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Existence Theory and Stability Analysis of Solutions of Perturbed Fractional Differential Equations

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Abstract: This paper proposes a novel approach to proving the existence of solutions for perturbed fractional differential equations. We introduce a new derivative operator along with an associated equation, denoted as $P(x)$. The purpose of $P(x)$ is to characterize the ball $\mathfrak{B}\varphi = \{\mu \in \mathfrak{C}/\|\mu\|_\infty \leq \|\varphi\|_\infty\}$, within which the solutions to our problem (1) are contained. This is achieved through the application of Schauder's and Boyd-Wong's fixed point theorems. Additionally, we explore Ulam-Hyers stability and its generalizations using tools from nonlinear analysis. The paper concludes with three illustrative examples, complete with numerical results and figures, demonstrating the validity of our findings.

Keywords: Banach space, Riemann-Liouville fractional derivative, Boundary value problem, Schauder's fixed point theorem.

1 Introduction

Fractional calculus has emerged as a powerful mathematical tool for modeling various processes in diverse fields such as engineering, physics, economics, and many others. The application of fractional-order models has proven advantageous over traditional integer-order models, especially in describing systems with memory and hereditary properties. This is due to the unique ability of fractional derivatives to capture the dynamics of processes where past states significantly influence future states, a phenomenon common in materials and natural processes. For example, fractional calculus is particularly useful in modeling viscoelastic materials, diffusion processes with anomalous diffusion, and the complex behavior of electromagnetic waves in certain media.

In recent years, fractional differential equations (FDEs) have found widespread applications in various domains such as material science, signal processing, biology, electromagnetics, fluid dynamics, thermodynamics, and control theory, among others. These equations have been shown to outperform integer-order models, especially when dealing with non-local behaviors and memory effects. For a comprehensive understanding of this topic, we refer to foundational monographs like [8,9,10,11,12,14,15,16,17]. This versatility is a key advantage of fractional calculus compared to classical integer-order models, making it a valuable tool in scientific modeling.

Key areas of fractional calculus revolve around several types of fractional derivatives, such as Riemann-Liouville, Caputo, Hilfer, Erdelyi-Kober, and Hadamard. Among these, recent advancements in solving boundary value problems for nonlinear fractional differential equations have garnered significant attention. These equations play a crucial role in describing real-world phenomena with inherent non-homogeneities and complex boundary conditions.

Many studies have been devoted to various fractional derivatives, such as the Riemann-Liouville [2,21], the Hilfer [9], the Caputo [22,23], the Hadamard [19,20], the Erdelyi-Kober [24,25], and the ψ-fractional derivatives [2,26]. The importance of these derivatives is reflected in the significant research efforts focused on proving the existence, uniqueness, and stability of solutions for fractional differential equations, which are crucial for modeling real-world systems involving fractional-order dynamics [29,30,31,32,33].

For example, in [2], the authors studied the existence, uniqueness, and continuous dependence of solutions for fractional differential equations with the Riemann-Liouville derivative in Banach spaces, considering the following

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boundary value problem:

$$\begin{cases} {}^{RL}D_{\delta}^{\kappa}\mu(\gamma) = \mathfrak{h}(\gamma,\mu(\gamma)), \quad \gamma \in [\delta,\beta], \\ \mu^{(k)}(\delta) = \mu_k, \quad k = 0,1,\ldots,n-2; \quad \mu^{(n-1)}(\beta) = \mu_\beta \end{cases}$$

In a similar vein, Sun *et al.* [27] established results concerning the existence and uniqueness of solutions to nonlinear fractional differential equations involving the Caputo fractional derivative:

$$\begin{cases} {}^{c}D_{\delta}^{\kappa}\mu(\gamma) = \mathfrak{h}(\gamma,\mu(\gamma)), \quad \gamma \in [\delta,\beta], \\ \mu^{(k)}(\delta) = \mu_k, \quad k = 0,1,\ldots,n-2; \quad \mu^{(n-1)}(\beta) = \mu_\beta \end{cases}$$

Jarad *et al.* [13] investigated the existence, uniqueness, and Ulam-Hyers stability of solutions for equations involving the Caputo Atangana-Baleanu fractional derivative, given by:

$$\begin{cases} {}^{ABC}D_{\delta}^{\kappa}\mu(\gamma) = \mathfrak{h}(\gamma,\mu(\gamma)), \quad \gamma \in [\delta,\beta], \\ \mu(\delta) = \mu_0. \end{cases}$$

In this paper, we present a detailed study of the existence of solutions to a perturbed fractional differential equation of the form:

$$\begin{cases} {}_{\varphi}D_{\delta}^{\kappa}\mu(\gamma) = \mathfrak{h}(\gamma,\mu(\gamma)), \quad \gamma \in J = [\delta,\beta], \\ \mu(\delta) = \mu_0, \end{cases} \tag{1}$$

where ${}_{\varphi}D_{\delta}^{\kappa}$ is the perturbed fractional derivative operator of the Riemann-Liouville derivative, given by

$${}_{\varphi}D_{\delta}^{\kappa}f(\gamma) = \frac{1}{\varphi(\gamma)}{}^{RL}D_{\delta}^{\kappa}f(\gamma),$$

with φ being a nonzero continuous function from J into $\mathbb{R}$, and $f : J \times \mathbb{R} \to \mathbb{R}$ satisfying Carathéodory-type conditions. Furthermore, there exist constants A and B (non-negative real constants) such that:

$$|\mathfrak{h}(\gamma,\mu)| \leq A|\mu| + B \quad \text{for all} \quad \mu \in \mathbb{R}, \quad \gamma \in J.$$

To analyze the existence of solutions, we introduce an auxiliary equation related to problem (1), denoted by $P(x)$, which is defined as:

$$P(x) = x^2\frac{A(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} + x\left[\frac{B(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} - 1\right] + |\mu_0|.$$

The goal is to identify the ball $\mathfrak{B}\varphi = \{\mu \in \mathfrak{C} : \|\mu\|_{\infty} \leq \|\varphi\|_{\infty}\}$ in which the solutions to the problem (1) lie.

To establish the existence of solutions, we first employ Schauder's fixed-point theorem. This theorem provides a framework for demonstrating the existence of a solution under certain conditions. Next, we apply the Boyd-Wong fixed-point theorem to establish not only the existence but also the uniqueness of the solution.

In Section 5, we discuss the Ulam-Hyers stability of the solutions and explore its generalized form. Stability is a crucial property, as it ensures that small perturbations in the initial conditions do not lead to large deviations in the solution. In Section 6, we provide several numerical examples, accompanied by graphical representations, to validate the theoretical findings of the paper.

2 Preliminary

In this section, we recall some basic definitions and results that will be used in this paper.

Denote by $L^1(J,\mathbb{R})$ the space of Lebesgue integrable real-valued functions on J and by $\mathfrak{C} = C(J,\mathbb{R})$ the Banach space of continuous functions $\mu : J \longrightarrow \mathbb{R}$, with the norm $\|\mu\|_{\infty} = \sup_{\gamma\in J}|\mu(\gamma)|$.

Definition 1.*([2, 11, 7]) Let $\kappa > 0$ and $\mathfrak{h} \in L^1(J,\mathbb{R})$. The Riemann-Liouville fractional integral of order κ for the function $\mathfrak{h}$ is expressed as*

$${}^{RL}I_{\delta}^{\kappa}\mathfrak{h}(\gamma) = \frac{1}{\Gamma(\kappa)}\int_{\delta}^{\gamma}(\gamma-\eta)^{\kappa-1}\mathfrak{h}(\eta)\,d\eta.$$

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Definition 2.*([2, 11, 7]) Let $\kappa > 0$ and $\mathfrak{h} \in C^n(J,\mathbb{R})$. The Riemann-Liouville fractional derivative of order κ for the function $\mathfrak{h}$ is given by*

$$
\begin{aligned}
{}^{RL}D_{\delta}^{\kappa}\mathfrak{h}(\gamma) &= \frac{1}{\Gamma(n-\kappa)}\frac{d^n}{d\gamma^n}\int_{\delta}^{\gamma}(\gamma-\eta)^{n-\kappa-1}\mathfrak{h}(\eta)\,d\eta \\
&= I_{\delta}^{n-\kappa}\left(\frac{d^n}{d\gamma^n}\mathfrak{h}(\gamma)\right),
\end{aligned}
$$

where $n = [\kappa]+1$ and $[\kappa]$ denotes the integer part of the real number κ.

Proposition 1.*([2, 21]) Let $\kappa > 0$, and $n = [\kappa]+1$.*

1.If $\mathfrak{h} \in L^1(J,\mathbb{R})$, then the relation

$$
{}^{RL}D_{\delta}^{\kappa}\left({}^{RL}I_{\delta}^{\kappa}\mathfrak{h}(\gamma)\right) = \mathfrak{h}(\gamma)
$$

holds almost everywhere on J.

2.If $\mathfrak{h} \in C^n(J,\mathbb{R})$, then for all $\gamma \in J$, we have:

$$
{}^{RL}I_{\delta}^{\kappa}\left({}^{RL}D_{\delta}^{\kappa}\mathfrak{h}(\gamma)\right) = \mathfrak{h}(\gamma) - \sum_{k=0}^{n-1}\frac{\mathfrak{h}^{(k)}(\delta)}{k!}(\gamma-\delta)^k.
$$

Theorem 1.*(Schauder's Fixed Point Theorem [4]) Assume that $\mathfrak{K}$ is a convex, closed, bounded, and nonempty subset in a Banach space $\mathfrak{X}$ and that $\mathfrak{T} : \mathfrak{K} \to \mathfrak{K}$ is a compact continuous mapping. Then, $\mathfrak{T}$ has a fixed point.*

Theorem 2.*(Boyd-Wong [3]) If X is a complete metric space and $\mathfrak{T} : X \longrightarrow X$ is a mapping, under the conditions that there exists a function*

$$
\phi \in \left\{ g : [0,\infty) \to [0,\infty) : g(r) < r \text{ for all } r > 0 \text{ and } \limsup_{s \to r^+} g(\eta) < r \text{ for all } r > 0 \right\},
$$

such that

$$
\|\mathfrak{T}(x_1) - \mathfrak{T}(x_2)\| \leq \phi(\|x_1 - x_2\|) \quad \text{for all } x_1, x_2 \in X,
$$

then $\mathfrak{T}$ has a unique fixed point in X.

3 Perturbed fractional operators

In this section, by the function φ we define our perturbed fractional operators of the Riemann-Liouville fractional integral operator and the Riemann-Liouville fractional derivative operator.

Definition 3. *Let $\kappa > 0$, $\mathfrak{h} \in L^1(J,\mathbb{R})$ and $\varphi : J \longrightarrow \mathbb{R}$ is a nonzero continuous function. We define new fractional integral that we called $\varphi-$fractional integral. For all $\gamma \in J$ we have*

$$
\begin{aligned}
{}_{\varphi}I_{\delta}^{\kappa}\mathfrak{h}(\gamma) &= {}^{RL}I_{\delta}^{\kappa}(\varphi\mathfrak{h})(\gamma) \\
&= \frac{1}{\Gamma(\kappa)}\int_{\delta}^{\gamma}(\gamma-z)^{\kappa-1}\varphi(z)\mathfrak{h}(z)\,dz
\end{aligned}
$$

The associated fractional derivative operator is defined, for any $\gamma \in J$,

$$
\begin{aligned}
{}_{\varphi}D_{\delta}^{\kappa}f(\gamma) &= \frac{1}{\varphi(\gamma)}D_{\delta}^{\kappa}\mathfrak{h}(\gamma) \\
&= \frac{1}{\varphi(\gamma)\Gamma(n-\kappa)}\frac{d^n}{d\gamma^n}\int_{\delta}^{\gamma}(\gamma-z)^{n-\kappa-1}\mathfrak{h}(z)\,dz
\end{aligned}
$$

We show in the following result that the Proposition 1 is true for operators ${}_{\varphi}I_{\delta}^{\kappa}$ and ${}_{\varphi}D_{\delta}^{\kappa}$.

Theorem 3. *Let $\kappa > 0$, and $n = [\kappa]+1$.*

1. If $\mathfrak{h} \in L^1(J,\mathbb{R})$, then the formula

$$
{}_{\varphi}D_{\delta}^{\kappa}\cdot{}_{\varphi}I_{\delta}^{\kappa}\mathfrak{h}(\gamma) = \mathfrak{h}(\gamma)
$$

holds almost everywhere on J.

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2. *If* $\mathfrak{h} \in C^n(J,\mathbb{R})$, *then for all* $\gamma \in J$ *we have:*

$$ {}_{\varphi}I_{\delta}^{\kappa} \cdot {}_{\varphi}D_{\delta}^{\kappa}\mathfrak{h}(\gamma) = \mathfrak{h}(\gamma) - \sum_{k=0}^{n-1} \frac{\mathfrak{h}^{(k)}(\delta)}{k!}(\gamma-\delta)^k $$

Proof. To prove this result we use the Proposition 1.

(1)

$$\begin{aligned} {}_{\varphi}D_{\delta}^{\kappa} \cdot {}_{\varphi}I_{\delta}^{\kappa}\mathfrak{h}(\gamma) &= \frac{1}{\varphi(\gamma)} {}^{RL}D_{\delta}^{\kappa} {}^{RL}I_{\delta}^{\kappa}(\varphi\mathfrak{h})(\gamma) \\ &= \frac{1}{\varphi(\gamma)} {}^{RL}I_{\delta}^{\kappa-\kappa}(\varphi\mathfrak{h})(\gamma) \\ &= \frac{1}{\varphi(\gamma)}.(\varphi\mathfrak{h})(\gamma) \\ &= \mathfrak{h}(\gamma) \end{aligned}$$

(2)

$$\begin{aligned} {}_{\varphi}I_{\delta}^{\kappa} \cdot {}_{\varphi}D_{\delta}^{\kappa}\mathfrak{h}(\gamma) &= {}^{RL}I_{\delta}^{\kappa}\left(\varphi(\gamma)\frac{1}{\varphi(\gamma)} {}^{RL}D_{\delta}^{\kappa}f(\gamma)\right) \\ &= {}^{RL}I_{\delta}^{\kappa}.{}^{RL}D_{\delta}^{\kappa}\mathfrak{h}(\gamma) \\ &= \mathfrak{h}(\gamma) - \sum_{k=0}^{n-1} \frac{\mathfrak{h}^{(k)}(\delta)}{k!}(\gamma-\delta)^k \end{aligned}$$

Remark.

1) For $\varphi(\gamma) = 1$, for any $\gamma \in J$, we have ${}_{\varphi}I_{\delta}^{\kappa} = {}^{RL}I_{\delta}^{\kappa}$, and $\varphi D_{\delta}^{\kappa} = D_{\delta}^{\kappa}$.

2) We have μ is solution of the problem (1) if and only if μ solution of

$$\begin{cases} {}^{RL}D_{\delta}^{\kappa}\mu(\gamma) = \varphi(\gamma)\mathfrak{h}(\gamma,\mu(\gamma)), & \gamma \in J, \\ \mu(\delta) = \mu_0, \end{cases}$$

4 Existence and uniqueness result of the Problem (1)

We will start this section by firstly, presenting some hypotheses. Secondly, introducing the equation $P(x)$ associate to the problem (1).

$\mathscr{H}_1$) $f : J \times \mathfrak{C} \to \mathfrak{C}$ satisfies the Carathéodory type conditions: the function $\gamma \longmapsto \mathfrak{h}(\gamma,\mu)$ is measurable on J for each $\mu \in \mathfrak{C}$ and $\mu \longmapsto \mathfrak{h}(\gamma,\mu)$ is continuous on $\mathfrak{C}$ for all $\gamma \in J$.

$\mathscr{H}_2$) There exist A and B two non-negative real constants satisfying

i)

$$\frac{B(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} > 4\frac{A(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)}|\mu_0| + 1$$

ii)

$$\frac{B(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} \leq 1$$

iii)

$$|\mathfrak{h}(\gamma,\mu)| \leq A|\mu| + B \tag{2}$$

for all $\mu \in \mathfrak{C}$, and $\gamma \in J$.

$\mathscr{H}_3$) There exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that for each $\mu, \vartheta \in \mathfrak{C}$ and $\gamma \in J$

$$|\mathfrak{h}(\gamma,\mu(\gamma)) - \mathfrak{h}(\gamma,\vartheta(\gamma))| \leq \psi(|\mu(\gamma) - \vartheta(\gamma)|)$$

with $\psi(r)\frac{(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} < \frac{r}{\|\varphi\|_{\infty}}$ for all $r > 0$.

To solve the problem (1), we introduce its associate equation defined by

$$P(x) = x^2\frac{A(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} + x\left[\frac{B(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} - 1\right] + |\mu_0|.$$

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The goal of $P(x)$ is to define the ball $\mathfrak{B}\varphi = \{\mu \in \mathfrak{C}/\|\mu\|_\infty \leq \|\varphi\|_\infty\}$ to which solutions of the problem (1) belong.

On the other hand, $P(x)$ does not depend on the function φ. Also, in our study, the existence of solutions of the problem (1) depends only on $\|\varphi\|_\infty$ solution of the inequation

$$P(x) \leq 0.$$

The following lemmas are used to show Theorem 4 below.

Lemma 1. *$u \in \mathfrak{C}$ solution of (1) if and only if*

$$\mu(\gamma) = \mu_0 + \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma-\eta)^{\kappa-1} \varphi(\eta) \mathfrak{h}(\eta, \mu(\eta))\, d\eta \tag{3}$$

Proof. Through direct computation, using Theorem 3, it becomes evident that if μ satisfies integral equation (3), then the problem (1) holds true, and by integrating the problem (1) we get the equation (3), thus concluding the proof.

Lemma 2. *Under the hypothesis $\mathscr{H}_2$) the set of the following inequation*

$$P(x) \leq 0 \tag{4}$$

is $[x_1, x_2]$, where $0 \leq x_1 < x_2$ are roots of the equation

$$P(x) = 0.$$

Proof. Consider the equation $P(x) = 0$, we have

$$\Delta = \left(\frac{B(\beta-\delta)^\kappa}{\Gamma(\kappa+1)} - 1 \right)^2 - 4\frac{A(\beta-\delta)^\kappa}{\Gamma(\kappa+1)} |\mu_0|.$$

So, by the hypothesis $\mathscr{H}_2$. a), we have $\Delta > 0$. Therefore, roots of the equation $P(x) = 0$ are given by

$$x_1 = \frac{\Gamma(\kappa+1) - B(\beta-\delta)^\kappa - \Gamma(\kappa+1)\sqrt{\Delta}}{2A(\beta-\delta)^\kappa}$$

and

$$x_2 = \frac{\Gamma(\kappa+1) - B(\beta-\delta)^\kappa + \Gamma(\kappa+1)\sqrt{\Delta}}{2A(\beta-\delta)^\kappa}.$$

On the other hand, we have $x_1 x_2 = \frac{|\mu_0|\Gamma(\kappa+1)}{A(\beta-\delta)^\kappa} \geq 0$, hence x_1 and x_2 have the same sign. And by the hypothesis $\mathscr{H}_2$. b)

$$(x_1 + x_2) = \frac{\Gamma(\kappa+1)}{A(\beta-\delta)^\kappa} \left[\frac{1 - B(\beta-\delta)^\kappa}{\Gamma(\kappa+1)} \right] \geq 0.$$

So, $0 \leq x_1 < x_2$. Therefore, the set of solutions of the inequation $P(x) \leq 0$ is $[x_1, x_2]$.

In the following result, we show that the problem (1) has a solution.

Theorem 4. *Assume that hypotheses $\mathscr{H}_1$) and $\mathscr{H}_2$) are satisfied, and that $\|\varphi\|_\infty$ is a solution of the inequality*

$$P(x) \leq 0.$$

Then the problem (1) has at least one solution $\mu \in \mathfrak{B}_\varphi \subset \mathfrak{C}$.

Proof. We consider the following operator and show that it has a fixed point in $\mathfrak{B}_\varphi$:

$$\mathfrak{D}(\mu)(\gamma) = \mu_0 + \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma-\eta)^{\kappa-1} \varphi(\eta) \mathfrak{h}(\eta, \mu(\eta))\, d\eta \tag{5}$$

FIRST STEP. We claim that $\mathfrak{B}_\varphi$ is invariant under the operator $\mathfrak{D}$.

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Let $\mu \in \mathfrak{B}_\varphi$, then we have

$$\begin{aligned}|\mathfrak{D}(\mu)(\gamma)| &\leq |\mu_0| + \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma-\eta)^{\kappa-1} |\varphi(\eta)\mathfrak{h}(\eta,\mu(\eta))|\, d\eta \\ &\leq |\mu_0| + \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma-\eta)^{\kappa-1} |\varphi(\eta)|(A|\mu(\eta)|+B)\, d\eta \\ &\leq |\mu_0| + \|\varphi\|_\infty \left(\frac{(A\|\varphi\|_\infty + B)(\beta-\delta)^\kappa}{\Gamma(\kappa+1)} \right).\end{aligned}$$

Since

$$P(\|\varphi\|_\infty) \leq 0$$

and

$$0 < |\mu_0| + \|\varphi\|_\infty \left(\frac{(A\|\varphi\|_\infty + B)(\beta-\delta)^\kappa}{\Gamma(\kappa+1)} \right) = P(\|\varphi\|_\infty) + \|\varphi\|_\infty,$$

we obtain

$$|\mathfrak{D}(\mu)(\gamma)| \leq \|\varphi\|_\infty.$$

Hence,

$$\|\mathfrak{D}(\mu)\|_\infty \leq \|\varphi\|_\infty,$$

which implies that $\mathfrak{D}(\mathfrak{B}_\varphi) \subset \mathfrak{B}_\varphi$.

SECOND STEP. We show that the operator $\mathfrak{D}$ is continuous in $\mathfrak{B}_\varphi$.

Let (μ_n) be a sequence such that μ_n converges to μ in $\mathfrak{B}_\varphi$. Then we have:

$$\begin{aligned}|\mathfrak{D}(\mu_n)(\gamma) - \mathfrak{D}(\mu)(\gamma)| &= \frac{1}{\Gamma(\kappa)} \left| \int_\delta^\gamma (\gamma-z)^{\kappa-1} \varphi(z) \left(\mathfrak{h}(z,\mu_n(z)) - \mathfrak{h}(z,\mu(z))\right) dz \right| \\ &\leq \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma-z)^{\kappa-1} |\varphi(z)| |\mathfrak{h}(z,\mu_n(z)) - \mathfrak{h}(z,\mu(z))|\, dz.\end{aligned}$$

Using hypotheses $\mathscr{H}_1$) and $\mathscr{H}_2$), we get

$$|(\gamma-z)^{\kappa-1} |\varphi(z)|\, |\mathfrak{h}(z,\mu_n(z)) - \mathfrak{h}(z,\mu(z))| \leq 2\|\varphi\|_\infty (A\|\varphi\|_\infty + B)(\gamma-z)^{\kappa-1}.$$

By the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n\to+\infty} \left| \int_\delta^\gamma (\gamma-z)^{\kappa-1} |\varphi(z)| \left(\mathfrak{h}(z,\mu_n(z)) - \mathfrak{h}(z,\mu(z))\right) dz \right| = 0.$$

Hence, $\mathfrak{D}$ is continuous.

THIRD STEP. We show that $\mathfrak{D}(\mathfrak{B}_\varphi)$ is equicontinuous.

Let $\mu \in \mathfrak{D}(\mathfrak{B}_\varphi)$ and $\gamma_1, \gamma_2 \in J$ $(\gamma_1 < \gamma_2)$. Then we have:

$$\begin{aligned}|\mathfrak{D}(\mu)(\gamma_2) - \mathfrak{D}(\mu)(\gamma_1)| &= \frac{1}{\Gamma(\kappa)} \left| \int_\delta^{\gamma_2} (\gamma_2-z)^{\kappa-1} \varphi(z)\mathfrak{h}(z,\mu(z))\, dz - \int_\delta^{\gamma_1} (\gamma_1-z)^{\kappa-1} \varphi(z)\mathfrak{h}(z,\mu(z))\, dz \right| \\ &\leq \frac{1}{\Gamma(\kappa)} \left| \int_\delta^{\gamma_1} \left((\gamma_2-z)^{\kappa-1} - (\gamma_1-z)^{\kappa-1} \right) \varphi(z)\mathfrak{h}(z,\mu(z))\, dz \right| \\ &\quad + \frac{1}{\Gamma(\kappa)} \left| \int_{\gamma_1}^{\gamma_2} (\gamma_2-z)^{\kappa-1} \varphi(z)\mathfrak{h}(z,\mu(z))\, dz \right| \\ &\leq \frac{\|\varphi\|_\infty (A\|\varphi\|_\infty + B)}{\Gamma(\kappa+1)} \left(|(\gamma_2-\delta)^\kappa - (\gamma_2-\gamma_1)^\kappa - (\gamma_1-\delta)^\kappa| \right) \\ &\quad + \frac{\|\varphi\|_\infty (A\|\varphi\|_\infty + B)}{\Gamma(\kappa+1)} \left(|\gamma_2-\gamma_1|^\kappa \right) \\ &\leq \frac{\|\varphi\|_\infty (A\|\varphi\|_\infty + B)}{\Gamma(\kappa+1)} \left(2|\gamma_2-\gamma_1|^\kappa + (\gamma_2-\delta)^\kappa - (\gamma_1-\delta)^\kappa \right).\end{aligned}$$

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Hence,

$$\lim_{\gamma_2 \to \gamma_1} |\mathfrak{D}(\mu)(\gamma_2) - \mathfrak{D}(\mu)(\gamma_1)| = 0$$

for all $\mu \in \mathfrak{D}(\mathfrak{B}_\varphi)$, which shows that $\mathfrak{D}(\mathfrak{B}_\varphi)$ is equicontinuous.

Finally, since $\mathfrak{D}(\mathfrak{B}_\varphi)$ is uniformly bounded and equicontinuous, by the Arzelà-Ascoli Theorem (see [1]), $\mathfrak{D}(\mathfrak{B}_\varphi)$ is relatively compact. Therefore, the operator $\mathfrak{D}$ is compact from $\mathfrak{B}_\varphi$ into $\mathfrak{B}_\varphi$. By Schauder's fixed point theorem, $\mathfrak{D}$ has at least one fixed point in $\mathfrak{B}_\varphi$, which is a solution of the problem (1) in $\mathfrak{B}_\varphi \subset \mathfrak{C}$.

If we change condition iii) in hypothesis $\mathscr{H}_2$) to:
iv) There exist positive and continuous functions $\kappa_1, \kappa_2 : J \longrightarrow \mathbb{R}$ such that

$$|\mathfrak{h}(\gamma, \mu)| \leq \kappa_1(\gamma)|\mu| + \kappa_2(\gamma)$$

for all $\mu \in \mathbb{C}$ and $\gamma \in J$, we obtain the following corollary:

Corollary 1. *Let $A = \|\kappa_1(\gamma)\|_\infty$ and $B = \|\kappa_2(\gamma)\|_\infty$; if A and B satisfy conditions a) and b) in $\mathscr{H}_2$), then Theorem 4 remains valid.*

Now, we introduce the result for the existence and uniqueness of the solution to problem (1).

Theorem 5. *Under the hypothesis $\mathscr{H}_3$), the problem (1) has a unique solution $\mu \in \mathbb{C}$.*

Proof. We show that the operator $\mathfrak{D}$ defined in (5) has a unique fixed point in $\mathbb{C}$.

Let $\gamma \in J$. We have:

$$\begin{aligned}
|\mathfrak{D}(\mu)(\gamma) - \mathfrak{D}(\vartheta)(\gamma)| &= \frac{1}{\Gamma(\kappa)} \left| \int_\delta^\gamma (\gamma - z)^{\kappa-1} \varphi(z) \left(\mathfrak{h}(z, \mu(z)) - \mathfrak{h}(z, \vartheta(z))\right) dz \right| \\
&\leq \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma - z)^{\kappa-1} |\varphi(z)| \, |\mathfrak{h}(z, \mu(z)) - \mathfrak{h}(z, \vartheta(z))| \, dz \\
&\leq \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma - z)^{\kappa-1} \|\varphi\|_\infty \psi(|\mu(z) - \vartheta(z)|) \, dz \\
&\leq \|\varphi\|_\infty \psi(\|\mu - \vartheta\|_\infty) \frac{1}{\Gamma(\kappa)} \int_\delta^\gamma (\gamma - z)^{\kappa-1} \, dz \\
&\leq \|\varphi\|_\infty \psi(\|\mu - \vartheta\|_\infty) \frac{(\beta - \delta)^\kappa}{\Gamma(\kappa + 1)}.
\end{aligned}$$

Hence,

$$\|\mathfrak{D}(\mu) - \mathfrak{D}(\vartheta)\|_\infty \leq \|\varphi\|_\infty \psi(\|\mu - \vartheta\|_\infty) \frac{(\beta - \delta)^\kappa}{\Gamma(\kappa + 1)} = \Psi(\|\mu - \vartheta\|_\infty),$$

where

$$\Psi(r) = \|\varphi\|_\infty \psi(r) \frac{(\beta - \delta)^\kappa}{\Gamma(\kappa + 1)}.$$

Using hypothesis $\mathscr{H}_3$, we have

$$\Psi(r) < r.$$

Then, by the Boyd-Wong fixed point theorem, we conclude that $\mathfrak{D}$ has a unique fixed point. Thus, the problem (1) has a unique solution.

5 Stability analysis

Lemma 3.*(A generalized Gronwall inequality [28]) Let $\mathfrak{u}$ and $\mathfrak{v}$ positive functions such that*

$$\mathfrak{u}(\gamma) \leq \mathfrak{v}(\gamma) + \mathfrak{b}(\gamma) \int_\delta^\gamma (\gamma - s)^{\kappa-1} \mathfrak{u}(s) \, ds.$$

Then

$$\mathfrak{u}(\gamma) \leq \mathfrak{v}(\gamma) + \int_\delta^\gamma \sum_{n=1}^{+\infty} \frac{(\mathfrak{b}(\gamma)\Gamma(\kappa))^n}{\Gamma(n\kappa)} (\gamma - s)^{n\kappa-1} \mathfrak{v}(s) \, ds. \tag{6}$$

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Definition 4. *We say that (1) is Ulam-Hyers stable if there exists $c_{\mathfrak{f}} > 0$, such that for each $\varepsilon > 0$ and $\mathfrak{v} \in \mathfrak{C}$ verifying*

$$|{}_{\varphi}D_{\delta}^{\kappa}\mathfrak{v}(\gamma) - \mathfrak{f}(\gamma, \mathfrak{v}(\gamma))| < \varepsilon, \quad \gamma \in J \tag{7}$$

there exists a solution $\mathfrak{u} \in \mathfrak{C}$ of (1) such that:

$$|\mathfrak{u}(\gamma) - \mathfrak{v}(\gamma)| < \varepsilon c_f, \quad \gamma \in J.$$

Furthermore, (1) is said to be generalized Ulam-Hyers if there exists $\phi_f : [0, +\infty) \longrightarrow [0, +\infty)$ a continuous function, with $\phi_f(0) = 0$, and for every $\mathfrak{v} \in \mathfrak{C}$ solution of the inequality

$$|{}_{\varphi}D_{\delta}^{\kappa}\mathfrak{v}(\gamma) - \mathfrak{f}(\gamma, \mathfrak{v}(\gamma))| < \varepsilon, \quad \gamma \in J \tag{8}$$

there exists a unique solution of (1) such that:

$$|\mathfrak{u}(\gamma) - \mathfrak{v}(\gamma)| < \phi_f(\varepsilon), \quad \gamma \in J.$$

Theorem 6. *Under the hypothesis $\mathscr{H}_3$, the problem (1) is Ulam-Hyers stable.*

Proof. Let $\mathfrak{v} \in \mathfrak{C}$ be a solution of the inequality (7) and $\mathfrak{u}$ the unique solution of (1) with $\mathfrak{u}(\delta) = \mathfrak{v}(\delta)$.

$$\begin{aligned}
|\mathfrak{u}(\gamma) - \mathfrak{v}(\gamma)| &= \left| \frac{1}{\Gamma(\kappa)} \int_{\delta}^{\gamma} (\gamma - s)^{\kappa - 1} \varphi(s) \left(\mathfrak{f}(s, \mathfrak{u}(s)) - \mathfrak{f}(s, \mathfrak{v}(s)) \right) \right| \\
&\quad + \left| \frac{1}{\Gamma(\kappa)} \int_{\delta}^{\gamma} (\gamma - s)^{\kappa - 1} \mathfrak{h}(s)\, ds \right| \\
&\leq \frac{1}{\Gamma(\kappa)} \int_{\delta}^{\gamma} (\gamma - s)^{\kappa - 1} |\varphi(s)| \, |\mathfrak{f}(s, \mathfrak{u}(s)) - \mathfrak{f}(s, \mathfrak{v}(s))| \, ds \\
&\quad + \frac{1}{\Gamma(\kappa)} \int_{\delta}^{\gamma} (\gamma - s)^{\kappa - 1} |\mathfrak{h}(s)| \, ds \\
&\leq \frac{\|\varphi\|_{\infty}}{\Gamma(\kappa)} \int_{\delta}^{\gamma} (\gamma - s)^{\kappa - 1} \psi \left(|\mathfrak{u}(s) - \mathfrak{v}(s)| \right) ds + \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)} \\
&\leq \frac{\|\varphi\|_{\infty}}{\Gamma(\kappa)} \int_{\delta}^{\gamma} (\gamma - s)^{\kappa - 1} |\mathfrak{u}(s) - \mathfrak{v}(s)| \, ds + \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)}.
\end{aligned}$$

Using Lemma 3, we have:

$$\begin{aligned}
|\mathfrak{u}(\gamma) - \mathfrak{v}(\gamma)| &\leq \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)} + \int_{\delta}^{\gamma} \sum_{n=1}^{+\infty} \frac{\|\varphi\|_{\infty}^{n}}{\Gamma(n\kappa)} (\gamma - s)^{n\kappa - 1} \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)} ds \\
&\leq \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)} \left[1 + \int_{\delta}^{\gamma} \sum_{n=1}^{+\infty} \frac{\|\varphi\|_{\infty}^{n}}{\Gamma(n\kappa)} (\gamma - s)^{n\kappa - 1} ds \right] \\
&\leq \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)} \left[1 + \sum_{n=1}^{+\infty} \frac{\|\varphi\|_{\infty}^{n}}{\Gamma(n\kappa)} \frac{(\beta - \delta)^{n\kappa}}{n\kappa} \right] \\
&\leq \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)} \left[1 + \sum_{n=1}^{+\infty} \frac{\|\varphi\|_{\infty}^{n}}{\Gamma(n\kappa + 1)} (\beta - \delta)^{n\kappa} \right] \\
&\leq \varepsilon \frac{(\beta - \delta)^{\kappa}}{\Gamma(\kappa + 1)} E_{\kappa} \left(\|\varphi\|_{\infty} (\beta - \delta)^{\kappa} \right) := c\varepsilon.
\end{aligned}$$

The Mittag-Leffler function is given by:

$$E_{\kappa}(z) = \sum_{k=0}^{+\infty} \frac{z^{k}}{\Gamma(n\kappa + 1)}.$$

This completes the proof of the theorem. Moreover, if we set $\Psi(c) = c\varepsilon$, we have $\Psi(0) = 0$, and the result shows that the problem is generalized Ulam-Hyers stable.

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6 Examples and applications

In this section, we investigate the following three problems:

1.The problem described in (9) below, where $\varphi(\gamma)$ is one of the following functions:

$$\varphi(\gamma)=\frac{1}{1+\gamma^2},\quad \varphi(\gamma)=\frac{-1}{1+\gamma},\quad \text{or}\quad \varphi(\gamma)=e^{-\gamma}.$$

2.The problem outlined in (10) below, where $\varphi_1(\gamma)=\frac{5}{1+\ln(1+\gamma)}$, $\varphi_2(\gamma)=\frac{-7}{1+\gamma\sqrt{\gamma}}$, or $\varphi_3(\gamma)=8e^{-\gamma}$.

3.The problem specified in (11) below, with $\varphi(t)=\frac{1}{\log(e+t^2)}$ or $\varphi(t)=e^{t-1}$.

Example 1:

We now consider the following fractional problem:

$$\begin{cases} {}_{\varphi}D_0^{\frac{1}{2}}\mu(\gamma)=\frac{|\mu(\gamma)|}{\gamma+4+|\mu(\gamma)|}+\frac{\ln(\gamma^2+1)}{6+\gamma}, & \gamma\in[0,2],\\ \mu(0)=0. \end{cases} \tag{9}$$

Here, we set $\delta=0$, $\beta=2$, and $\kappa=\frac{1}{2}$, and define the function $\mathfrak{h}(\gamma,\mu)$ as:

$$\mathfrak{h}(\gamma,\mu)=\frac{|\mu|}{\gamma+4+|\mu|}+\frac{\ln(\gamma^2+1)}{6+\gamma},\quad (\gamma,\mu)\in[0,2]\times\mathbb{R}.$$

We verify the assumptions $\mathscr{H}_1$) and $\mathscr{H}_2$), with the values $\Gamma(\kappa+1)=\frac{\sqrt{\pi}}{2}$, $A=\frac{1}{4}$, and $B=\frac{\ln(5)}{8}$.

Next, for the equation $P(x)=0$ associated with the problem (9), we find that $\Delta=0.46>0$, $x_1=0$, and $x_2=1.7$. Using Lemma 2, the set of solutions to the inequality $P(x)\leq 0$ (i.e., the solution to (4)) is $[x_1,x_2]$.

Moreover, if we take φ to be one of the following functions, for example:

$$\varphi(x)=\frac{1}{1+x^2},\quad \varphi(x)=\frac{-1}{1+x},\quad \text{or}\quad \varphi(x)=e^{-x},$$

we observe that $\|\varphi\|_\infty=1$, which lies within the interval $[x_1,x_2]$.

Therefore, by applying Theorem 4, we conclude that the problem (9) has at least one bounded solution μ defined on the interval $[0,2]$, with $\|\mu\|_\infty\leq 1$.

Example 2:

We consider the following fractional problem:

$$\begin{cases} {}_{\varphi}D_0^{\kappa}\mu(\gamma)=\frac{\sqrt{\gamma+1}\sin(\mu(\gamma))}{\exp(2+\gamma)+(6+\gamma)^4+19}+\frac{\cos(\gamma)\mu(\gamma)}{20\exp(\gamma^3+2\gamma+1)+\gamma+50}\\ \quad+\frac{\sin(\gamma^2+\gamma-1)}{\ln(2\gamma^2+5\gamma+13)+7}, \quad \gamma\in[0,5],\\ \mu(0)=1. \end{cases} \tag{10}$$

Let

$$\mathfrak{h}(\gamma,\mu)=\frac{\sqrt{\gamma+1}\sin(\mu)}{\exp(2+\gamma)+(6+\gamma)^4+19}+\frac{\cos(\gamma)\mu}{20\exp(\gamma^3+2\gamma+1)+\gamma+50}+\frac{\sin(\gamma^2+\gamma-1)}{\ln(2\gamma^2+5\gamma+13)+7}.$$

For any $(\gamma,\mu)\in[0,5]\times\mathbb{R}$, we have:

$$\begin{aligned} |\mathfrak{h}(\gamma,\mu)| &\leq |\mu|\left[\frac{\sqrt{\gamma+1}}{\exp(2+\gamma)+(6+\gamma)^4+19}+\frac{1}{20\exp(\gamma^3+2\gamma+1)+\gamma+50}\right]\\ &\quad+\frac{|\sin(\gamma^2+\gamma-1)|}{\ln(2\gamma^2+5\gamma+13)+7}\\ &\leq A|\mu|+B, \end{aligned}$$

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where

$$A = \sup_{\gamma\in[0,5]}\left[\frac{\sqrt{\gamma+1}}{\exp(2+\gamma)+(6+\gamma)^4+19}+\frac{1}{20\exp(\gamma^3+2\gamma+1)+\gamma+50}\right] = \frac{1}{e^2+1315}+\frac{1}{20e+50},$$

and

$$B = \sup_{\gamma\in[0,5]}\left[\frac{1}{\ln(2\gamma^2+5\gamma+13)+7}\right] = \frac{1}{\ln(13)+7}.$$

Table 1: Numerical results of Δ, x_1, x_2 at various values of $\kappa \in (0,1)$ for Example 2.

κ	Δ	x_1	x_2
0.1	0.7074	1.1682	67.0636
0.2	0.6483	1.2135	53.0452
0.3	0.5834	1.2698	42.1840
0.4	0.5130	1.3406	33.6313
0.5	0.4377	1.4308	26.7939
0.6	0.3582	1.5489	21.2449
0.7	0.2756	1.7093	16.6667
0.8	0.1910	1.9413	12.8067
0.9	0.1056	2.3193	9.4232

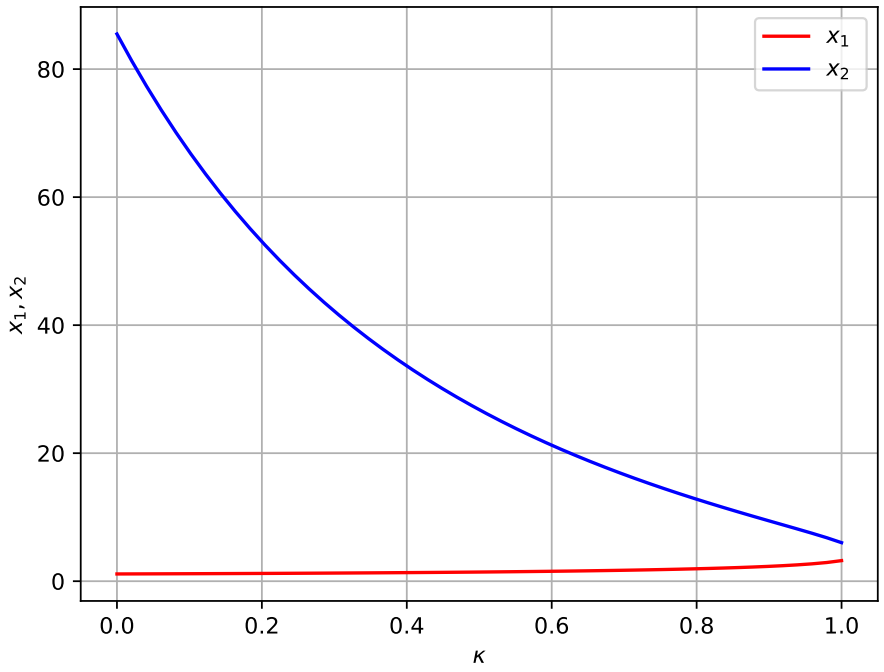


Fig. 1: Graphical representation of x_1 and x_2, when $\kappa \in (0,1)$, for Example 2.

Figure 1 and Table 1 show the numerical results and graphical representation of x_1 and x_2, the roots of $P(x)$ associated with the problem (10). We have $\Delta > 0$, and $0 \le x_1 < x_2$ for any $\kappa \in (0,1)$. Using Lemma 2, the set of solutions of $P(x) \le 0$ (i.e., the solution to (4)) is $[x_1, x_2]$.

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Now, let φ be one of the following functions, for example:

$$\varphi_1(\gamma) = \frac{5}{1+\ln(1+\gamma)}, \quad \varphi_2(\gamma) = \frac{-7}{1+\gamma\sqrt{\gamma}}, \quad \varphi_3(\gamma) = 8e^{-\gamma}.$$

We have $\|\varphi_1\|_\infty = 5$, $\|\varphi_2\|_\infty = 7$, $\|\varphi_3\|_\infty = 8$, and since $\{5,7,8\} \subset [x_1, x_2]$, by Theorem 4, the problem (10) has at least one solution.

Example 3:

We consider the following fractional problem:

$$\begin{cases} {}_{\varphi}D_{-1}^{\frac{1}{7}}\mu(\gamma) = \frac{\ln(2+\sin(\mu(\gamma)))}{5+\gamma} + \frac{\cos(\gamma)|\mu(\gamma)|}{2+\gamma} + \frac{\ln(\gamma^2+1)}{e^{\gamma}(2+\gamma)}, \quad \gamma \in [-1,1], \\ \mu(0) = 1, \end{cases} \tag{11}$$

In this example, $\delta = -1$ and $\beta = 1$.

Let

$$\mathfrak{h}(\gamma,\mu) = \frac{\ln(2+\sin(\mu))}{5+\gamma} + \frac{\cos(\gamma)|\mu|}{3+\gamma} + \frac{\ln(\gamma^2+1)}{e^{\gamma}(2+\gamma)}, \quad (\gamma,\mu) \in [-1,1]\times\mathbb{R}.$$

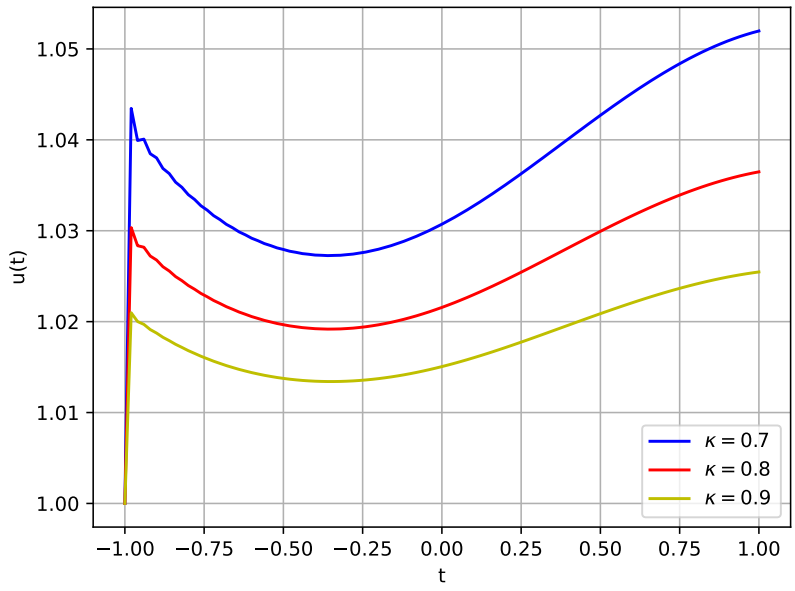


Fig. 2: Exposed Illustration of the Solution for Different Values of κ for $\varphi(t) = e^{t-1}$

For all $\gamma \in [-1,1]$ and $\mu, \vartheta \in \mathbb{R}$, we can prove:

$$\begin{aligned} |\mathfrak{h}(\gamma,\mu) - \mathfrak{h}(\gamma,\vartheta)| &\leq \frac{1}{4}|\ln(2+\sin(\mu)) - \ln(2+\sin(\vartheta))| + \frac{|\mu-\vartheta|}{2} \\ &\leq \frac{1}{4}|\sin(\mu) - \sin(\vartheta)| + \frac{|\mu-\vartheta|}{2} \\ &\leq \psi(|\mu-\vartheta|) \end{aligned}$$

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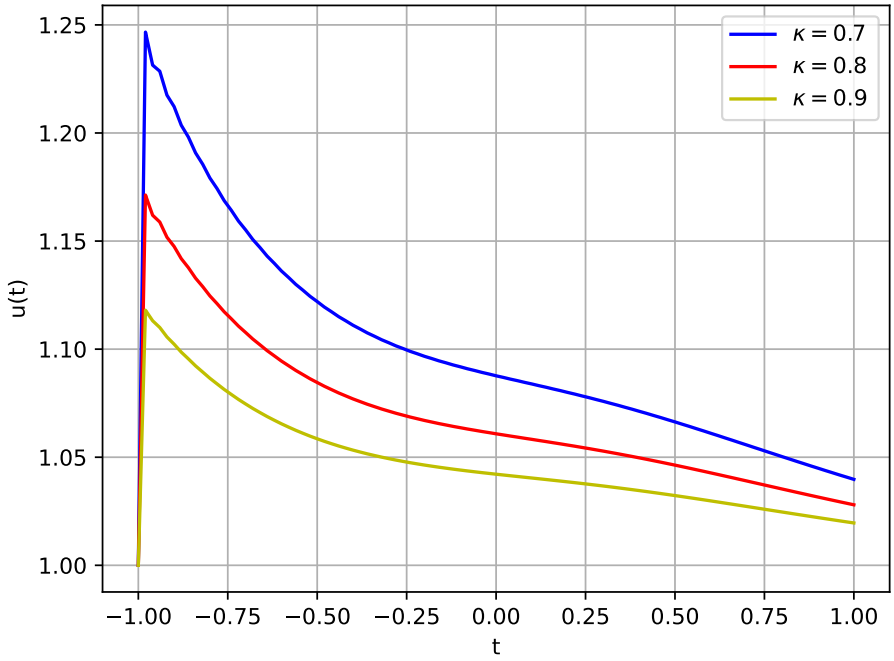


Fig. 3: Exposed Illustration of the Solution for Different Values of κ for $\varphi(t) = \frac{1}{\log(e+t^2)}$

where $\psi(r) = \frac{3}{4}r$ for any $r > 0$. The function ψ is continuous and non-decreasing, and satisfies the condition

$$\psi(r)\frac{(\beta-\delta)^{\kappa}}{\Gamma(\kappa+1)} = \frac{3}{4\Gamma(8/7)}r < r \quad \text{for all } r > 0.$$

Thus, the hypothesis $\mathscr{H}_3$) is satisfied. Consequently, by Theorem 5, the nonlinear fractional problem (11) admits a unique solution on the interval $[-1,1]$. Furthermore, applying Theorem 6, we conclude that the solution of (1) is Ulam-Hyers stable.

Additionally, Figures 2 and 3 display the numerical solutions for different values of κ, with $\varphi(t) = e^{t-1}$ and $\varphi(t) = \frac{1}{\log(e+t^2)}$, respectively. These computations were carried out using the Adams-Bashforth-Moulton method, as described in [5,6]. Theoretically, we have $\|\varphi_1\| = \|\varphi_2\| = 1$, which confirms the existence of solutions in both cases. However, Figures 2 and 3 show the differences between the solutions.

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