A Retrial Inventory System with Impatient Customers

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Abstract: This article presents a stochastic inventory system under continuous review at a service facility consisting of a finite waiting room and a single server, in which two types of customers arrive in Poisson processes with arrival rates \( \lambda_1 \) for high priority and \( \lambda_2 \) for low priority customers. The low priority customers arrive only for repair. The inventory is replenished according to an \((s, S)\) policy and the replenishing times are assumed to be exponential. The service times follow exponential distributions with parameters \( \mu_1 \) and \( \mu_2 \) for high and low priority customers respectively. Retrial and impatience are introduced for low priority customers only. The orbiting customers independently renege the system after an exponentially distributed time with parameter \( \alpha > 0 \). The orbiting customers compete for service by sending signals that are exponentially distributed. The joint probability distribution of the number of customers in the waiting area, the number of customers in the orbit and the inventory level is obtained for the steady state case. Some important system performance measures in the steady state are derived. Several numerical examples are presented to illustrate the effect of the system parameters and costs on these measures.

Keywords: \((s, S)\) policy, Continuous review, Inventory with service time, Markov process, Priority customers, Retrial, Impatient customers.

1 Introduction

Most inventory models in the literature make the assumption that inter demand times are independently and identically distributed random variables ([3], [5]). Some models make a more general assumption and consider the case of state-dependent demand rates. In all these models, the major assumption is that all customers are treated alike and if there is stock on-hand, the demand by a customer is met. However, in certain cases some customers may have financial resources to pay higher prices than others and hence are treated as preferential customers by retailers. The retailers would not like to turn away a preferential customer and as such when there is stock on-hand, they may inform an ordinary customer that the product is sold out thus hoarding items for preferential customers. In assembly manufacturing system customers with long-term supply contracts have been given higher priority than the other ordinary customers. In multi-speciality hospitals patients with serious illness are given higher priority than the other patients opting for routine checks or else.

The concept of retrial demands in inventory was introduced by Artalejo et al. [1]. They assumed Poisson demand, exponentially distributed lead time and retrial time. In their work, the authors proceeded with an algorithmic analysis of the system. Ushakumari [11] considered a retrial inventory system with a classical retrial policy. The author has assumed a Markovian setup for the time between consecutive arrivals, replenishment, and retrials. Krishnamoorthy and Jose [4] analyzed and compared three \((s, S)\) inventory systems with positive service time and retrial of customers.

An important issue in the queueing - inventory system with two classes of customers is the priority assignment problem. Ning Zhao and Zhaotong Lian [6] analyzed a queueing - inventory system with two classes of customers. The authors have assumed Poisson arrival and exponential service times. Each service using one item in the attached inventory supplied by an outside supplier with exponentially distributed lead time. Choi and Chang [2] analyzed single server retrial queues with priority calls. Sapna [10] analyzed a continuous review \((s, S)\) inventory system with priority customers and arbitrarily

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distributed lead times. The author has assumed that the arrival of the two-types of customers form independent Poisson processes. Real life problems stimulate one to study the queueing - inventory system with two types of customers.

In all the above models, the authors assumed that at the service completion epoch the customer receives an item and hence the inventory level was decreased by one. But in real life situations, an arriving customer may demand an item or alternatively may overhaul the item. For example, in a car sales and service shop, a customer may buy a new car or demand repair only.

Reneging is an important feature in many real - world queueing contexts. In fact, Palm [7] introduced reneging as a means of modelling the behaviour of telephone switchboard customers more than 70 years ago. However, due to the explosive growth of the call centre industry, there has been renewed interest in such models in recent years. In the call centre setting, customer impatience (amplified by large customer loads) leads naturally to large amounts of reneging. Ignoring the presence of reneging can lead to inappropriate sizing of the system and poor staffing allocation. Models in which reneging is present are also potentially valuable in problem contexts, Ward and Glynn [12]. Paul Manual et al. [9] analyzed a service facility inventory system with impatient customers. The authors have assumed the arrival time points of customers form a Poisson process. The service time, life time of items in stock and the lead time of orders are all assumed to be independently distributed as exponential. The waiting customer independently reneges the system after an exponentially distributed amount of time. An item demanded by a customer is issued after performing service on the item. Patrick Wiichner et al. [8] analyzed a finite source M/M/S retrial queue with search for balking and impatient customers from the orbit. The authors have assumed that the arriving customers may either join the queue or go to the orbit. Moreover, the requests becomes impatient and abandon the buffer after a random time and enter the orbit, too.

This paper considers the two types of customers arriving in Poisson fashion and the service times follow exponential distributions. Retrial and impatience are introduced for low priority customers only. A non-pre-emptive priority service rule is assumed. The joint probability distribution of the number of customers in the waiting area, the number of customers in the orbit and the inventory level is obtained for the steady state case. Various measures of system performance are computed in the steady state case and the results are illustrated with numerical examples. The last section is meant for conclusion.

2 Problem formulation

Consider an inventory system with a maximum capacity of $S$ for stocking units at a service facility. Customers arriving at the service station belong to any one of the two types such that the high and the low priority customers and their arrivals belong to independent Poisson processes with parameters $\lambda_1$ and $\lambda_2$ respectively. The low priority customers request overhaul of the item only and the high priority customers demand unit item, which is delivered after performing service on the item. The high and low priority customers receive their service one-by-one. The service times follow exponential distributions with parameters $\mu_1$ and $\mu_2$ for high and low priority customers respectively. The waiting area is limited to accommodate a maximum number $N$ of high priority customers including the one at the service point. The retrial and impatience are introduced for low priority customers only. Whenever the server is idle, an arriving low priority customer is immediately taken for service by the server. Otherwise, that customer enters into the orbit of finite size $M$. The orbiting customers compete for service by sending out signals that are exponentially distributed. The customers in the orbit may either retry or may leave the orbit. In this article the latter type of customers are described as impatient (reneging) customers. An impatient customer in the orbit leaves the orbit independently after a random time which is distributed as negative exponential with parameter $\alpha(>0)$.

When the server is busy (high/low) and there are spaces available in the waiting area, the arriving high priority customer can enter into the service station and waits for service. If the server is idle and the inventory level is positive then the arriving high priority customer can enter into the service station and taken for service immediately by the server. When the server is idle, the inventory level is zero and there is space available in the waiting area, then the arriving high priority customer can enter into the service station and waits for his turn. When the inventory level is zero and server is idle, the arriving low priority customer (primary/retrial) will be taken for service. If the server is busy with a low priority customer, the inventory level is zero with no high priority customers in the waiting area, then after the service completion the server becomes idle. Otherwise, if the server is busy with a low priority customer, the inventory level is positive with at least one high priority customer in the waiting area, then after the completion of service, the server will immediately become busy with a high priority customer. Any arriving high priority customer, who finds the waiting room full is considered to be lost. It is also assumed that any arriving low priority customer, who finds that the server is busy and there is no space in the orbit, is considered to be lost. The reorder level for the
commodity is fixed as \( s \) and an order is placed when the inventory level reaches the reorder level \( s \). The ordering quantity for the commodity is \( Q(= S - s + 1) \) items. The requirement \( S - s > s + 1 \) ensures that after a replenishment the inventory level will always be above the reorder level. Otherwise it may not be possible to place reorders which leads to a perpetual shortage. The lead time is assumed to be distributed as negative exponential with parameter \( \beta (>0) \).

In this article the classical retrial policy is followed. More explicitly, when there are \( i \geq 1 \) low priority customers in the orbit, a signal is sent out according to an exponential distribution with parameter \( \theta > 0 \). That is, the probability of a repeated attempt in an interval \((t, t + dt)\) given that \( t \) customers are in the orbit at time \( t \) is \( i\theta + o(dt) \). Also here a non-preemptive priority service rule is assumed, that is, when the server is engaging with a low priority customer at time \( t \) the high priority customer will get service only after completion of the service of the low priority customer who is in service. Further, it is assumed that the inter arrival times between high and low priority customers, intervals between repeated attempts of the retrial times, service times of high and low priority customers, and the lead times are mutually independent exponential distributions.

**Notations:**

- \( I \) : Identity matrix
- \( e^T \) : \((1, 1, \ldots, 1)\)
- \( 0 \) : Zero matrix
- \([A]_{ij} \) : entry at \((i, j)^{th} \) position of a matrix \( A \)
- \( \delta_{ij} \) : \(1 \) if \( j = i \)
- \( \delta_{ij} \) : \( 1 - \delta_{ij} \)
- \( H(x) \) : \(1 \) if \( x \geq 0 \), \( 0 \) otherwise.
- \( k \in V_i^j : k = i, i + 1, \ldots, j \)
- \( Y(t) \) : \(2 \) if the server is busy with a low priority customer at time \( t \)
- \( 1 \) if the server is busy with a high priority customer at time \( t \)
- \( 0 \) if the server is idle at time \( t \)

### 3 Analysis

Let \( L(t), Y(t), X_1(t) \) and \( X_2(t) \), respectively, denote the inventory level, the server status, the number of high priority customers (waiting and being served) in the waiting area and the number of low priority customers in the orbit at time \( t \). From the assumptions made on the input and output processes, it can be shown that the quadruplet \( \{(L(t), Y(t), X_1(t), X_2(t), t \geq 0)\} \) is a continuous time Markov chain with state space given by \( E = E_1 \cup E_2 \cup E_3 \cup E_4 \) with

- \( E_1 : \{(0, 0, i_3, i_4) \mid i_3 = 0, 1, 2, \ldots, N\} \)
- \( E_2 : \{(i_1, 0, 0, i_4) \mid i_1 = 1, 2, \ldots, S\} \)
- \( E_3 : \{(i_1, 1, i_3, i_4) \mid i_1 = 1, 2, \ldots, S, i_3 = 1, 2, \ldots, N, i_4 = 0, 1, 2, \ldots, M\} \)
- \( E_4 : \{(i_1, 2, i_3, i_4) \mid i_1 = 1, 2, \ldots, S, i_3 = 0, 1, 2, \ldots, N, i_4 = 0, 1, 2, \ldots, M\} \)

By ordering the set of states of \( E \) lexicographically, the infinitesimal generator \( A = (A(i_1, i_2, i_3, i_4), (j_1, j_2, j_3, j_4)) \), \((i_1, i_2, i_3, i_4, (j_1, j_2, j_3, j_4)) \in E \), can be conveniently expressed in a block partitioned matrix with entries.

\[
[A]_{i_1,j_1} = \begin{cases} 
A_{2}, j_1 = i_1 - 1, i_1 = 1 \\
A_{2}, j_1 = i_1 - 1, i_1 \in V_2^S \\
C, j_1 = i_1 + Q, i_1 \in V_1^S \\
C_1, j_1 = i_1 + Q, i_1 = 0 \\
B_0, j_1 = i_1, i_1 = 0 \\
B_1, j_1 = i_1, i_1 \in V_1^S \\
B_2, j_1 = i_1, i_1 \in V_1^{S+1} \\
0, \text{ otherwise}
\end{cases}
\]

with

\[
[A_2]_{i_2,j_2} = \begin{cases} 
A_{20}, j_2 = 0, i_2 = 1 \\
0, \text{ otherwise}
\end{cases}
\]

\[
[A_2^{(2)}]_{i_3,j_3} = \begin{cases} 
F_1, j_3 = i_3 - 1, i_3 \in V_1^N \\
0, \text{ otherwise}
\end{cases}
\]

\[
[F_1]_{i_4,j_4} = \begin{cases} 
\mu_1, j_4 = i_4, i_4 \in V_0^M \\
0, \text{ otherwise}
\end{cases}
\]

\[
[A_1^{(2)}]_{j_2,j_2} = \begin{cases} 
A_1^{(2)}(j_2 = 0, i_2 = 1) \\
A_1^{(2)}(j_2 = i_2, i_2 = 1) \\
0, \text{ otherwise}
\end{cases}
\]

\[
[A_1^{(1)}]_{i_3,j_3} = \begin{cases} 
F_1, j_3 = i_3 - 1, i_3 = 1 \\
0, \text{ otherwise}
\end{cases}
\]

\[
[A_1^{(1)}]_{i_3,j_3} = \begin{cases} 
F_1, j_3 = i_3 - 1, i_3 \in V_2^N \\
0, \text{ otherwise}
\end{cases}
\]

\[
[C_2]_{i_2,j_2} = \begin{cases} 
C_{00}, j_2 = i_2, i_2 = 0 \\
C_{11}, j_2 = i_2, i_2 = 1 \\
C_{22}, j_2 = i_2, i_2 = 2 \\
0, \text{ otherwise}
\end{cases}
\]

\[
[C_0]_{i_3,j_3} = \begin{cases} 
W_1, j_3 = 0, i_3 = 0 \\
0, \text{ otherwise}
\end{cases}
\]

\[
[W]_{i_4,j_4} = \begin{cases} 
\beta, j_4 = i_4, i_4 \in V_0^M \\
0, \text{ otherwise}
\end{cases}
\]
\[ C_{11}^{(1)}_{j_3,j_3} = \begin{cases} W_1, & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & \text{otherwise,} \end{cases} \quad [B_{22}^{(0)}]_{j_3,j_3} = \begin{cases} G_2, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ G_3, & j_3 = i_3, \quad i_3 = N, \\ F, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-1}, \\ 0, & \text{otherwise,} \end{cases} \]

\[ C_{22}^{(1)}_{j_3,j_3} = \begin{cases} W_1, & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & \text{otherwise,} \end{cases} \quad [G_{24}]_{j_4} = \begin{cases} (\lambda_1 + \lambda_2 \delta_{i_3} + \beta + j_4 = i_4, \quad i_4 \in V_0^M, \\ i_4 \alpha, \quad j_4 = i_4 - 1, \quad i_4 \in V_1^M, \\ \lambda_2, \quad j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ 0, \quad \text{otherwise,} \end{cases} \]

\[ [C_{01}^{(1)}]_{j_3,j_3} = \begin{cases} W_1, & j_3 = i_3, \quad i_3 \in V_1^N, \\ 0, & \text{otherwise,} \end{cases} \quad [G_{34}]_{j_4} = \begin{cases} (\lambda_2 \delta_{i_3} + \beta + j_4 = i_4, \quad i_4 \in V_0^M, \\ i_4 \alpha, \quad j_4 = i_4 - 1, \quad i_4 \in V_1^M, \\ \lambda_2, \quad j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ 0, \quad \text{otherwise,} \end{cases} \]

\[ [C_{00}^{(1)}]_{j_3,j_3} = \begin{cases} W_1, & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & \text{otherwise,} \end{cases} \quad [B_{1}]_{j_2,j_2} = \begin{cases} B_{00}^{(0)} & j_2 = i_2, \quad i_2 = 0, \\ B_{01}^{(0)} & j_2 = 1, \quad i_2 = 0, \\ B_{02}^{(0)} & j_2 = 2, \quad i_2 = 0, \\ B_{20}^{(0)} & j_2 = 0, \quad i_2 = 2, \\ B_{21}^{(0)} & j_2 = 1, \quad i_2 = 2, \\ B_{22}^{(0)} & j_2 = i_2, \quad i_2 = 2, \\ 0, & \text{otherwise,} \end{cases} \]

\[ [B_{00}^{(0)}]_{j_3,j_3} = \begin{cases} G, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ G_1, & j_3 = i_3, \quad i_3 = N, \\ F, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-1}, \\ 0, & \text{otherwise,} \end{cases} \quad [B_{01}^{(1)}]_{j_3,j_3} = \begin{cases} G_0, & j_3 = i_3, \quad i_3 = 0 \\ 0, & \text{otherwise,} \end{cases} \]

\[ [B_{02}^{(0)}]_{j_3,j_3} = \begin{cases} G_0, & j_3 = i_3, \quad i_3 \in V_0^N, \\ 0, & \text{otherwise,} \end{cases} \]

\[ [B_{11}^{(1)}]_{j_3,j_3} = \begin{cases} D_0, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ D_0, & j_3 = i_3, \quad i_3 = N, \\ F, & j_3 = i_3 + 1, \quad i_3 \in V_0^{N-1}, \\ 0, & \text{otherwise,} \end{cases} \]

\[ [F]_{j_4} = \begin{cases} \lambda_1, & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise,} \end{cases} \]

\[ [D_{j_4}]_{j_4} = \begin{cases} (\lambda_1 + \lambda_2 \delta_{i_4} + \beta + j_4 = i_4, \quad i_4 \in V_0^M, \\ \lambda_2, \quad j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ i_4 \alpha, \quad j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ 0, \quad \text{otherwise,} \end{cases} \]

\[ \begin{align*}
[C_{12}]_{j_3,j_3} &= \{ W_1, j_3 = i_3, \quad i_3 \in V_0^N, \\
& \quad 0, \quad \text{otherwise,} \}
\end{align*} \]
$$\begin{align*}
[B_{21}^{(1)}]_{i,j} &= \begin{cases} 
F_0, & j_3 = i_3, \ i_3 = 0 \\
0, & \text{otherwise,}
\end{cases} \\
[B_{21}^{(2)}]_{i,j} &= \begin{cases} 
F_0, & j_3 = i_3, \ i_3 \in V_1^N \\
0, & \text{otherwise,}
\end{cases} \\
[B_{22}^{(1)}]_{i,j} &= \begin{cases} 
G_2, & j_3 = i_3, \ i_3 \in V_0^{N-1} \\
G_3, & j_3 = i_3, \ i_3 = N \\
F, & j_3 = i_3 + 1, \ i_3 \in V_0^{N-1} \\
0, & \text{otherwise,}
\end{cases} \\
[B_{22}^{(2)}]_{i,j} &= \begin{cases} 
B_{11}^{(2)}, & j_2 = i_2, \ i_2 = 0 \\
B_{12}^{(2)}, & j_2 = i_2, \ i_2 = 2 \\
B_{21}^{(2)}, & j_2 = i_2, \ i_2 = 2 \\
B_{22}^{(2)}, & j_2 = i_2, \ i_2 = 2 \\
0, & \text{otherwise,}
\end{cases} \\
[B_0^{(2)}]_{i,j} &= \begin{cases} 
W, & j_3 = i_3, \ i_3 = 0 \\
0, & \text{otherwise,}
\end{cases} \\
[B_1^{(2)}]_{i,j} &= \begin{cases} 
G_0, & j_3 = i_3, \ i_3 = 0 \\
0, & \text{otherwise,}
\end{cases} \\
[B_2^{(2)}]_{i,j} &= \begin{cases} 
G_4, & j_3 = i_3, \ i_3 \in V_0^{N-1} \\
G_5, & j_3 = i_3, \ i_3 = N \\
F, & j_3 = i_3 + 1, \ i_3 \in V_1^{N-1} \\
0, & \text{otherwise,}
\end{cases} \\
[G_4]_{i,j} &= \begin{cases} 
-(\lambda_1 + \lambda_2 \delta_{i,M} + \mu_2 + i_4 \alpha), & j_4 = i_4, \ i_4 \in V_0^M \\
\lambda_2, & j_4 = i_4 + 1, \ i_4 \in V_0^{M-1} \\
0, & \text{otherwise,}
\end{cases} \\
[G_5]_{i,j} &= \begin{cases} 
-(\lambda_2 \delta_{i,M} + i_4 \alpha + \mu_2), & j_4 = i_4, \ i_4 \in V_0^M \\
\lambda_2, & j_4 = i_4 + 1, \ i_4 \in V_0^{M-1} \\
0, & \text{otherwise,}
\end{cases} \\
[B_{20}^{(2)}]_{i,j} &= \begin{cases} 
F_0, & j_3 = i_3, \ i_3 = 0 \\
0, & \text{otherwise,}
\end{cases} \\
[B_{21}^{(2)}]_{i,j} &= \begin{cases} 
G_6, & j_3 = i_3, \ i_3 \in V_0^{N-1} \\
G_7, & j_3 = i_3, \ i_3 = N \\
F, & j_3 = i_3 + 1, \ i_4 \in V_0^{N-1} \\
0, & \text{otherwise,}
\end{cases} \\
[G_6]_{i,j} &= \begin{cases} 
-(\lambda_1 + \lambda_2 \delta_{i,M} + \mu_2 + i_4 \alpha), & j_4 = i_4, \ i_4 \in V_0^M \\
\lambda_2, & j_4 = i_4 + 1, \ i_4 \in V_0^{M-1} \\
0, & \text{otherwise,}
\end{cases} \\
[G_7]_{i,j} &= \begin{cases} 
-(\lambda_2 \delta_{i,M} + i_4 \alpha + \mu_2), & j_4 = i_4, \ i_4 \in V_0^M \\
\lambda_2, & j_4 = i_4 + 1, \ i_4 \in V_0^{M-1} \\
0, & \text{otherwise,}
\end{cases}
\end{align*}$$

It may be noted that the matrices $A_1$, $A_2$, $B_0$, $B_1$, $B_2$, $C$ and $C_1$ are square matrices of size $2(N+1)(M+1)$. $B_0^{(0,0)}$, $B_0^{(0,1)}$, $B_0^{(1,0)}$, $B_0^{(1,1)}$, $B_0^{(2,0)}$, $B_0^{(2,1)}$, and $C_0$ are square matrices of size $(N+1)(M+1)$. $A_1^{(1,0)}$, $B_1^{(0,0)}$, $B_1^{(0,1)}$, $B_1^{(1,0)}$, $B_1^{(1,1)}$, $B_1^{(2,0)}$, $B_1^{(2,1)}$, and $C_1$ are square matrices of size $N(M+1)$. $D$, $D_0$, $F$, $F_0$, $G_0$, $i \in V_0^7$, $W$, $W_1$, $B_0^{(0,0)}, B_0^{(0,1)}$, and $C_0$ are square matrices of size $(M+1)$. $B_2^{(0,0)}$, $B_2^{(0,1)}$, and $C_2$ are matrices of size $(N+1)(M+1)$, $B_2^{(1,0)}$ and $C_1^{(1)}$ are matrices of size $(N+1)(M+1) \times (M+1)$, and $B_2^{(1,1)}$ and $B_2^{(2,0)}$ are matrices of size $(M+1) \times (N+1)(M+1)$. $A_{10}^{(1)}$ is of size $N(M+1) \times (M+1)$. $A_2^{(2)}$ is of size $N(M+1) \times (N+1)(M+1)$.

### 3.1 Steady state analysis

It can be seen from the structure of $A$ that the homogeneous Markov process $\{L(t), Y(t), X_1(t), X_2(t) : t \geq 0\}$ on the finite space $E$ is irreducible, aperiodic and persistent non-null. Hence the limiting distribution $\pi(i_1,i_2,i_3,i_4) = \lim_{t \to \infty} P(L(t) = i_1, Y(t) = i_2, X_1(t) = i_3, X_2(t) = i_4 | L(0), Y(0), X_1(0), X_2(0))$ exists. Let $\Pi = (\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(S)})$

$$\Pi^{(0)} = (\Pi^{(0,0)}, \Pi^{(0,2)}),$$

$$\Pi^{(i)} = (\Pi^{(i,0)}, \Pi^{(i,1)}, \Pi^{(i,2)}), \quad i_1 = 1, \ldots, S;$$

which is partitioned as follows:

$$\Pi^{(0,0)} = (\Pi^{(0,0,0)}, \ldots, \Pi^{(0,0,N)}),$$

$$\Pi^{(0,2)} = (\Pi^{(0,2,0)}, \ldots, \Pi^{(0,2,N)}),$$

$$\Pi^{(i,0)} = (\Pi^{(i,0,0)}, \ldots, \Pi^{(i,0,N)}), \quad i_1 = 1, 2, \ldots, S;$$

$$\Pi^{(i,1)} = (\Pi^{(i,1,0)}, \Pi^{(i,1,1)}, \ldots, \Pi^{(i,1,N)}), \quad i_1 = 1, 2, \ldots, S;$$

$$\Pi^{(i,2)} = (\Pi^{(i,2,0)}, \Pi^{(i,2,1)}, \ldots, \Pi^{(i,2,N)}), \quad i_1 = 1, 2, \ldots, S;$$

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Further the above vectors are also partitioned as follows:

\[ \Pi^{(0,0,i_0)} = (\Pi^{(0,0,i_0),0,0}, \Pi^{(0,0,i_0),1}, \ldots, \Pi^{(0,0,i_0),M}) \]

\[ i_3 = 0, 1, 2, \ldots, N; \]

\[ \Pi^{(0,2,i_0)} = (\Pi^{(0,2,i_0),0,0}, \Pi^{(0,2,i_0),1}, \ldots, \Pi^{(0,2,i_0),M}) \]

\[ i_3 = 0, 1, 2, \ldots, N; \]

\[ \Pi^{(i_1,0,0)} = (\Pi^{(i_1,0,0),0,0}, \Pi^{(i_1,0,0),1}, \ldots, \Pi^{(i_1,0,0),M}) \]

\[ i_1 = 1, 2, \ldots, S; \]

\[ \Pi^{(i_1,1,i_0)} = (\Pi^{(i_1,1,i_0),0,0}, \Pi^{(i_1,1,i_0),1}, \ldots, \Pi^{(i_1,1,i_0),M}) \]

\[ i_1 = 1, 2, \ldots, S; i_3 = 1, 2, \ldots N; \]

\[ \Pi^{(i_1,2,i_0)} = (\Pi^{(i_1,2,i_0),0,0}, \Pi^{(i_1,2,i_0),1}, \ldots, \Pi^{(i_1,2,i_0),M}) \]

\[ i_1 = 1, 2, \ldots, S; i_3 = 0, 1, 2, \ldots, N; \]

The vector of limiting probabilities \( \Pi \) then satisfies

\[ \Pi A = 0 \text{ and } \Pi e = 1. \tag{1} \]

The first equation of the above yields the following set of equations:

\[ \Pi^{(i_1)} a_2 = 0, \quad i_1 = 1, \ldots, S. \]

\[ \Pi^{(i_1)} a_S = \Pi^{(i_1)} \Omega_i, \quad i_1 = 0, 1, \ldots, S. \]

where

\[ \Omega_i = \begin{cases} 0 & i_1 = 0, 1, \ldots, S, \\ \begin{multline} 0 \times (\Pi^{(Q)} (A_1 B_2^{-1}) (Q-(x+1)) (A_1 B_1^{-1}) (A_2 B_0^{-1}) + \\ \Pi^{(Q)} \sum_{i_1=1}^{S} (-i_1) (A_1 B_2^{-1}) (Q-(x+1)) (A_1 B_1^{-1}) (A_2 B_0^{-1}) \end{multline} \end{cases} \]

After lengthy simplifications, the above equations, (except *), yield

\[ \Pi^{(i_1)} = \Pi^{(Q)} \Omega_i, \quad i_1 = 0, 1, \ldots, S. \]

and

\[ \Pi^{(Q)} (1) = \Pi^{(Q)} (A_1 B_2^{-1}) (Q-(x+1)) (A_1 B_1^{-1}) (A_2 B_0^{-1}) + \\ \Pi^{(Q)} \sum_{i_1=1}^{S} (-i_1) (A_1 B_2^{-1}) (Q-(x+1)) (A_1 B_1^{-1}) (A_2 B_0^{-1}) \]

4 System performance measures

In this section some performance measures of the system under consideration in the steady state are derived.

4.1 Expected inventory level

Let \( \eta_i \) denote the average inventory level in the steady state. Then

\[ \eta_i = \sum_{i_1=1}^{S} \sum_{i_2=0}^{M} \frac{\pi^{(i_1,0,0,i_2)}}{\pi^{(i_1,1,i_0,i_2)}} + \sum_{i_3=1}^{N} \frac{\pi^{(1,2,i_3,i_2)}}{\pi^{(1,1,i_0,i_2)}} + \sum_{i_3=0}^{N} \frac{\pi^{(i_2,2,i_3,i_2)}}{\pi^{(i_1,1,i_0,i_2)}} \]

4.2 Expected reorder rate

Let \( \eta_r \) denote the expected reorder rate in the steady state. Then

\[ \eta_r = \sum_{i_3=1}^{N} \sum_{i_2=0}^{M} \mu_1 \pi^{(x+1,1,i_3,i_2)} \]

4.3 Expected loss rate for high priority customers

Let \( \eta_{hh} \) denote the expected loss rate for high priority customers in the waiting area. Then

\[ \eta_{hh} = \sum_{i_2=0}^{M} \lambda_1 \pi^{(0,0,i_2)} + \sum_{i_1=1}^{S} \frac{\pi^{(i_1,1,i_0,i_2)}}{\pi^{(i_1,1,i_0,i_2)}} + \sum_{i_2=0}^{M} \frac{\pi^{(i_1,2,i_3,i_2)}}{\pi^{(i_1,1,i_0,i_2)}} \]

4.4 Expected loss rate for primary low priority customers

Let \( \eta_{bl} \) denote the expected loss rate for primary low priority customers in the orbit. Then

\[ \eta_{bl} = \lambda_2 \sum_{i_1=1}^{S} \sum_{i_3=1}^{N} \pi^{(i_1,1,i_0,i_2)} + \lambda_2 \sum_{i_1=0}^{S} \sum_{i_3=0}^{N} \pi^{(i_1,2,i_3,i_2)} \]
4.5 Expected number of high priority customers in the waiting area

Let $\eta_{gh}$ denote the expected number of high priority customers in the waiting area. Then

$$\eta_{gh} = \sum_{i_3=0}^{N} \sum_{i_1=1}^{N} i_3 \pi^{(0,0,i_3,i_4)} + \sum_{i_3=0}^{N} \sum_{i_1=1}^{N} (i_3 - 1) \pi^{(1,1,i_3,i_4)} + \sum_{i_3=0}^{N} \sum_{i_1=1}^{N} \sum_{i_2=0}^{i_3} i_2 \pi^{(1,2,i_3,i_4)}$$

4.6 Expected number of low priority customers in the orbit

Let $\eta_{ql}$ denote the expected number of low priority customers in the orbit. Then

$$\eta_{ql} = \sum_{i_3=1}^{S} \sum_{i_1=1}^{S} i_4 \pi^{(i_1,0,0,i_3)} + \sum_{i_3=0}^{N} \sum_{i_1=1}^{N} \sum_{i_2=0}^{i_3} i_2 \pi^{(i_1,1,i_3,i_4)} + \sum_{i_3=1}^{N} \sum_{i_1=1}^{N} \sum_{i_2=0}^{i_3} i_2 \pi^{(i_1,2,i_3,i_4)}$$

4.7 Effective arrival rate for high priority customers

Let $\lambda_{ah}$ denote the effective arrival rate for high priority customers in the waiting area. Then

$$\lambda_{ah} = \sum_{i_3=0}^{N} \sum_{i_1=0}^{N} \lambda_1 \pi^{(0,0,i_3,i_4)} + \sum_{i_3=1}^{N} \sum_{i_1=1}^{N} \lambda_1 \pi^{(i_1,1,i_3,i_4)} + \sum_{i_3=0}^{N} \sum_{i_1=0}^{N} \sum_{i_2=0}^{i_3} \lambda_1 \pi^{(i_1,2,i_3,i_4)}$$

4.8 Effective arrival rate for primary low priority customers

Let $\lambda_{al}$ denote the effective arrival rate for primary low priority customers in the orbit. Then

$$\lambda_{al} = \sum_{i_3=0}^{N} \sum_{i_1=0}^{N} \lambda_2 \pi^{(i_1,0,0,i_3)} + \sum_{i_3=1}^{N} \sum_{i_1=1}^{N} \lambda_2 \pi^{(i_1,1,i_3,i_4)} + \sum_{i_3=0}^{N} \sum_{i_1=0}^{N} \sum_{i_2=0}^{i_3} \lambda_2 \pi^{(i_1,2,i_3,i_4)}$$

4.9 Effective reneging rate for orbiting customers

Let $\alpha_{ro}$ denote the effective reneging rate for orbiting customers in the steady state. Then

$$\alpha_{ro} = \sum_{i_3=1}^{M} \sum_{i_1=1}^{M} i_4 \alpha \pi^{(i_1,0,0,i_3)} + \sum_{i_3=0}^{M} \sum_{i_1=1}^{M} \sum_{i_2=0}^{i_3} i_2 \alpha \pi^{(i_1,1,i_3,i_4)} + \sum_{i_3=1}^{M} \sum_{i_1=1}^{M} \sum_{i_2=0}^{i_3} i_2 \alpha \pi^{(i_1,2,i_3,i_4)}$$

4.10 Expected waiting time for a high priority customer

Let $\eta_{wh}$ denote the expected waiting time for a high priority customer in the waiting area. Then

$$\eta_{wh} = \frac{\eta_{gh}}{\lambda_{ah}}$$

4.11 Expected waiting time for a low priority customer in the orbit

Let $\eta_{wl}$ denote the expected waiting time for a low priority customer in the orbit. Then

$$\eta_{wl} = \frac{\eta_{ql}}{\lambda_{al}}$$

4.12 The overall rate of retrials

Let $\eta_{or}$ denote the overall rate of retrials in the steady state. Then

$$\eta_{or} = \sum_{i_3=0}^{M} \sum_{i_1=0}^{M} i_4 \theta \pi^{(0,0,i_3,i_4)} + \sum_{i_3=0}^{M} \sum_{i_1=0}^{M} \sum_{i_2=0}^{i_3} i_2 \theta \pi^{(i_1,1,i_3,i_4)} + \sum_{i_3=1}^{M} \sum_{i_1=1}^{M} \sum_{i_2=0}^{i_3} i_2 \theta \pi^{(i_1,2,i_3,i_4)}$$

4.13 The successful retrial rate

Let $\eta_{sr}$ denote the successful retrial rate in the steady state. Then

$$\eta_{sr} = \sum_{i_3=0}^{M} i_4 \theta \left( \sum_{i_3=0}^{N} \pi^{(0,0,i_3,i_4)} + \sum_{i_1=1}^{N} \pi^{(i_1,0,0,i_4)} \right)$$

4.14 The fraction of successful rate of retrial

Let $\eta_{fr}$ denote the fraction of successful retrial rate in the steady state. Then

$$\eta_{fr} = \frac{\eta_{sr}}{\eta_{or}}$$

5 Cost Analysis

To compute the total expected cost per unit time (total expected cost rate), the following costs are considered,

- $c_h$: The inventory carrying cost per unit item per unit time.
- $c_r$: Setup cost per order.
- $c_{bh}$: Cost per high priority customer lost.
- $c_{bl}$: Cost per low priority customer lost.
- $c_{wh}$: Waiting cost of a high priority customer per unit time.
- $c_{wl}$: Waiting cost of a low priority customer per unit time.
- $c_r$: Reneging cost per orbiting customer per unit time.

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The long run total expected cost rate is given by

\[ TC(s, S) = c_h \eta_1 + c_i \eta_1^* + c_{bh} \eta_{bh} + c_{bl} \eta_{bl} + c_{wh} \eta_{wh} + c_w \eta_{wl} + c_r \eta_{ro}, \]

where \( \eta^* \)s are as given in (4.1) – (4.11).

Due to the complex form of the limiting distribution, it is difficult to discuss the properties of the cost function analytically. Hence, a detailed computational study of the cost function is carried out.

### 5.1 Numerical Illustrations

To study the behaviour of the model developed in this work, several examples were dealt with and a set of representative results are shown. Although not showing the convexity of \( TC(s, S) \) analytically, experience with considerable numerical examples indicates the function \( TC(s, S) \), to be convex. Some of the results are presented in tables 1 and 2. Simple numerical search procedures are used to obtain the optimal values of \( TC \), \( s \) and \( S \) (say \( TC^* \), \( s^* \) and \( S^* \)). A typical three dimensional plot of the expected cost function is given in figure 1. The effect of varying the system parameters and costs on the optimal values have been studied and the results agreed with as expected. Some of the results are presented in tables 1 and 2 where the lower entry in each cell gives the optimal expected cost rate and the upper entries the corresponding \( S^* \) and \( s^* \).

**Example 1.**

Here, we study the impact of arrival rates \( \lambda_1 \) and \( \lambda_2 \) of high and low priority customers respectively, service rates \( \mu_1 \) and \( \mu_2 \) of high and low priority customers respectively; the lead time parameter, \( \beta \), the reneging rate, \( \alpha \) and the retrial rate, \( \theta \), on the optimal values \((s^*, S^*)\) and the corresponding total expected cost rate \( TC^* \) towards this end, first by fixing parameter \( \alpha = 2.4 \) and the cost values as \( c_r = 2, c_b = 1, c_s = 15, c_{bh} = 6, c_{bl} = 3, c_{wh} = 7 \) and \( c_{wl} = 1.9 \) is carried out. Observe the following from table 1.

1. If \( \lambda_1 \) increases monotonically and other parameters are fixed then the total expected cost rate \( TC^* \), \( S^* \) and \( s^* \) increase. From this one may observe that, to maintain the maximum inventory level and reorder level is substantial for avoiding frequent ordering.
2. If \( \lambda_2 \) increases monotonically and other parameters are fixed then the total expected cost rate \( TC^* \) increases.
3. If \( \beta \) increases monotonically and other parameters are fixed then the total expected cost rate \( TC^* \), \( S^* \) and \( s^* \) decrease.
4. If any one of the parameters \( \mu_1 \) and \( \mu_2 \) increases monotonically and other parameters are fixed then the total expected cost rate decreases. Also when \( \theta \) increases monotonically then \( TC^* \) increases.

**Example 2.**

The impact of the setup cost \( c_s \), holding cost \( c_h \), lost (waiting) cost \( c_{bh} \) (\( c_{wh} \)) and \( c_{bl} \) (\( c_{wl} \)) of high and low priority customers, respectively, on the optimal values \((s^*, S^*)\) are studied and the corresponding total expected cost rate \( TC^* \) towards this end, first by fixing the parameter values as \( \lambda_1 = 9, \lambda_2 = 7, \mu_1 = 8, \mu_2 = 11.9, \theta = 5, \alpha = 2.4 \) and \( \beta = 6 \), and cost values \( cr = 2 \).

Observe the following from table 2:

1. If any one of the parameters \( c_h, c_s, c_{bh}, c_{bl}, c_{wh} \) and \( c_{wl} \) increases monotonically whereas other parameters are fixed then the total expected cost rate \( TC^* \) increases.
2. If any one of the parameters \( c_s, c_{wh} \) and \( c_{bl} \) increases monotonically whereas other parameters are fixed then \( S^* \) increases. This is because if the setup cost increases and one has to maintain high inventory to avoid frequent ordering. Similarly the waiting cost of high priority customers and balking cost of the high priority customers increase, one has to maintain high inventory to reduce the number of waiting customers.
3. If \( c_b \) increases monotonically then \( S^* \) and \( s^* \) decrease. This is to be expected since the holding cost increases, and one resorts to maintain low stock in the inventory. We also note that \( s^* \) monotonically increases when \( c_{wh} \) increases.

![Fig. 1: A three dimensional plot of the cost function TC(s,S)](image-url)
Table 1: Deviation in the Total expected cost rate at $N = 15$, $M = 11$ and $\alpha = 2.4$

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Table 2: Deviation in the Total expected cost rate at $N = 15$, $M = 11$ and $c_r = 2$

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6 Conclusion

The stochastic model discussed here is useful in studying a retrial inventory system with impatient customers in which two types of customers arrive say high priority and low priority customers. The joint probability distribution of the number of customers in the waiting area, the number of customers in the orbit and the inventory level are derived in the steady state and the stationary measures of system performances have been computed. Illustration has also been provided to the existence of local optimums when the total expected cost function is treated as a function of two variables \( s \) and \( S \). The authors are working in the direction of MAP arrival for the two types of customers.

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