1(\alpha+n_1-1)+n_3(\beta'+n_3-1)}}{\langle 1,1;q\rangle_{n_3+n_4}\langle 1;q\rangle_{n_1+n_4}\langle 1;q\rangle_{n_2+n_4}\langle 1;q\rangle_{n_2+n_3}\langle 1;q\rangle_{n_1+n_3}\langle 1,\gamma';q\rangle_{n_4}}$$

$$.\frac{(-z_1)^{n_1+n_2}(-z_2)^{n_2+n_3+n_4}(-z_3)^{n_1+n_3+n_4}\langle 1,1,\alpha,\beta';q\rangle_{n_1+n_3+n_4}\langle 1,1,\beta,\beta';q\rangle_{n_2+n_3+n_4}}{\langle 1,1,\beta;q\rangle_{n_2}\langle 1,1,\alpha;q\rangle_{n_1}\langle 1,\beta';q\rangle_{n_3}\langle\gamma;q\rangle_{n_1+n_2}\langle\gamma';q\rangle_{n_1+n_2+2n_3+2n_4}}$$

$$._4\Phi_3\left[\begin{matrix}1,\alpha+n_1+n_2,\beta+n_1+n_2,1+n_1+n_2\\ 1+n_2,\gamma+n_1+n_2,1+n_1\end{matrix}\middle|q;-z_1\right]$$

$$.\Phi_3\left[\begin{matrix}\infty:a\\ \gamma'+n_1+n_2+2n_3+2n_4:b\end{matrix}\middle|q;-z_2,-z_3\right]$$

where

$$a=(1,1+n_2+n_3+n_4,1+n_2+n_3+n_4,\beta+n_2+n_3+n_4,\beta'+n_2+n_3+n_4);$$
$$(1,1+n_1+n_3+n_4,1+n_1+n_3+n_4,\alpha+n_1+n_3+n_4,\beta'+n_1+n_3+n_4)$$

$$b=(1+n_2+n_3,1+n_3+n_4,1+n_2+n_4,\infty);(1+n_3+n_4,1+n_1+n_4,1+n_1+n_3,\infty)$$

Proof:
By the relation (21), the computation goes as follows:

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma'|q;-z_1,-z_2,-z_3)$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6,n_7=0}^{\infty}\frac{\langle -n_5,-n_7;q\rangle_{n_1}q^{n_1(\alpha+n_5+n_7)}\langle -n_5,-n_6;q\rangle_{n_2}q^{n_2(\beta+n_5+n_6)}\langle -n_6,-n_7;q\rangle_{n_3}}{\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}\langle 1,\beta';q\rangle_{n_3}\langle 1,\gamma';q\rangle_{n_4}}$$
$$.\frac{\langle\alpha,\beta;q\rangle_{n_5}\langle\beta,\beta';q\rangle_{n_6}\langle\alpha,\beta';q\rangle_{n_7}\langle -n_6,-n_7;q\rangle_{n_4}(-z_1)^{n_5}(-z_2)^{n_6}(-z_3)^{n_7}q^{n_3(\beta'+n_6+n_7)+n_4(\gamma'+n_6+n_7)}}{\langle 1,\gamma;q\rangle_{n_5}\langle\gamma';q\rangle_{n_6+n_7}\langle 1;q\rangle_{n_6}\langle 1;q\rangle_{n_7}}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6,n_7=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_5}q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)+n_3(\beta'+n_3-1)+n_4(\gamma'+n_4-1)}}{\langle 1;q\rangle_{n_5-n_1}\langle 1,\alpha;q\rangle_{n_1}\langle 1;q\rangle_{n_5-n_2}\langle 1;q\rangle_{n_7-n_1}\langle 1;q\rangle_{n_7-n_3}\langle 1,\beta;q\rangle_{n_2}\langle 1,\beta';q\rangle_{n_3}}$$

$$.\frac{(-z_1)^{n_5}(-z_2)^{n_6}(-z_3)^{n_7}\langle 1,1,\beta,\beta';q\rangle_{n_6}\langle 1,1,\alpha,\beta';q\rangle_{n_7}}{\langle 1;q\rangle_{n_6-n_2}\langle 1;q\rangle_{n_6-n_3}\langle 1;q\rangle_{n_6-n_4}\langle 1;q\rangle_{n_7-n_4}\langle\gamma;q\rangle_{n_5}\langle\gamma';q\rangle_{n_6+n_7}\langle 1,\gamma';q\rangle_{n_4}}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6,n_7=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_1+n_2+n_5}\langle 1,1,\alpha,\beta';q\rangle_{n_1+n_3+n_4+n_7}}{\langle 1;q\rangle_{n_5+n_2}\langle 1;q\rangle_{n_5+n_1}\langle 1;q\rangle_{n_3+n_4+n_7}\langle 1;q\rangle_{n_1+n_4+n_7}\langle 1;q\rangle_{n_1+n_3+n_7}}$$
$$.\frac{\langle 1,1,\beta,\beta';q\rangle_{n_2+n_3+n_4+n_6}(-z_1)^{n_1+n_2+n_5}(-z_2)^{n_2+n_3+n_4+n_6}(-z_3)^{n_1+n_3+n_4+n_7}}{\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}\langle 1,\beta';q\rangle_{n_3}\langle 1;q\rangle_{n_2+n_3+n_6}\langle 1;q\rangle_{n_3+n_4+n_6}}$$
$$.\frac{q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)+n_3(\beta'+n_3-1)+n_4(\gamma'+n_4-1)}}{\langle\gamma;q\rangle_{n_1+n_2+n_5}\langle\gamma';q\rangle_{n_1+n_2+2n_3+2n_4+n_6+n_7}\langle 1,\gamma';q\rangle_{n_1}\langle 1;q\rangle_{n_2+n_4+n_6}}=RHS.$$

Therefore the relation (26) is true.

Theorem (4):

$$H_B(\alpha;\alpha,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3)$$
$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_3}\langle\alpha,q\rangle_{n_1+n_2}\langle\beta',q\rangle_{n_2+n_3}}{\langle 1,\gamma;q\rangle_{n_1}\langle 1,\gamma';q\rangle_{n_2}\langle 1,\gamma'';q\rangle_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

Then by using the operator given in (22) the following relation is true:

$$H_B(\alpha,\alpha,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3) \tag{27}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}$$
$$.\frac{q^{n_1(\alpha+n_1-1)+n_2(\alpha+n_2-1)}q^{\binom{3}{2}\left((n_3)^2-n_3\right)+n_3\alpha}\langle 1,\alpha,\alpha;q\rangle_{n_1+n_2}(-z_1)^{n_1+n_2}(-z_2)^{n_3+n_2}(-z_3)^{n_1+n_3}}{\langle\alpha+n_3;q\rangle_{n_2+n_3}\langle\alpha+n_3;q\rangle_{n_1+n_3}\langle\gamma';q\rangle_{n_2+n_3}\langle\gamma'';q\rangle_{n_1+n_3}}$$

$$.\frac{\langle 1,\alpha;q\rangle_{n_3+n_2}\langle\alpha,\beta';q\rangle_{n_1+n_2+2n_3}(-1)^{n_3}\langle\alpha;q\rangle_{2n_3}\langle 1,\alpha;q\rangle_{n_1+n_3}\langle 1+n_1+n_3,\alpha+n_1+n_3;q\rangle_{n_3}}{\langle 1,1,1,\alpha;q\rangle_{n_1}\langle 1,1,1,\alpha;q\rangle_{n_2}\langle\gamma;q\rangle_{n_1+n_2}\langle 1,1,\alpha+n_3-1,\alpha,\alpha;q\rangle_{n_3}}$$

$$._4\Phi_3\left[\begin{matrix}1,1+n_1+n_2,\alpha+n_1+n_2,\alpha+n_1+n_2\\ 1+n_1,1+n_2,\gamma+n_1+n_2\end{matrix}\middle|q;-z_1\right]$$

$$.\Phi_4\left[\begin{matrix}\alpha+n_1+n_2+2n_3,\beta'+n_1+n_2+2n_3;a\\ \infty,\infty,b\end{matrix}\middle|q;-z_2,-z_3\right]$$

where

$$a=(1,1+n_3+n_2,\alpha+n_3+n_2,\infty,\infty);\ (1,1+n_1+n_3,\alpha+n_1+n_3,\infty,\infty)$$

$$b=(\alpha+n_2+2n_3,\gamma'+n_2+n_3,1+n_3,1+n_2);\ (\alpha+n_1+2n_3,\gamma''+n_1+n_3,1+n_1,1+n_3)$$

Proof:
By the relation (22), the computation goes as follows:

$$H_B(\alpha,\alpha,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3)$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{\langle -n_4,-n_6;q\rangle_{n_1} q^{n_1(\alpha+n_4+n_6)}\langle -n_4,-n_5;q\rangle_{n_2} q^{n_2(\alpha+n_4+n_5)}(-1)^{n_3}}{\langle 1,\alpha;q\rangle_{n_1}\langle 1,\alpha;q\rangle_{n_2}}$$

$$.\,Z^M\frac{\langle -n_5,-n_6;q\rangle_{n_3}\langle\alpha;q\rangle_{2n_3}}{\langle 1,\alpha+n_3-1,\alpha+n_5,\alpha+n_6;q\rangle_{n_3}}\;\frac{\langle\alpha,\alpha;q\rangle_{n_4}\langle\alpha,\beta';q\rangle_{n_5+n_6}}{\langle 1,\gamma;q\rangle_{n_4}\langle 1,\gamma';q\rangle_{n_5}\langle 1,\gamma'';q\rangle_{n_6}}q^{\binom{n_3}{2}+n_3(\alpha+n_5+n_6)}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{\langle 1,\alpha,\alpha;q\rangle_{n_4} q^{n_1(\alpha+n_1-1)+n_2(\alpha+n_2-1)}\langle 1,\alpha;q\rangle_{n_6}(-1)^{n_3}\langle\alpha;q\rangle_{2n_3}}{\langle\gamma';q\rangle_{n_5}\langle\gamma'';q\rangle_{n_6}\langle 1;q\rangle_{n_4-n_1}\langle 1;q\rangle_{n_4-n_2}\langle 1,\alpha;q\rangle_{n_1}\langle 1,\alpha;q\rangle_{n_2}\langle 1;q\rangle_{n_6-n_1}}$$

$$.\,Z^M\frac{\langle\alpha,\beta';q\rangle_{n_5+n_6}\langle\alpha,1;q\rangle_{n_5} q^{\left(\frac{3}{2}\right)\left((n_3)^2-n_3\right)+n_3\alpha}}{\langle 1;q\rangle_{n_5-n_2}\langle 1;q\rangle_{n_5-n_3}\langle\gamma;q\rangle_{n_4}\langle\alpha+n_3;q\rangle_{n_5}\langle\alpha+n_3;q\rangle_{n_6}\langle 1;q\rangle_{n_6-n_3}\langle 1,\alpha+n_3-1,\alpha,\alpha;q\rangle_{n_3}}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{q^{n_1(\alpha+n_1-1)+n_2(\alpha+n_2-1)}\langle 1,\alpha,\alpha;q\rangle_{n_4+n_1+n_2} q^{\left(\frac{3}{2}\right)\left((n_3)^2-n_3\right)+n_3\alpha}}{\langle\alpha+n_3,\gamma';q\rangle_{n_5+n_2+n_3}\langle\alpha+n_3,\gamma'';q\rangle_{n_6+n_1+n_3}\langle 1,\alpha;q\rangle_{n_1}\langle 1,\alpha;q\rangle_{n_2}}$$

$$.\,(-z_1)^{n_4+n_1+n_2}(-z_2)^{n_5+n_3+n_2}(-z_3)^{n_6+n_1+n_3}\;\frac{\langle\alpha,\beta';q\rangle_{n_5+n_6+n_1+n_2+2n_3}(-1)^{n_3}\langle\alpha;q\rangle_{2n_3}}{\langle 1,\alpha+n_3-1,\alpha,\alpha;q\rangle_{n_3}}$$

$$.\,\frac{\langle 1,\alpha;q\rangle_{n_5+n_3+n_2}\langle 1,\alpha;q\rangle_{n_6+n_3+n_1}}{\langle 1;q\rangle_{n_4+n_2}\langle 1;q\rangle_{n_4+n_1}\langle 1;q\rangle_{n_6+n_3}\langle 1;q\rangle_{n_6+n_1}\langle 1;q\rangle_{n_5+n_3}\langle 1;q\rangle_{n_5+n_2}\langle\gamma;q\rangle_{n_4+n_1+n_2}}=RHS$$

Therefore the relation (27) is true.
Theorem (5):
Let the Srivastava qH_B- hypergeometric functions

$$H_B(\alpha;\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3)$$

$$=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{\langle\alpha,q\rangle_{n_1+n_3}\langle\beta,q\rangle_{n_1+n_2}\langle\beta',q\rangle_{n_2+n_3}}{\langle 1,\gamma;q\rangle_{n_1}\langle 1,\gamma';q\rangle_{n_2}\langle 1,\gamma'';q\rangle_{n_3}}(-z_1)^{n_1}(-z_2)^{n_2}(-z_3)^{n_3}$$

Then by using the operator given in (22) the following relation is true:

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3) \qquad (28)$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6,n_7=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_1+n_2} q^{n_2(\beta+n_2-1)}\langle 1,\alpha,\beta';q\rangle_{n_1+n_3} q^{n_1(\alpha+n_1-1)+n_3(\beta+n_3-1)}}{\langle 1,1,1,\beta;q\rangle_{n_2}\langle 1,1,1,\alpha;q\rangle_{n_1}\langle 1,1,1,\beta';q\rangle_{n_3}}$$

$$.\,\frac{\langle 1,\beta,\beta';q\rangle_{n_2+n_3}(-z_1)^{n_1+n_2}(-z_2)^{n_2+n_3}(-z_3)^{n_1+n_3}}{\langle\gamma;q\rangle_{n_1+n_2}\langle\gamma';q\rangle_{n_2+n_3}\langle\gamma'';q\rangle_{n_1+n_3}}$$

$$.\,{}_4\Phi_3\left[\begin{matrix}1,\alpha+n_1+n_2,\beta+n_1+n_2,1+n_1+n_2\\ 1+n_2,\gamma+n_1+n_2,1+n_1\end{matrix}\middle|q;-z_1\right]\Phi^{0:4}_{0:3}\left[\begin{matrix}-:a\\ -:b\end{matrix}\middle|q;-z_2,-z_3\right]$$

where

$$a=(1,\beta+n_2+n_3,1+n_2+n_3,\beta'+n_2+n_3);(1,\alpha+n_1+n_3,1+n_1+n_3,\beta'+n_1+n_3)$$

$$b=(1+n_3,1+n_2,\gamma'+n_2+n_3);\ (1+n_3,1+n_1,\gamma''+n_1+n_3)$$

Proof:
By the relation (23), the computation goes as follows:

$$H_B(\alpha,\beta,\beta';\gamma,\gamma',\gamma''|q;-z_1,-z_2,-z_3)=$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{\langle -n_4,-n_6;q\rangle_{n_1} q^{n_1(\alpha+n_4+n_6)}\langle -n_4,-n_5;q\rangle_{n_2} q^{n_2(\beta+n_4+n_5)}\langle -n_5,-n_6;q\rangle_{n_3} q^{n_3(\beta'+n_5+n_6)}}{\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}\langle 1,\beta';q\rangle_{n_3}}$$

$$.\,\frac{(-z_1)^{n_4}(-z_2)^{n_5}(-z_3)^{n_6}\langle\alpha,\beta;q\rangle_{n_4}\langle\beta,\beta';q\rangle_{n_5}\langle\alpha,\beta';q\rangle_{n_6}}{\langle 1,\gamma;q\rangle_{n_4}\langle 1,\gamma';q\rangle_{n_5}\langle 1,\gamma'';q\rangle_{n_6}}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)+n_3(\beta'+n_3-1)}}{\langle 1;q\rangle_{n_4-n_1}\langle 1,\alpha;q\rangle_{n_1}\langle 1;q\rangle_{n_4-n_2}\langle 1;q\rangle_{n_6-n_1}\langle 1;q\rangle_{n_6-n_3}\langle 1,\beta;q\rangle_{n_2}\langle 1,\beta';q\rangle_{n_3}}$$

$$.\,\frac{(-z_1)^{n_4}(-z_2)^{n_5}(-z_3)^{n_6}\langle 1,\alpha,\beta;q\rangle_{n_4}\langle 1,\beta,\beta';q\rangle_{n_5}\langle 1,\alpha,\beta';q\rangle_{n_6}}{\langle 1;q\rangle_{n_5-n_2}\langle 1;q\rangle_{n_5-n_3}\langle\gamma;q\rangle_{n_4}\langle\gamma';q\rangle_{n_5}\langle\gamma'';q\rangle_{n_6}}$$

$$=\sum_{n_1,n_2,n_3,n_4,n_5,n_6=0}^{\infty}\frac{\langle 1,\alpha,\beta;q\rangle_{n_1+n_2+n_4}\langle 1,\alpha,\beta';q\rangle_{n_1+n_3+n_6}\langle 1,\beta,\beta';q\rangle_{n_2+n_3+n_5}}{\langle 1;q\rangle_{n_4+n_2}\langle 1;q\rangle_{n_4+n_1}\langle 1;q\rangle_{n_3+n_6}\langle 1;q\rangle_{n_1+n_6}\langle 1,\alpha;q\rangle_{n_1}\langle 1,\beta;q\rangle_{n_2}\langle 1,\beta';q\rangle_{n_3}}$$

$$.\,\frac{(-z_1)^{n_1+n_2+n_4}(-z_2)^{n_2+n_5+n_3}(-z_3)^{n_1+n_3+n_6} q^{n_1(\alpha+n_1-1)+n_2(\beta+n_2-1)+n_3(\beta'+n_3-1)}}{\langle 1;q\rangle_{n_2+n_5}\langle 1;q\rangle_{n_3+n_5}\langle\gamma;q\rangle_{n_1+n_2+n_4}\langle\gamma';q\rangle_{n_5+n_2+n_3}\langle\gamma'';q\rangle_{n_6+n_1+n_3}}$$

$$=RHS.$$

Therefore the relation (28) is true.

3 Conclusion

The expansion formulas often have a symmetric form. If a number of ∞ appear up or down in the first place of the Φ-function, an equal number of ∞ always appear opposite in the two second places of the Φ-function. The sum of the number of indices of q-shifted factorials in numerator and denominator is always the same for every step in the proofs. The number of variables in the exponent in this case is equal to the number of $\Delta_{z_1,z_2;q}$- operators.

References

[1] G. Andrew, R. Askey, R. Roy "Special functions." Cambridge University press (1999).
[2] A. Hasanov, H. M. Srivastava, and M. Turaev, “Decomposition formulas for some triple hypergeometric functions,” Journal of Mathematical Analysis and Applications, vol. 324, no. 2, pp. 955– 969, (2006).
[3] M. G. Bin-Saad, “Symbolic operational images and decomposition formulas for hypergeometric functions,” Journal of Mathematical Analysis and Applications, vol. 376, no. 2, pp.451–468, (2011).
[4] J.L. Burchnall, T.W. Chaundy, "Expansions of Appell's double hypergeometric functions," Quart. J. Math. Oxford Ser. 11, pp. 249–270, (1940).
[5] J.L. Burchnall, T.W. Chaundy, "Expansions of Appell's double hypergeometric functions. II," Quart. J. Math. Oxford Ser. 12 pp. 112–128, (1941).
[6] T. Ernst, "A Comprehensive Treatment of q-Calculu," Birkhauser, (2012).
[7] G. Gasper and M. Rahman, "Basic Hypergeometric Series," Cambridge University Press, Cambridge, UK, (1990).
[8] T.W. Chaundy, "Expansions of hypergeometric functions," Quart. J. Math. Oxford Ser. 13 pp. 159–171, (1942).
[9] T.W. Chaundy, "On Appell's Fourth Hypergeometric Functions." The Quart. J. Mathematical oxford (2) 17 pp. pp.81-85, (1966).
[10] T. Ernst, "Decomposition Formulas for Triple q-Hypergeometric Functions". International Journal of Combinatorics. Volume 2014,Article ID 712321,14 pages.

[11] T. Ernst, "Convergence aspects for q-Appell functions I," The Journal of the Indian Mathematical Society, vol.81, no. 1-2, pp. 67–77, (2014).
[12] F. H. Jackson, "On basic double hypergeometric functions," Quarterly Journal of Mathematics, vol. 13, no. 1, pp.69–82, (1942).

Mosaed M. Makky is currently an Associate Professor of Mathematics at Faculty of Science (Qena), South Valley Univeristy - Egypt His research interests are mainly in the areas of pure mathematics including Complex Analysis and Special functions . He has published over 22 research articles in reputable international journals of mathematics.