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Laplace Integral Representation of Solution to a Stochastic Heat-type Equation

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Abstract: Consider the following stochastic heat-type equation $Lu = \lambda\sigma(u)\dot{w}(t,x)$, $x \in \mathbb{R}^d$, $t > 0$; $u(0,x) = u_0(x)$, $x \in \mathbb{R}^d$. The constant $\lambda > 0$ is a noise level and σ is a Lipschitz continuous function and a differential operator $L := \partial_t - D^2$ with its adjoint given by $L^* = -\partial_t - D^2$. We propose a probabilistic representation of the solution to the above equation in terms of a Laplace integral as follows:

$$e^{tD^2} = \int_{\mathbb{R}^d} e^{-yD} k_t(y)\, dy,$$

where $k_t(x)$ is the integral kernel of the transform with D an 'operational symbol'. The result establishes the existence and uniqueness of the solution, and give some growth and the second moment upper bound estimate applying the properties of a 'good kernel' and 'approximation to the identity'.

Keywords: Approximation to identity, good kernel, growth moment, Laplace integral, operational symbol, mild solution

1 Introduction

The classical stochastic heat equation $\partial_t u(x,t) = \Delta u(x,t) + \lambda\sigma(u)\dot{w}(x,t)$ and subsequently space-fractional stochastic heat equation

$$\partial_t u(x,t) = -(-\Delta)^{\alpha/2} u(x,t) + \lambda\sigma(u)\dot{w}(x,t),\ x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \quad (1)$$

with $u(0,x) = u_0(x)$, $x \in \mathbb{R}^d$ a non-random initial datum, where the operator $-(-\Delta)^{\alpha/2}$, $\alpha \in (0,2]$ is the generator of an isotropic stable process and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, have been extensively studied, see [1, 2] and their references. The fundamental solution ($\sigma = 0$) of the above equation (1) is

$$u(x,t) = \int_{\mathbb{R}^d} p(t, x-y) u_0(y)\, dy,$$

where $p(t,x,y)$ is a fractional heat kernel with the following properties and estimates, see [3]:

$$p(t,x) = t^{-d/\alpha} p(1, t^{-1/\alpha} x)$$
$$p(st,x) = t^{-d/\alpha} p(s, t^{-1/\alpha} x).$$

The Chapman-Kolmogorov equation,

$$\int_{\mathbb{R}^d} p(t,x) p(s,x)\, dx = p(t+s, 0).$$

Proposition 1([3,4]). *Let $p(t,x)$ be the transition density of a strictly α-stable process. If $p(t,0) \le 1$ and $a \ge 2$, then*

$$p\big(t, \frac{1}{a}(x-y)\big) \ge p(t,x) p(t,y)\ \forall x,y \in \mathbb{R}^d.$$

Lemma 1([3]). *Suppose that $p(t,x)$ denotes the heat kernel for a strictly stable process of order α. Then the following estimate holds.*

$$p(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \quad \text{for all} \quad t > 0 \quad \text{and} \quad x,y \in \mathbb{R}^d.$$

Here, for two non-negative functions f, g, $f \asymp g$ means that there exists a positive constant $c > 1$ such that $c^{-1} g \le f \le c g$ on their common domain of definition.

Motivated by the work of Mazzucchi, in [5], which proposes a particular probabilistic representation for the solution of a complex-valued stochastic process

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$u_t = -\alpha\Delta^2 u + V$, $\alpha \in \mathbb{C}$ in terms of a Feynman-Kac type formula where the class of potentials V can be handled by requiring the probabilistic integrals to be defined in Lebesgue sense, see also [6,7,8,9] and their references; we therefore, rather than following the probabilistic approach, that is, using the above properties and estimates on the heat kernel, generalize by employing the properties of "good kernel" and "approximation to the identity" to study some properties and existence and uniqueness result of solution to the following class of stochastic partial differential equation

$$\partial_t u(x,t) = D^2 u(t,x) + \lambda\sigma(u(t,x))\dot{w}(t,x) \tag{2}$$

with the initial function $u(x,0) = u_0(x)$. $\lambda > 0$ is the level of noise, $\dot{w}$ denotes space-time white noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function satisfying some linear growth condition and D some "operational symbol". In an effort to finding a suitable kernel that solves equation (2), we consider a solution of the form

$$u(x,t) = \exp(tD^2)u_0(x);$$

where in particular $D = \partial_x$ and $D^2 = \Delta$. If the operational symbol, D for differentiation were a number, then

$$\mathrm{e}^{cD} = \sum_{n=0}^{\infty} \frac{c^n}{n!} D^n.$$

Applying the operator to a function u_0 and interpreting D^n as an n^{th} derivative, one obtains

$$\mathrm{e}^{cD}u_0(x) = \sum_{n=0}^{\infty} \frac{c^n}{n!} u_0^{(n)}(x) = u_0(x+c),$$

which follows by MacLaurine's theorem for suitable function u_0. We now seek to give a suitable meaning to $\exp(tD^2)$. Suppose D were a number, then the function

$$u(x,t) = \mathrm{e}^{t\,D^2} u_0(x)$$

satisfies a heat equation with $u(x,0) = u_0(x)$. Now define this operator in terms of a Laplace integral as follows:

$$\mathrm{e}^{tD^2} = \int_{\mathbb{R}} \mathrm{e}^{-yD} k_t(y)\,dy,$$

where $k_t(x)$ is the integral kernel of the transform. Thus the fundamental solution to the heat-type equation $u_t - D^2 u = 0$ is given by

$$\begin{aligned} u(x,t) &= \mathrm{e}^{tD^2} u_0(x) \\ &= \int_{\mathbb{R}^d} \mathrm{e}^{-yD} u_0(x) k_t(y)\,dy \\ &= \int_{\mathbb{R}^d} u_0(x-y) k_t(y)\,dy. \end{aligned}$$

Define the mild solution to equation (2) in sense of Walsh [10] as follows:

Definition 1. *We say that a process $\{u(x,t)\}_{x\in\mathbb{R}^d,t>0}$ is a mild solution of (2) if a.s, the following is satisfied*

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^d} k_t(x-y)u_0(y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} k_{t-s}(x-y)\sigma(u(s,y))w(dy\,ds). \end{aligned}$$

If in addition to the above, $\{u(x,t)\}_{x\in\mathbb{R}^d,t>0}$ satisfies the following condition

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d} \mathrm{E}|u(x,t)| < \infty,$$

for all $T > 0$, then we say that $\{u(x,t)\}_{x\in\mathbb{R}^d,t>0}$ is a random field solution to (2).

Our aim is to study some properties of the solution to (2) defined in terms of the integral kernel of Laplace transform via a regularization/mollification argument. The regularization procedure allows to approximate function $u \in L^q(\Omega)$ by smooth functions.

Definition 2([11,12]). *Let $\omega(x), x \in \mathbb{R}^d$, be a function such that*

$$\omega \in C_0^\infty(\mathbb{R}^d), \quad \omega(x) \ge 0, \quad \omega(x) = 0, \quad \text{if} \quad |x| \ge 1,$$

and $\int_{\mathbb{R}^d} \omega(x)\,dx = 1$. The functions are regularized by convolution as follows

$$\omega * u = \int_{\mathbb{R}^d} \omega(x-y)u(y)\,dy.$$

For $\rho > 0$, we put $\omega_\rho(x) = \rho^{-d}\omega(\frac{x}{\rho})$, $x \in \mathbb{R}^d$. Then

$$\omega_\rho \in C_0^\infty(\mathbb{R}^d), \quad \omega_\rho(x) \ge 0, \quad \omega_\rho(x) = 0, \quad \text{if} \quad |x| \ge \rho,$$

$\int_{\mathbb{R}^d} \omega_\rho(x)\,dx = 1$, and ω_ρ is called a mollifier.

Let $\Omega \subset \mathbb{R}^d$ be a domain, and let $u \in L^q(\Omega)$ with some $1 \le q \le \infty$. One extends $u(x)$ by zero on $\mathbb{R}^d \setminus \Omega$ and consider the convolution

$$\omega_\rho * u = \int_{\mathbb{R}^d} \omega_\rho(x-y)u(y)\,dy.$$

In particular, the function $k_t(x) := \omega_t(x) = t^{-d}\omega(\frac{x}{t})$, $t > 0$ is an integral kernel which satisfies an approximation to identity property. Also, the heat kernel $k_t(x) = \frac{1}{(2\pi t)^{d/2}}\exp(-\frac{|x|^2}{2t})$ for $\alpha = 2$ is an example of the Gaussian approximate identity. We now introduce the concept of 'good kernels' and approximations to the identity, see chapter 3, section 2 of [13]. Quickly recall that averages of functions given as convolutions, can be defined by

$$(f * k_t)(x) = \int_{\mathbb{R}^d} f(x-y)k_t(y)\,dy,$$

where f is a general integrable function.

Definition 3. *A function $k_t(x)$ is called a "good kernel" if it is integrable and satisfies the following for $t>0$:*

$$\int_{\mathbb{R}^d} k_t(x)\,dx = 1 \tag{3}$$

$$\int_{\mathbb{R}^d} |k_t(x)|\,dx \le A \tag{4}$$

For every $\eta > 0$,

$$\int_{|x|\ge\eta} |k_t(x)|\,dx \to 0 \text{ as } t\to 0. \tag{5}$$

Here A is a constant independent of t.

Next, we state the conditions that k_t satisfies to be an approximation to identity.

Definition 4. *The function k_t is also called an approximation to identity if it is integrable and satisfy condition (3) but, instead of conditions (4) and (5), we assume:*

$$|k_t(x)| \le At^{-d}, \text{ for all } t>0 \tag{6}$$

$$|k_t(x)| \le \frac{At^{\varepsilon}}{|x|^{d+\varepsilon}}, \varepsilon>0, \text{ for all } t>0 \text{ and } x\in\mathbb{R}^d. \tag{7}$$

The special case where $\varepsilon = 1$ suffices in most circumstances. We now verify conditions (4) and (5) using the following integral estimate:

$$\int_{|x|\ge\varepsilon} |k_t(x)|\,dx \le \frac{C}{\varepsilon}, \text{ for some } C>0, \text{ and } \forall\varepsilon>0,$$

which is based on the following dilation-invariance property of Lebesgue measure:

$$\int_{|x|\ge\varepsilon} \frac{dx}{|x|^{d+1}} = \frac{1}{\varepsilon}\int_{|x|\ge 1}\frac{dx}{|x|^{d+1}} \le \frac{C}{\varepsilon}.$$

Now using estimates (6) and (7) when $|x|<t$ and $|x|\ge t$, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^d} |k_t(x)|\,dx\\
&= \int_{|x|<t}|k_t(x)|\,dx + \int_{|x|\ge t}|k_t(x)|\,dx\\
&\le A\int_{|x|<t}\frac{dx}{t^d} + At\int_{|x|\ge t}\frac{dx}{|x|^{d+1}}\\
&\le 2At^{1-d} + At\frac{C}{t} < \infty.
\end{aligned}$$

Secondly, in a similar manner, we have that for $\eta>0$,

$$\int_{|x|\ge\eta}|k_t(x)|\,dx \le At\int_{|x|\ge\eta}\frac{dx}{|x|^{d+1}} \le At\frac{C}{\eta} \to 0 \text{ as } t\to 0.$$

The present paper is organised as follows. The introduction including the problem formulation and some preliminary concepts and definitions forms Section 1. Some auxiliary results and estimates that will be used in the proofs of main results given in Section 2; and Section 3 gives statements and proofs of the main results. A short summary of results obtained were given in Section 4.

2 Some Estimates

Let

$$(\mathscr{A}u)(t,x) = \lambda\int_0^t\int_{\mathbb{R}} \sigma(u(y,s))k_{t-s}(x-y)w(dyds)$$

and define the following norm on $L^2(\mathrm{P})$ by

$$\|u\|_{2,T}^2 = \sup_{0\le t\le T}\sup_{x\in\mathbb{R}^d}\mathrm{E}|u(t,x)|^2.$$

Proposition 2. *Let k_t satisfies conditions (4) and (6). Let d be some positive number chosen such that $1-d>0$ and u some random field solution such that $\|u\|_{2,T}<\infty$. Then there exists some positive constant $C(\lambda,T,A,d) := \frac{1}{1-d}\lambda^2\mathrm{Lip}_\sigma^2A^2T^{1-d}$ such that*

$$\|\mathscr{A}u\|_{2,T} \le \sqrt{C(\lambda,T,A,d)}\mathrm{Lip}_\sigma\|u\|_{2,T}.$$

Proof. By Itô isometry, we obtain using the conditions on the integral kernel k:

$$\begin{aligned}
&\mathrm{E}|\mathscr{A}u(t,x)|^2\\
&= \lambda^2\int_0^t\int_{\mathbb{R}^d}|k_{t-s}(x-y)||k_{t-s}(x-y)|\mathrm{E}|\sigma(u(s,y))|^2dyds\\
&\le \lambda^2\mathrm{Lip}_\sigma^2\int_0^t\int_{\mathbb{R}^d}|k_{t-s}(x-y)||k_{t-s}(x-y)|\mathrm{E}|u(s,y)|^2dyds\\
&\le \lambda^2\mathrm{Lip}_\sigma^2\int_0^t ds\sup_{z\in\mathbb{R}^d}|k_{t-s}(z)|\sup_{y\in\mathbb{R}^d}\mathrm{E}|u(s,y)|^2\int_{\mathbb{R}}|k_{t-s}(x-y)|dy\\
&\le \lambda^2\mathrm{Lip}_\sigma^2A^2\int_0^t ds(t-s)^{-d}\sup_{y\in\mathbb{R}^d}\mathrm{E}|u(s,y)|^2\\
&\le \lambda^2\mathrm{Lip}_\sigma^2A^2\sup_{s\ge0}\sup_{y\in\mathbb{R}^d}\mathrm{E}|u(s,y)|^2\int_0^t(t-s)^{-d}ds\\
&\le \frac{1}{1-d}\lambda^2\mathrm{Lip}_\sigma^2A^2T^{1-d}\|u\|_{2,T}^2,
\end{aligned}$$

and the result follows.

Proposition 3. *Let k_t satisfies conditions (4) and (6). Let $1-d>0$ for some positive number d and let u and v be some random field solutions such that $\|u\|_{2,T}+\|v\|_{2,T}<\infty$. Then there exists some positive constant $C(\lambda,T,A,d) := \frac{1}{1-d}\lambda^2\mathrm{Lip}_\sigma^2A^2T^{1-d}$ such that*

$$\|\mathscr{A}u-\mathscr{A}v\|_{2,T} \le \sqrt{C(\lambda,T,A,d)}\mathrm{Lip}_\sigma\|u-v\|_{2,T}.$$

Proof. The proof follows same steps as above.

We use the above estimates to prove the existence and uniqueness result.

3 Main Results

We state the main results of this paper. Firstly, is the existence and uniqueness result which follows by condition: The function σ satisfies the following conditions:

$$|\sigma(0)| = 0 \quad \text{with} \quad |\sigma(x) - \sigma(y)| \leq \text{Lip}_\sigma |x-y|. \tag{8}$$

Theorem 1. *Given conditions (4) and (6) on the kernel k_t and $0 < \sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma < 1$, then there exists a unique random field solution to (2), where $C(\lambda,T,A,d) := \frac{1}{1-d}\lambda^2 \text{Lip}_\sigma^2 A^2 T^{1-d}$ with $1 > d$.*

Proof. We prove the existence of the solution via Banach-Picard iteration schemes. Define $v_0(x,t) = u_0(x)$ for all $t \geq 0$ and set

$$\begin{cases} v_{n+1} = \mathscr{A} v_n(x,t) + (u_0 * k_t)(x) \\ v_n = \mathscr{A} v_{n-1}(x,t) + (u_0 * k_t)(x). \end{cases}$$

By the condition on k_t and measurable and boundedness of u_0, we have that

$$|(u_0 * k_t)(x)| \leq \sup_{z\in\mathbb{R}} |u_0(z)| \int_{\mathbb{R}} |k_t(y)| dy \leq A \sup_{z\in\mathbb{R}} |u_0(z)| \leq const,$$

where the constant depends on the initial datum and the kernel. Let the constant be arbitrarily chosen and fixed so small such that $\|\mathscr{A} v_{n+1}\|_{2,T} = \|\mathscr{A}(\mathscr{A} v_n)\|_{2,T}$. Then by the proposition 2, we have that

$$\|\mathscr{A} v_{n+1}\|_{2,T} = \|\mathscr{A}(\mathscr{A} v_n)\|_{2,T} \leq \sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma \|\mathscr{A} v_n\|_{2,T}.$$

Let T be taken sufficiently small (that is, $C(\lambda,T,A,d) < 1/\text{Lip}_\sigma$), then it follows that $\sup_{n\geq 0} \|\mathscr{A} v_{n+1}\|_{2,T} = \sup_{n\geq 0} \|\mathscr{A} v_n\|_{2,T}$ and

$$\sup_{n\geq 0} \|\mathscr{A} v_n\|_{2,T} [1 - \sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma] \leq 0.$$

Therefore $\sup_{n\geq 0} \|\mathscr{A} v_n\|_{2,T} < \infty$ since $1 - \sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma > 0$. Also, by uniform bound on $(u_0 * k_t)(x)$,

$$\begin{aligned} \|v_k\|_{2,T} &= \|\mathscr{A} v_{k-1}\|_{2,T} + \|u_0 * k_t\|_{2,T} \\ &\leq \|\mathscr{A} v_{k-1}\|_{2,T} + A \sup_{z\in\mathbb{R}} |u_0(z)|. \end{aligned}$$

Then taking sup of both sides, we have that

$$\begin{aligned} \sup_{k\geq 1} \|v_k\|_{2,T} \leq \sup_{k\geq 1} \|\mathscr{A} v_{k-1}\|_{2,T} \\ + A \sup_{z\in\mathbb{R}} |u_0(z)| < \infty. \end{aligned}$$

More so, by proposition 3 and for $j = 1,2,\ldots,n$, we have

$$\begin{aligned} &||v_{n+1} - v_n||_{2,T} \\ &= \|\mathscr{A} v_n - \mathscr{A} v_{n-1}\|_{2,T} \\ &\leq \sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma \|v_n - v_{n-1}\|_{2,T} \\ &\vdots \\ &\leq (\sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma)^j \|v_{n-(j-1)} - v_{n-j}\|_{2,T} \\ &\leq (\sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma)^n \|v_1 - v_0\|_{2,T}. \end{aligned}$$

Given $\varepsilon > 0$, we can find $N(\varepsilon) \in \mathbb{N}$ such that $(\sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma)^{N(\varepsilon)} \|v_1 - v_0\|_{2,T} < \varepsilon$. Given that $\sqrt{C(\lambda,T,A,d)} < 1$, then for any $n+1 > n \geq N(\varepsilon)$, we have that

$$\begin{aligned} \|v_{n+1} - v_n\|_{2,T} &\leq (\sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma)^n \|v_1 - v_0\|_{2,T} \\ &\leq (\sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma)^{N(\varepsilon)} \|v_1 - v_0\|_{2,T} < \varepsilon. \end{aligned}$$

Thus $\{v_n\}_{n\geq 1}$ is a Cauchy sequence in $(L^2, \|.\|_{2,T})$ and since the space $(L^2, \|.\|_{2,T})$ is complete, then the sequence has its point-wise limit u in $(L^2, \|.\|_{2,T})$. Next, show that the equation has a unique solution up to modification. Suppose for contradiction that there exist solutions u_1 and u_2 with $u_1 \neq u_2$ such that

$$\begin{cases} u_1 = \mathscr{A} u_1(t,x) + (u_0 * k_t)(x) \\ u_2 = \mathscr{A} u_2(t,x) + (u_0 * k_t)(x). \end{cases}$$

Then $\|u_1 - u_2\|_{2,T} = \|\mathscr{A} u_1 - \mathscr{A} u_2\|_{2,T} \leq \sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma \|u_1 - u_2\|_{2,T}$. That's $\|u_1 - u_2\|_{2,T}[1 - \sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma] \leq 0$ and since $\sqrt{C(\lambda,T,A,d)}\text{Lip}_\sigma < 1$, we have that $\|u_1 - u_2\|_{2,T} = 0$ and consequently, $u_1 = u_2$.

Now for the energy growth bounds of the solution, we consider the following renewal inequality.

Lemma 2([14]). *Suppose $b \geq 0, \beta > 0$ and $a(t)$ is a non-negative function locally integrable on $0 \leq t \leq T$, $(T < \infty)$, and suppose that $f(t)$ is non-negative and locally integrable on $0 \leq t \leq T$ with*

$$f(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1} f(s) ds$$

on this interval; then

$$f(t) \leq a(t) + \theta \int_0^t E'_\beta(\theta(t-s)) a(s) ds, \quad 0 \leq t \leq T,$$

where $\theta = (b\Gamma(\beta))^{1/\beta}$, $E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta+1)}$, $E'_\beta(z) = \frac{d}{dz}E_\beta(z)$, $E'_\beta(\beta) = \frac{z^{\beta-1}}{\Gamma(\beta)}$ as $z \to 0^+$, $E'_\beta(\beta) = \frac{1}{\beta}e^z$ as $z \to \infty$ (and $E_\beta(z) \approx \frac{1}{\beta}e^z$ as $z \to \infty$). If $a(t) \equiv a$, constant, then

$$f(t) \leq aE_\beta(\theta t).$$

Rather than finding the asymptotic properties of the function, we get bounds on the functions involved and state as follows:

Proposition 4([2]). *Let $\rho > 0$ and suppose $f(t)$ is a non-negative and locally integrable function satisfying*

$$f(t) \leq c_1 + \kappa \int_0^t (t-s)^{\rho-1} f(s)ds, \text{ for all } t > 0,$$

where c_1 is some positive number. Then we have that:

$$f(t) \leq c_2 \exp\left(c_3 (\Gamma(\rho))^{1/\rho} \kappa^{1/\rho} t\right) \text{ for all } t > 0.$$

*Proof.*See [1,2] for the proof of the proposition.

We now apply the above proposition to prove the moment growth of the solution.

Theorem 2.*Suppose conditions (4) and (7) on the kernel hold, then a.e,*

$$\mathrm{E}|u(t,x)|^2 \leq c_2 \exp\left(c_3 \lambda^{\frac{1}{1+\varepsilon}} t\right)$$

for some positive constants c_2, c_3.

For now, we are able to prove the upper bound for the moment estimate; the lower bound is open for further research.

*Proof.*Applying It ô isometry,

$$\begin{aligned} &\mathrm{E}|u(t,x)|^2 \\ &\leq \left| \int_{\mathbb{R}^d} u_0(x-y) k_t(y) dy \right|^2 \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \mathrm{E}|\sigma(u(s,y))|^2 |k_{t-s}(x-y)|^2 dyds \\ &\leq A^2 \left(\sup_{z\in\mathbb{R}^d} |u_0(z)| \right)^2 \\ &\quad + \mathrm{Lip}_\sigma^2 \int_0^t \int_{\mathbb{R}^d} \mathrm{E}|u(s,y)|^2 |k_{t-s}(x-y)|^2 dyds. \end{aligned}$$

Let $F_t = \sup_{x\in\mathbb{R}^d} \mathrm{E}|u(t,x)|^2$, then since u_0 is a bounded function,

$$\mathrm{E}|u(t,x)|^2 \leq c_1 + \mathrm{Lip}_\sigma^2 \int_0^t dsF_s \int_{\mathbb{R}^d} |k_{t-s}(x-y)||k_{t-s}(x-y)|dy.$$

Therefore by the conditions (4) and (7) on the kernel, we obtain

$$\mathrm{E}|u(t,x)|^2 \leq c_1 + \mathrm{Lip}_\sigma^2 A^2 \sup_{z\in\mathbb{R}^d} \frac{1}{|z|^{d+\varepsilon}} \int_0^t (t-s)^\varepsilon F_s ds.$$

Assume that almost surely, $\sup_{z\in\mathbb{R}^d} \frac{1}{|z|^{d+\varepsilon}} < \infty$, then

$$\begin{aligned} F_t &\leq c_1 + c\,\mathrm{Lip}_\sigma^2 A^2 \int_0^t (t-s)^\varepsilon F_s ds \\ &= c_1 + c\,\mathrm{Lip}_\sigma^2 A^2 \int_0^t (t-s)^{\rho-1} F_s ds \end{aligned}$$

with $\rho = 1+\varepsilon$. The result follows by applying proposition 4 for $\kappa = c\,\mathrm{Lip}_\sigma^2 A^2$.

4 Conclusion

An integral representation to a solution of the equation were given, existence and uniqueness result established and we were able to give an upper energy growth bound for the integral solution.

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