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Generalized Difference Sequence Spaces of Fractional-Order via Orlicz-Functions Sequence

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Abstract: In this article, a new type of sequence spaces is defined using an Orlicz-functions sequence and the generalized fractional-order difference operator combined together. This article is devoted to examine some general algebraic and topological properties of these new spaces. Furthermore, we give some inclusion theorems of those spaces.

Keywords: Difference sequence spaces, Generalized difference operators, Orlicz functions, Paranorm

1 Definitions and Preliminaries

This section is devoted to mention some definitions, notations and preliminaries and to give a background about our topic (for more details, one can refer to [1], [2], [3], [4], and [5]). Assume that s is the space containing all real-valued sequences, or all complex-valued sequences. Every linear subspace X of s is said to be a space of sequences (shortly, a sequence space). Suppose that ℓ_∞ is the space of all bounded sequences $x = (x_k)$ with real terms (or complex terms), c is the space of all convergent sequences $x = (x_k)$ with real terms (or complex terms), and c_0 is the space of all sequences vanishing at infinity $x = (x_k)$ with real terms (or complex terms). Each of them is normed by $\|x\|_\infty = \sup_k |x_k|$, for $k \in \mathbb{N} = \{1,2,3,...\}$. If X is a sequence space, then the sequence space $X((p_k))$ is defined as

$$X((p_k)) = \{x = (x_k) \in s : (|x_k|^{p_k}) \in X\},$$

where (p_k) is a bounded sequence of positive real numbers.

Definition 1.1. Based on a real number m, assume that the Gamma function is $\Gamma(m)$ for $m \neq 0,-1,-2,....$ Then, $\Gamma(m)$ may be expressed in terms of an improper integral as follows:

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} \mathrm{d}t.$$

Suppose that $\mathbb{N}_0 = \{0,1,2,...\}$ is the set of nonnegative integers. We can list some properties of Gamma functions as follows $\Gamma(m+1) = m!$ if $m \in \mathbb{N}_0$, and $\Gamma(m+1) = m\Gamma(m)$ if $m \in \mathbb{R}$ and $m \neq 0,-1,-2,....$ In addition, we have some particular cases as follows $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!$, $\Gamma(4) = 3!$, $\cdots$.

Definition 1.2. Assume that any function $\phi : [0,\infty) \to [0,\infty)$ is satisfying non-decreasing property, continuity property and convexity property along with $\phi(0) = 0$, $\phi(x) > 0$ when $x > 0$ and $\phi(x) \to \infty$ when $x \to \infty$, then one can call it an Orlicz function.

For instance, in all the following definitions, we assume that X and Y are sequence linear spaces.

Definition 1.3. Let g be a function which is real subadditive on X. If $g(x) \geq 0$ for all $x \in X$, but for the zero vector θ in X, we have $g(\theta) = 0$. In addition, if we have $g(-x) = g(x)$ for all $x \in X$ and the scalar multiplication is satisfying continuity property, i.e.,

$\forall\ (x_n) \in X$: $g(x_n - x) \to 0$ as $n \to \infty$ and $\forall\ (\eta_n) \in \mathbb{R}$: $|\eta_n - \eta| \to 0$ as $n \to \infty$, we get $g(\eta_n x_n - \eta x) \to 0$ as $n \to \infty$.

Then, one can call $X = (X, g)$ a paranormed space if it is a topological linear space in which the paranorm g itself defines the topology.

Definition 1.4. One can write, for $X, Y \subset s$, the following:

$$\mathscr{M}(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{z \in s : zx \in Y \ for\ all\ x \in X\}.$$

$X^\delta = \mathscr{M}(X, \ell_1)$ is said to be $K\ddot{o}the - Toeplitz$ dual of X. In case that $X \subset Y$, then we have $Y^\delta \subset X^\delta$. Note that $X \subset$

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$(X^{\delta})^{\delta} = X^{\delta\delta}$. In particular, if $X = X^{\delta\delta}$, then we can call X a perfect sequence space (or a *Köthe* space).

Definition 1.5. X is said to be normal (or solid), if the following condition is satisfied:

if $|y_n| \leq |x_n|$, for all $n \in \mathbb{N}$, for some $x = (x_n) \in X$, then we have $y = (y_n) \in X$.

The condition of normality (or solidity) property of X is also equivalent to:

If $x = (x_n) \in X$, for all $\beta = (\beta_n) \in \mathbb{R}$ along with $|\beta_n| \leq 1$, for all $n \in \mathbb{N}$, then $(\beta_n x_n) \in X$.

Definition 1.6. X is said to be sequence algebra, if we have $x = (x_n) \in X$ and $y = (y_n) \in X$ implies $(x_n y_n) \in X$.

Definition 1.7. X is said to be symmetric, if $x = (x_n) \in X$ implicates $(x_{\pi(n)}) \in X$, for any permutation $\pi(n)$ of $\mathbb{N}$.

The following inequality is very important to be used throughout the paper:

Lemma 1.1. Suppose that (p_n) is a positive real-numbers sequence satisfying

$$0 < h = \inf_n p_n \leq p_n \leq \sup_n p_n = H < \infty, \tag{1}$$

then we get that

$$|a_n + b_n|^{p_n} \leq D\{|a_n|^{p_n} + |b_n|^{p_n}\}, \tag{2}$$

where $a_n, b_n \in \mathbb{C}$ and $D = \max\{1, 2^{H-1}\}$.

Lemma 1.2. Let (ϕ_k) be a sequence of Orlicz functions. The case of $\sup_k \phi_k(z) < \infty$, $z > 0$ is satisfied if and only if there is a point $z_0 > 0$ such that $\sup_k \phi_k(z_0) < \infty$.

Lemma 1.3. Let (ϕ_k) be a sequence of Orlicz functions. The case of $\inf_k \phi_k(z) > 0$, $z > 0$ is satisfied if and only if there is a point $z_0 > 0$ such that $\inf_k \phi_k(z_0) > 0$.

2 Motivation and Introduction

First, Kizmaz [6] introduced, for $\Delta x = (x_k - x_{k+1})$ and $X \in \{\ell_\infty, c, c_0\}$, the difference sequence spaces $X(\Delta)$ by the following formula:

$$X(\Delta) = \{x = (x_k) \in s : (\Delta x_k) \in X\}.$$

Note that those spaces are Banach spaces along with the induced norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$. After that, many results were established by a lot of researchers satisfying various extensions of difference sequence spaces initiated by Kizmaz. Later, Colak and Et [7] used

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i},$$

where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, and $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$ to define new sequence spaces $X(\Delta^m)$ as follows:

$$X(\Delta^m) = \{x = (x_k) \in s : (\Delta^m x_k) \in X\},$$

for $X \in \{\ell_\infty, c, c_0\}$. In addition, these spaces are Banach spaces along with the induced norm

$$\|x\|_{\Delta^m} = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty.$$

After that, Et and Esi [8] utilized

$$\Delta_\nu^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \nu_{k+i} x_{k+i},$$

for a fixed non-zero complex-numbers sequence $\nu = (\nu_k)$, where $m \in \mathbb{N}$ is a fixed number, $\Delta_\nu^0 x_k = (\nu_k x_k)$, $\Delta_\nu x_k = (\nu_k x_k - \nu_{k+1} x_{k+1})$ and $\Delta_\nu^m x_k = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+1})$ to introduce the sequence spaces $X(\Delta_\nu^m)$ by the following formula:

$$X(\Delta_\nu^m) = \{x = (x_k) \in s : (\Delta_\nu^m x_k) \in X\}.$$

Recently, Dutta and Baliarsingh [2] defined a generalization of all difference sequence spaces introduced earlier by extending those results to the fractional case. They investigated new difference sequence spaces of fractional-order $X(\Delta^\alpha, (p_k))$ as follows:

$$X(\Delta^\alpha, (p_k)) = \{x = (x_k) \in s : (\Delta^\alpha x_k) \in X((p_k))\},$$

where (p_k) is a bounded sequence of positive real numbers and α is a positive proper fraction. For instance, Δ^α is said to be the difference operator of fractional-order and is introduced by the following formula:

$$\Delta^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} x_{k+i}.$$

To construct a new sequence space ℓ_ϕ, Lindenstrauss and Tzafriri applied the idea of Orlicz function [4] and defined it as follows:

$$\ell_\phi = \{x = (x_k) \in s : \sum_{k=1}^{\infty} \phi(\frac{|x_k|}{\rho}) < \infty \; for\; some\; \rho > 0\}.$$

Note that ℓ_ϕ along with the induced Luxemburg norm

$$\|x\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} \phi(\frac{|x_k|}{\rho}) \leq 1\}$$

is a Banach space, named an Orlicz sequence space. In addition, we have that ℓ_ϕ is closely related to ℓ_p which is an Orlicz sequence space along with $\phi(x) = x^p$, for $1 \leq p < \infty$. Parashar and Choudhary [9] gave an extension of this sequence space which is

$$X(\phi, (p_k)) = \{x = (x_k) \in s : (\phi(\frac{|x_k|}{\rho})) \in X((p_k))\},$$

for $X \in \{\ell, \ell_\infty, c, c_0\}$. After then, Tripathy and Sarma [10] introduced, for $X \in \{\ell_\infty, c, c_0\}$, the sequence spaces $X(\phi, (p_k), \Delta)$ as:

$$X(\phi, (p_k), \Delta) = \{x = (x_k) \in s : (\Delta x_k) \in X(\phi, (p_k))\}.$$

Recently, Bektas [11] extended these sequence spaces by using an Orlicz-functions sequence (ϕ_k) and Δ^m to

$$X((\phi_k),(p_k),\Delta^m) = \{x=(x_k)\in s : (\Delta^m x_k)\in X((\phi_k),(p_k))\},$$

for $X \in \{\ell_\infty, c, c_0\}$. On the other hand, many researchers used the modulus function and the fractional-order difference operator to construct new spaces and studied their properties, and to make the picture complete, we can say that many others studied the fractional-order difference operator itself and its applications (for more details, one can refer to [12], [13], [14], [15], [16], [17], [18], [19], [20] and [21]).

In this work, we combine the generalized fractional-order difference operator with an Orlicz-functions sequence to define a new type of sequence spaces. The rest of the paper introduces the main new results and is organized as the following. Section (3) is devoted to introduce the definition of the generalized fractional-order difference sequence spaces defined on an Orlicz-functions sequence. Moreover, in section (3), we obtain some algebraic and topological properties of those new-defined spaces. The aim of section (4) is to establish some inclusion relations involving those new spaces. Finally, section (5) introduces conclusions about our work.

3 Algebraic and Topological Properties

The aim of this section is to define a new type of sequence spaces using an Orlicz-functions sequence and the generalized difference operator of fractional-order combined together. Furthermore, many algebraic and topological characteristics of those spaces are investigated in this section.

Throughout the rest of the paper, we suppose that (ϕ_k) is an Orlicz-functions sequence, (p_k) is a bounded sequence and $1 \leq p_k < \infty$. For $X \in \{\ell_\infty, c, c_0\}$, the following spaces can be defined:

$$X((\phi_k),(p_k),\Delta_\nu^\alpha) = \{x=(x_k)\in s : (\Delta_\nu^\alpha x)\in X((\phi_k),(p_k))\},$$

in which

$$\Delta_\nu^\alpha x_k = \sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)}\nu_{k+i}x_{k+i}, \tag{3}$$

for any non-zero fixed complex-numbers sequence $\nu=(\nu_k)$, and for a proper fraction α. We call these new spaces the generalized fractional-order difference sequence spaces defined on an Orlicz-functions sequence.

Definition 3.1. Let ϕ be an Orlicz function. If $\exists$ a constant $K>0$ such that $\phi(2u)\leq K\phi(u)$, then one can say that ϕ satisfies $\Delta_2 - condition$ for all values of $u \geq 0$.

Remark 3.1. The $\Delta_2 - condition$ stated in Definition (3.1) implies that $\phi(lu)\leq K^{(\log_2 l)+1}\phi(u)$, for all values of u and for $l \geq 1$.

Proof. In fact, since for all $l\in\mathbb{R}$, we have:

$$2^{m-1}\leq l \leq 2^m,\ m\in\mathbb{N}, \tag{4}$$

then by hypothesis, we have:

$$\phi(lu)\leq\phi(2^m u)\leq K^m\phi(u). \tag{5}$$

By taking $\log_2$ to all sides of Inequality (4), we have:

$$m-1\leq \log_2 l \leq m \Rightarrow m \leq (\log_2 l)+1. \tag{6}$$

Hence from (5) and (6), we have $\phi(lu)\leq K^m\phi(u)\leq K^{(\log_2 l)+1}\phi(u)$.

Theorem 3.1. Assume that (ϕ_k) is an Orlicz-functions sequence. The sequence spaces $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$, $c((\phi_k),(p_k),\Delta_\nu^\alpha)$ and $c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$ are linear spaces over $\mathbb{R}$.

Proof. We omit the proof since it is easy.

Corollary 3.1. The sequence spaces $\ell_\infty((\phi_k),(p_k),\Delta_\nu^j)$, $c((\phi_k),(p_k),\Delta_\nu^j)$ and $c_0((\phi_k),(p_k),\Delta_\nu^j)$, for $j:0,1,2,\ldots,m$ and for $m\in\mathbb{N}$, are linear spaces over $\mathbb{R}$.

Proof. We omit the proof since it is straightforward.

Theorem 3.2. $\ell_\infty((\phi_k),\Delta_\nu^\alpha)$, $c((\phi_k),\Delta_\nu^\alpha)$ and $c_0((\phi_k),\Delta_\nu^\alpha)$ are Banach spaces with the norm

$$\|x\|_{\Delta_\nu^\alpha} = \inf\{\rho>0 : \sup_k \phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho})\leq 1\}.$$

Proof. We prove it for the case $\ell_\infty((\phi_k),\Delta_\nu^\alpha)$ and we can use the similar technique in all other cases. Assume that (x^i) is a Cauchy sequence in $\ell_\infty((\phi_k),\Delta_\nu^\alpha)$, where $x^i = (x_k^{(i)}) = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)},\ldots)\in\ell_\infty((\phi_k),\Delta_\nu^\alpha)$, $for\ all\ i\in\mathbb{N}$. Suppose that $r, x_0>0$ are fixed, then $\forall\ \frac{\varepsilon}{rx_0}>0, \exists$ a positive integer N such that

$$\|x^{(i)}-x^{(j)}\|_{\Delta_\nu^\alpha} < \frac{\varepsilon}{rx_0},\ for\ all\ i,j\geq N.$$

By applying the definition of the norm, the following is obtained:

$$\sup_k \phi_k(\frac{|\Delta_\nu^\alpha x_k^{(i)}-\Delta_\nu^\alpha x_k^{(j)}|}{\|x^{(i)}-x^{(j)}\|_{\Delta_\nu^\alpha}})\leq 1,$$

for all $i,j\geq N$.

$$\Rightarrow \phi_k(\frac{|\Delta_\nu^\alpha x_k^{(i)}-\Delta_\nu^\alpha x_k^{(j)}|}{\|x^{(i)}-x^{(j)}\|_{\Delta_\nu^\alpha}})\leq 1,$$

for all $k\geq 1$, and for all $i,j\geq N$. Therefore $r>0$ can be found with $\phi_k(rx_0)>1$, for all k such that

$$\phi_k(\frac{|\Delta_\nu^\alpha x_k^{(i)}-\Delta_\nu^\alpha x_k^{(j)}|}{\|x^{(i)}-x^{(j)}\|_{\Delta_\nu^\alpha}})\leq \phi_k(rx_0).$$

Since ϕ_k are nondecreasing functions for all k, we get:

$$\frac{|\Delta_\nu^\alpha x_k^{(i)} - \Delta_\nu^\alpha x_k^{(j)}|}{\|x^{(i)} - x^{(j)}\|_{\Delta_\nu^\alpha}} \leq rx_0.$$

Hence we have

$$|\Delta_\nu^\alpha x_k^{(i)} - \Delta_\nu^\alpha x_k^{(j)}| \leq rx_0\|x^{(i)} - x^{(j)}\|_{\Delta_\nu^\alpha} < rx_0\frac{\varepsilon}{rx_0}.$$

Then $|\Delta_\nu^\alpha x_k^{(i)} - \Delta_\nu^\alpha x_k^{(j)}| < \varepsilon$ for all $i, j \geq N$. Therefore $(\Delta_\nu^\alpha x_k^{(i)})_{i=1}^n$ is a Cauchy sequence in $\mathbb{R}$ for all k, i.e. $\Delta_\nu^\alpha x_k^{(j)} \to \Delta_\nu^\alpha x_k^{(0)}$ as $j \to \infty$ (say) $\in \mathbb{R}$ for all k. By Utilizing the continuity of ϕ_k for all k, we obtain that:

$$\sup_k \phi_k(\frac{|\Delta_\nu^\alpha x_k^{(i)} - \lim_{j\to\infty}\Delta_\nu^\alpha x_k^{(j)}|}{\rho}) \leq 1$$

$$\Rightarrow \sup_k \phi_k(\frac{|\Delta_\nu^\alpha x_k^{(i)} - \Delta_\nu^\alpha x_k^{(0)}|}{\rho}) \leq 1.$$

Therefore by taking the inf of those ρ's, the following is obtained:

$$\inf\{\rho > 0 : \sup_k \phi_k(\frac{|\Delta_\nu^\alpha x_k^{(i)} - \Delta_\nu^\alpha x_k^{(0)}|}{\rho}) \leq 1\} < \varepsilon,$$

for all $i \geq N$, $j \to \infty$. Since $(x^{(i)}) \in \ell_\infty((\phi_k), \Delta_\nu^\alpha)$ and ϕ_k are Orlicz functions (continuous) $\forall\, k$, we get $x^{(i)} \to x^{(0)} \in \ell_\infty((\phi_k), \Delta_\nu^\alpha)$ as $i \to \infty$.

Corollary 3.2. The sequence spaces $\ell_\infty((\phi_k), \Delta_\nu^j)$, $c((\phi_k), \Delta_\nu^j)$ and $c_0((\phi_k), \Delta_\nu^j)$ are Banach spaces with the norm $\|x\|_{\Delta_\nu^j} = \inf\{\rho > 0 : \phi_k(\frac{|\Delta_\nu^j x_k|}{\rho}) \leq 1\}$, for $j : 0, 1, 2, ..., m$ and for $m \in \mathbb{N}$.

Proof. The proof technique looks like that of Theorem (3.2), so one can omit it.

Theorem 3.3. Given that (ϕ_k) is an Orlicz-functions sequence satisfying $\Delta_2 - condition$. The sequence spaces $\ell_\infty((\phi_k), (p_k), \Delta_\nu^\alpha)$, $c((\phi_k), (p_k), \Delta_\nu^\alpha)$ and $c_0((\phi_k), (p_k), \Delta_\nu^\alpha)$ are paranormed spaces with the paranorm

$$g(x) = \inf\{\rho^{\frac{p_n}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}))^{p_k})^{\frac{1}{H}} \leq 1\},$$

where $H = \sup_k p_k$, and $1 \leq p_k < \infty$.

Proof. One can investigate it for the case $\ell_\infty((\phi_k), (p_k), \Delta_\nu^\alpha)$. We can use the similar technique in all other cases. We prove the conditions of the paranormed space as follows:

1. Suppose that $\rho_1 > 0$ and $\rho_2 > 0$ are such that

$$(\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho_1}))^{p_k})^{\frac{1}{H}} \leq 1,$$

and

$$(\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho_2}))^{p_k})^{\frac{1}{H}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then

$$(\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha (x_k + y_k)|}{\rho}))^{p_k})^{\frac{1}{H}}$$
$$\leq (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k| + |\Delta_\nu^\alpha y_k|}{\rho_1 + \rho_2}))^{p_k})^{\frac{1}{H}}.$$

Since ϕ_k are Orlicz functions (i.e. convex functions) for all k, we get:

$$(\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha (x_k + y_k)|}{\rho}))^{p_k})^{\frac{1}{H}}$$
$$\leq (\sup_k(\frac{\rho_1}{\rho_1+\rho_2}\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho_1}) + \frac{\rho_2}{\rho_1+\rho_2}\phi_k(\frac{|\Delta_\nu^\alpha y_k|}{\rho_2}))^{p_k})^{\frac{1}{H}}.$$

Using Inequality (2), we have:

$$(\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha (x_k + y_k)|}{\rho}))^{p_k})^{\frac{1}{H}}$$
$$\leq (\frac{\rho_1}{\rho_1+\rho_2})(\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho_1}))^{p_k})^{\frac{1}{H}}$$
$$+ (\frac{\rho_2}{\rho_1+\rho_2})(\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha y_k|}{\rho_2}))^{p_k})^{\frac{1}{H}}.$$

Since the ρ's are nonnegative, then

$$g(x+y) = \inf\{\rho^{\frac{p_k}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha (x_k + y_k)|}{\rho}))^{p_k})^{\frac{1}{H}} \leq 1\}$$
$$\leq \inf\{\rho_1^{\frac{p_k}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho_1}))^{p_k})^{\frac{1}{H}} \leq 1\}$$
$$+ \inf\{\rho_2^{\frac{p_k}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha y_k|}{\rho_2}))^{p_k})^{\frac{1}{H}} \leq 1\}.$$

Therefore, we get $g(x+y) \leq g(x) + g(y)$.

2. We have trivially that $g(\theta) = 0$, for the zero vector θ in $\ell_\infty((\phi_k), (p_k), \Delta_\nu^\alpha)$.
3. It is obvious that $g(x) = g(-x)\ for\ all\ x \in \ell_\infty((\phi_k), (p_k), \Delta_\nu^\alpha)$.
4. To prove the continuity of the scalar multiplication, assume that $\lambda \in \mathbb{R}$, then

$$g(\lambda x) = \inf\{\rho^{\frac{p_k}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha \lambda x_k|}{\rho}))^{p_k})^{\frac{1}{H}} \leq 1\}$$
$$= \inf\{(|\lambda d|)^{\frac{p_k}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{d}))^{p_k})^{\frac{1}{H}} \leq 1\},$$

where $d = \frac{\rho}{|\lambda|}$. From the fact that $|\lambda|^{p_k} \leq \max(1, |\lambda|^{\sup_k p_k})$, we obtain

$$g(\lambda x) = \max(1, |\lambda|^{\sup_k p_k})$$
$$\times \inf\{d^{\frac{p_k}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{d}))^{p_k})^{\frac{1}{H}} \leq 1\}.$$

Corollary 3.3. The sequence spaces $\ell_\infty((\phi_k),(p_k),\Delta_\nu^j)$, $c((\phi_k),(p_k),\Delta_\nu^j)$ and $c_0((\phi_k),(p_k),\Delta_\nu^j)$ are paranormed spaces with the paranorm

$$g(x) = \inf\{\rho^{\frac{p_n}{H}} : (\sup_k(\phi_k(\frac{|\Delta_\nu^j x_k|}{\rho}))^{p_k})^{\frac{1}{H}} \leq 1\},$$

where $H = \sup_k p_k$, $1 \leq p_k < \infty$, for $j : 0,1,2,...,m$ and for $m \in \mathbb{N}$.

Proof. The proof technique looks like that of Theorem (3.3), so one can omit it.

Theorem 3.4. $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$, $c((\phi_k),(p_k),\Delta_\nu^\alpha)$ and $c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$ are not solid (not normal) for $\alpha \geq 1$.

Proof. For $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$, suppose that $\nu = (1,1,1,...)$, $p_k = 1$, $\phi_k(x) = x$, $x = (x_k) = (k^\alpha)$ and $\beta_k = (-1)^k$, $\forall\, k \in \mathbb{N}$. Then $x = (k^\alpha) \in \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ but $(\beta_k x_k) \notin \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ is not solid in general. One can use similar examples to prove all other cases.

Remark 3.2. $\ell_\infty((\phi_k),(p_k))$, $c((\phi_k),(p_k))$ and $c_0((\phi_k),(p_k))$ are solid.

Proposition 3.1. $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$, $c((\phi_k),(p_k),\Delta_\nu^\alpha)$ and $c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$ are not perfect for $\alpha \geq 1$.

Theorem 3.5. $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ and $c((\phi_k),(p_k),\Delta_\nu^\alpha)$ are not symmetric for $\alpha \geq 1$.

Proof. For $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$, assume that $\nu = (1,1,1,...)$, $p_k = 1$, $\phi_k(x) = x$ and $x = (x_k) = (k^\alpha)$, $\forall\, k \in \mathbb{N}$. Therefore, we get that $x = (k^\alpha) \in \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$. Suppose that $y = (y_k)$ is a rearrangement of (x_k) given as the following:

$$(y_k) = \{x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},x_8,x_{49},x_{10},...\}.$$

Thus, we obtain $y \notin \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ is not symmetric in general. In addition, we can consider a similar example to show that $c((\phi_k),(p_k),\Delta_\nu^\alpha)$ is also not symmetric.

Remark 3.3. $c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$ is not symmetric for $\alpha \geq 2$.

Theorem 3.6. $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$, $c((\phi_k),(p_k),\Delta_\nu^\alpha)$ and $c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$ are not sequence algebras.

Proof. For $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$, let $\nu = (1,1,1,...)$, $p_k = 1$, $\phi_k(x) = x$, $\forall\, k \in \mathbb{N}$. By taking the sequences $x = (x_k) = (k^{\alpha-2})$ and $y = (y_k) = (k^{\alpha-2})$ and, we obtain $x,y \in \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ but $x \cdot y \notin \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ is not sequence algebra in general. One can use similar examples to prove all other cases.

4 Inclusion Relations

In this section, the necessary and sufficient conditions for the inclusion theorems involving relations between $X(\Delta_\nu^\alpha)$ and $Y((\phi_k),(p_k),\Delta_\nu^\alpha)$, with $X,Y = \ell_\infty$ *or* c_0 are obtained.

Theorem 4.1. The following statements are equivalent:

1.$\ell_\infty(\Delta_\nu^\alpha) \subseteq \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$.
2.$c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$.
3.$\sup_k(\phi_k(\frac{t}{\rho}))^{p_k} < \infty$, $t,\rho > 0$.

Proof. $(1) \Rightarrow (2)$: is obvious, since $c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$.

$(2) \Rightarrow (3)$: let $c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$. Assume that (3) is not true, therefore by Lemma (1.2), $\sup_k(\phi_k(\frac{t}{\rho}))^{p_k} = \infty$ for all $t > 0$ and for some $\rho > 0$. Then, there exists a positive integers sequence (k_i) such that

$$(\phi_{k_i}(\frac{i^{-1}}{\rho}))^{p_k} > i, for\ i = 1,2,.... \tag{7}$$

One can define $x = (x_k)$ as the following:

$$x_k = \begin{cases} i^{-1},\ if\ k = k_i\ for\ i = 1,2,..., \\ 0,\ otherwise. \end{cases}$$

Then $x \in c_0(\Delta_\nu^\alpha)$. But $x \notin \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ by (7) for $\nu_k = p_k = 1$ and $k \in \mathbb{N}$, that leads to a contradiction with (2). Therefore, (3) must be true.

$(3) \Rightarrow (1)$: assume that (3) is true and $x \in \ell_\infty(\Delta_\nu^\alpha)$. Suppose that $x \notin \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$. Then

$$\sup_k(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}))^{p_k} = \infty\ for\ \Delta_\nu^\alpha x \in \ell_\infty. \tag{8}$$

Let $t = |\Delta_\nu^\alpha x_k|$, then by (8) $\sup_k(\phi_k(\frac{t}{\rho}))^{p_k} = \infty$, that leads to a contradiction with (3). Hence (1) must be true.

Corollary 4.1. The following statements are equivalent:

1.$\ell_\infty(\Delta_\nu^j) \subseteq \ell_\infty((\phi_k),(p_k),\Delta_\nu^j)$.
2.$c_0(\Delta_\nu^j) \subseteq \ell_\infty((\phi_k),(p_k),\Delta_\nu^j)$.
3.$\sup_k(\phi_k(\frac{t}{\rho}))^{p_k} < \infty$, $t,\rho > 0$,

for $j : 1,2,3,...,m$ and for $m \in \mathbb{N}$.

Proof. The proof technique looks like that of Theorem (4.1), so one can omit it.

Theorem 4.2. The following statements are equivalent:

1.$c_0((\phi_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$.
2.$c_0((\phi_k),(p_k),\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$.
3.$\inf_k(\phi_k(\frac{t}{\rho}))^{p_k} > 0$, $t,\rho > 0$.

Proof. $(1) \Rightarrow (2)$: is obvious, since $c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$.

$(2) \Rightarrow (3)$: let $c_0((\phi_k),(p_k),\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$. Assume that (3) is not true, therefore by Lemma (1.3), $\inf_k(\phi_k(\frac{t}{\rho}))^{p_k} = 0$ for all $t > 0$ and for some $\rho > 0$. Thus we have that there exists a positive-integers sequence (k_i) such that

$$(\phi_{k_i}(\frac{i^2}{\rho}))^{p_k} < i^{-1}, for\ i = 1,2,.... \tag{9}$$

Define $x = (x_k)$ by

$$x_k = \begin{cases} i^2,\ if\ k = k_i\ for\ i = 1,2,..., \\ 0,\ otherwise. \end{cases}$$

Then $x \in c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$. But $x \notin \ell_\infty(\Delta_\nu^\alpha)$ by (9) for $\nu_k = p_k = 1$ and $k \in \mathbb{N}$, that leads to a contradiction with (2). Hence (3) must be true.

(3) $\Rightarrow$ (1): assume (3) is true and $x \in c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$, i.e.

$$\lim_k (\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}))^{p_k} = 0 \; and \; x \notin c_0(\Delta_\nu^\alpha). \qquad (10)$$

Therefore, for some $\varepsilon_0 > 0$ and for positive integer k_0, we obtain that $|\Delta_\nu^\alpha x_k| \geq \varepsilon_0$ for $k \geq k_0$. Then

$$\phi_k(\frac{\varepsilon_0}{\rho}) \leq \phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}),$$

for $k \geq k_0$. Then by (10) $\lim_k \phi_k(\frac{\varepsilon_0}{\rho}) = 0$, which contradicts (3). Therefore, we obtain that (1) must hold.

Corollary 4.2. The following statements are equivalent:

1.$c_0((\phi_k),(p_k),\Delta_\nu^j) \subseteq c_0(\Delta_\nu^j)$.
2.$c_0((\phi_k),(p_k),\Delta_\nu^j) \subseteq \ell_\infty(\Delta_\nu^j)$.
3.$\inf_k(\phi_k(\frac{t}{\rho}))^{p_k} > 0,\ t,\rho > 0$,

for $j : 1,2,3,...,m$ and for $m \in \mathbb{N}$.

Proof. The proof technique looks like that of Theorem (4.2), so one can omit it.

Theorem 4.3. $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$ if and only if

$$\lim_k (\phi_k(\frac{t}{\rho}))^{p_k} = \infty \; for \; t,\rho > 0. \qquad (11)$$

Proof. Let $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$. Suppose that (11) is not hold. Then, there exists a number $t_0 > 0$ and a positive integers index sequence k_i such that

$$\lim_k (\phi_{k_i}(\frac{t_0}{\rho}))^{p_k} \leq K < \infty, \; for \; some \; \rho > 0. \qquad (12)$$

One can introduce the sequence $x = (x_k)$ by the following formula:

$$x_k = \begin{cases} t_0, \; if \; k = k_i \; for \; i = 1,2,..., \\ 0, \; otherwise. \end{cases}$$

Therefore, we obtain $x \in \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ by (12). But $x \notin c_0(\Delta_\nu^\alpha)$ for $\nu_k = p_k = 1$ and $k \in \mathbb{N}$. Thus, (11) is hold if $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$.

Conversely, let (11) be satisfied. Suppose $x \in \ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha)$ and $x \notin c_0(\Delta_\nu^\alpha)$. For $k \in \mathbb{N}$,

$$(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}))^{p_k} \leq K < \infty.$$

Therefore, for some $\varepsilon_0 > 0$ and for positive integer k_0, we get that $|\Delta_\nu^\alpha x_k| \geq \varepsilon_0$ for $k \geq k_0$. Then

$$(\phi_k(\frac{\varepsilon_0}{\rho}))^{p_k} \leq (\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}))^{p_k} \leq K,$$

for $k \geq k_0$, which contradicts (11). Hence $\ell_\infty((\phi_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$.

Corollary 4.3. $\ell_\infty((\phi_k),(p_k),\Delta_\nu^j) \subseteq c_0(\Delta_\nu^j)$ if and only if

$$\lim_k (\phi_k(\frac{t}{\rho}))^{p_k} = \infty,$$

for $t,\rho > 0$, for $j : 1,2,3,...,m$ and for $m \in \mathbb{N}$.

Proof. The proof technique looks like that of Theorem (4.3), so one can omit it.

Theorem 4.4. $\ell_\infty(\Delta_\nu^\alpha) \subseteq c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$ if and only if

$$\lim_k (\phi_k(\frac{t}{\rho}))^{p_k} = 0 \; for \; t,\rho > 0. \qquad (13)$$

Proof. Let $\ell_\infty(\Delta_\nu^\alpha) \subseteq c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$. Suppose that (13) is not satisfied. Then

$$\lim_k (\phi_k(\frac{t_0}{\rho}))^{p_k} = l \neq 0, \; for \; some \; t_0,\rho > 0. \qquad (14)$$

We can give the sequence $x = (x_k)$ formula as follows:

$$x_k = t_0 \sum_{\nu=0}^{k-\nu} (-1)^{\lfloor\alpha\rfloor} \frac{\Gamma(\alpha+k-\nu)}{(k-\nu)!\Gamma(\alpha)}, \; for \; k \in \mathbb{N},$$

in which $\lfloor\alpha\rfloor = [\alpha]$ can be considered as the floor of α (the integral part of α). Therefore, we obtain $x \notin c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$ for $\nu_k = p_k = 1$ and $k \in \mathbb{N}$ by (14). Hence, (13) is satisfied.

Conversely, let (13) be satisfied. Let $x \in \ell_\infty(\Delta_\nu^\alpha)$. Then $|\Delta_\nu^\alpha x_k| \leq M < \infty$ for $k \in \mathbb{N}$. Therefore

$$(\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}))^{p_k} \leq (\phi_k(\frac{M}{\rho}))^{p_k}$$

and

$$\lim_k (\phi_k(\frac{|\Delta_\nu^\alpha x_k|}{\rho}))^{p_k} \leq \lim_k (\phi_k(\frac{M}{\rho}))^{p_k} = 0$$

by (13). Thus $x \in c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty(\Delta_\nu^\alpha) \subseteq c_0((\phi_k),(p_k),\Delta_\nu^\alpha)$.

Corollary 4.4. $\ell_\infty(\Delta_\nu^j) \subseteq c_0((\phi_k),(p_k),\Delta_\nu^j)$ if and only if

$$\lim_k (\phi_k(\frac{t}{\rho}))^{p_k} = 0,$$

for $t,\rho > 0$, for $j : 1,2,3,...,m$ and for $m \in \mathbb{N}$.

Proof. The proof technique looks like that of Theorem (4.4), so one can omit it.

5 Conclusions

In this study, new sequence spaces have been defined by combining an Orlicz-functions sequence and the difference operator of fractional-order. Furthermore, many algebraic and topological properties of these new spaces have been established. Finally, inclusion theorems including those new spaces have been introduced. In fact, those new spaces have generalized most of the spaces defined earlier on the Orlicz function. In addition, those new spaces can't be considered as a special type of any other space introduced before. By using similar techniques, the authors can introduce some new spaces as future work.

Conflict of interest

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References

[1] V.K. Bhardwaj and S. Gupta, Ces*á*ro summable difference sequence space, J. Inequal. Appl. **2013(1)**, 315-323 (2013).
[2] S. Dutta and P. Baliarsingh, A note on paranormed difference sequence spaces of fractional order and their matrix transformations, J. Egyptian Math. Soc. **22**, 249-253 (2014).
[3] P.K. Kamthan and M. Gupta, Sequence spaces and series, Marcel Dekker, New York, 1981.
[4] M.A. Krasnosel'skii and Y.B. Ruttckii, Convex functions and Orlicz spaces, Netherlands, 1961.
[5] I.J. Maddox, Elements of Functional Analysis, Cambridge Univ. Press, Cambridge, 1970.
[6] H. Kizmaz, On certain sequence spaces, Canad. Math. Bull. **24(2)**, 169-176 (1981).
[7] R. Colak and M. Et, On some generalized difference sequence spaces and related matrix transformations, Hakkaido Math. J. **26(3)**, 483-492 (1997).
[8] M. Et and A. Esi, On K*ö*the-Toeplitz duals of generalized difference sequence spaces, Bull. Malaysian Math. Sci. Soc. **23**, 25-32 (2000).
[9] S.D. Parashar and B. Choudhary, Sequence spaces defined by Orlicz functions, Indian J. Pure Appl. Math. **25(4)**, 419-428 (1994).
[10] B.C. Tripathy and B. Sarma, Some classes of difference paranormed sequence spaces defined by Orlicz functions, Thai. J. Math. **3(2)**, 209-218 (2005).
[11] C. Bektas, On some difference sequence spaces defined by a sequence of Orlicz functions, J. Zhejiang Univ. SCIENCE A **7(12)**, 2093-2096 (2006).
[12] N. Ahmad, S.K. Sharma and S.A. Mohiuddine, Generalized entire sequence spaces defined by fractional difference operator and sequence of modulus functions, TWMS J. App. Eng. Math. **10**, 63-72 (2020).
[13] P. Baliarsingh, Some new difference sequence spaces of fractional order and their dual spaces, Appl. Math. Comput. **219**, 9737-9742 (2013).
[14] P. Baliarsingh and S. Dutta, A unifying approach to the difference operators and their applications, Bol. Soc. Paran. Mat. **33(1)**, 49-57 (2015).
[15] O. Duyar and S. Demiriz, On some new generalized difference sequence spaces and their topological properties, J. New Result Sci. **6**, 1-14 (2014).
[16] N. Faried, M.S.S. Ali and H.H. Sakr, On generalized fractional order difference sequence spaces defined by a sequence of modulus functions, Math. Sci. Lett. **6(2)**, 163-168 (2017).
[17] N. Faried, M. Ali and H. Sakr, A note on FS isometry operators, Math. Sci. Lett. **10(1)**, 1-3 (2021).
[18] N. Faried, M. Ali and H. Sakr, On FS normal operators, Math. Sci. Lett. **10(2)**, 41-46 (2021).
[19] T. Yaying, On a new class of generalized difference sequence spaces of fractional order defined by modulus function, Proyecciones J. Math. **38(3)**, 485-497 (2019).
[20] T. Yaying, On a generalized difference sequence spaces of fractional order associated with multiplier sequence defined by a modulus function, Sakarya University J. Sci. **24(5)**, 1105-1114 (2020).
[21] T. Yaying, B. Hazarika and S.A. Mohiuddine, On difference sequence spaces of fractional-order involving Padovan numbers, Asian-European J. Math. **14(6)**, 2150095 (2021).