Some Mathematical Analytic Arguments for Determining Valid Optimal Lot Size for Deteriorating Items with Limited Storage Capacity under Permissible Delay in Payments

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Abstract: In the year 2011, Y. Liang and F. Zhou presented an inventory model with two levels of storages, in which one has finite dimension and the other has infinite dimension, and with conditionally permissible delay in payments. In essence, it concentrated on the establishment of the inventory model, but did not concentrate on the validity of the processes of finding the optimal solution from the viewpoint of logic. In addition, it ignored whether the case of the trade credit period, $M$, is greater than the time interval and whether the order quantity is greater than $W$ units or not, so the discussion of the optimal solution is questionable. The main purpose of this paper is to characterize the optimal solutions in accordance with the functional behavior of the total average cost under different circumstances, not only to overcome the shortcomings in the aforementioned work of Y. Liang and F. Zhou, but also to obtain accurate and reliable solution procedures. Finally, numerical examples are given to illustrate the theoretical results and the sensitivity analysis of the optimal solution with respect to the parameters of the system is carried out to reveal the exact results.

Keywords: Inventory modelling; Economic order quantity; Deteriorating items; Permissible delay in payments; Limited storage capacity; Mathematical solution procedures; Optimal solution; Sensitivity analysis.

1 Introduction

In modern business transactions, allowing a grace period for settling the amount owed is becoming ubiquitous. Usually, there is no charge if the outstanding amount is settled within the permitted fixed settlement period; beyond that the permitted fixed settlement period, interest is charged. Conversely, the retailer can sell items, accumulate revenues and finally earn interest during the permissible delay period. Furthermore, the main purpose of the permissible delay period is to encourage the retailer to buy more, to increase market share or to deplete inventories of certain items. Recently, a lot of articles consider inventory models for deteriorating items with permissible delay in payments. For example, [1] developed an EOQ model for deteriorating items under supplier credits linked to ordering quantity. [2], [3], [4], [5] provided lot-sizing decisions under permissible delay in payments depending on the ordering quantity in different circumstances. [6] established inventory ordering policies of delayed deteriorating items under permissible delay in payments. [7] established an EPQ

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model for deteriorating items with up-stream full trade credit and down-stream partial trade credit. [8] revealed the retailers optimal ordering policy for deteriorating items with maximum lifetime under suppliers trade credit financing. Many additional related developments can be found in [9,10,11,12,13,14,15,16,17,18,19,20,21,22] and [23], and in the references cited therein.

On the other hand, in many practical situations, when an attractive price discount for bulk purchase is available or the items are seasonal or the cost of procuring items is higher than other inventory related cost or demand of the items is very high, these items cannot be accommodated in the existing store facility located at busy market places. In this regard, for storing the excess items, one additional storage facility is hired on rental basis, which may be located a little away from it. Furthermore, a rented warehouse (RW) is used to store the excess units over the fixed capacity on the own warehouse (OW) and the retailer is served first RW, then from OW. [24] incorporated the concepts of the basic two-warehouse inventory model and the conditions of permissible delay in payments in order to generalize the earlier work [25].

Recently, [26] and [27] explored an inventory model for deteriorating items with two warehouses in which the deterioration rates of items in OW and RW are different. [28] studied lot-sizing decisions for deteriorating items with capacity constraints under an order-size-dependent trade credit. [29] studied a two-warehouse inventory model for deteriorating items and stock dependent demand under conditionally permissible delay in payment in imprecise environment. [30] studied a two-warehouse inventory model for deteriorating items with linear increasing demand under conditionally permissible delay in payment. [31] explored the optimal strategy of deteriorating items with capacity constraints under two-levels of trade credit policy. Many additional related investigations can be found in [32,33,34,35,36,37,38] and [39], and also in the references cited therein.

This paper explores an inventory model under the situation considered in the work of Liang and Zhou [40], who explored a two-warehouse inventory model for deterioration items under conditionally permissible delay in payment and its object was seen to find \((T^* , t_w^*)\) such that \(TC(T^*,t_w^*)\) is the minimum value. However, it is found that \(t_w^*\) is a function of \(T\) which results in observation that the total average cost \(TC\) is established by a decision variable, \(T\), not by both \(T\) and \(t_w^*\). In addition, Liang and Zhou [40] ignored whether \(M\) is greater than the time interval and that the order quantity is greater than \(W\) units or not, so this paper characterizes the validity of the optimal solutions in accordance with the functional behavior of the total average cost under different circumstances in order to overcome the shortcomings in the work of Liang and Zhou [40]. It also presents simple and easy-to-understand solution procedures. Finally, numerical examples to illustrate the theoretical results are presented and the sensitivity analysis with respect to the parameters of the system is performed.

2 Mathematical Formulation

The notation and assumptions in this paper are the same as that of [40] except for the last item. In addition, Liang and Zhou [40] focused on the assumption that the order quantity is always greater than the capacity of the owned warehouse, so this article denote by \(T_a\) the time interval in which the order quantity is greater than \(W\) units so that the inequality \(Q \geq W\) holds true if and only if \(T \geq T_a\).

On the other hand, because of the continuity of \(I_0(t)\) at time \(t_w\), Liang and Zhou [40] revealed that

\[
W \cdot e^{-a t_w} = \frac{D}{\alpha} \left( e^{\alpha(T-t_w)} - 1 \right).
\]

From Eq. (1), it is obvious that

\[
t_w = \frac{1}{\alpha} \ln \left( \frac{De^{\alpha T} - \alpha W}{D} \right). \tag{2}
\]

Thus, it can be easily seen that \(t_w\) is a function of \(T\), and that the total average cost, \(TC\), is a function of \(T\), not \(T\) and \(t_w\). However, Liang and Zhou [40] supposed that the total average cost was a function of \(T\) and \(t_w\), so their solution procedures are questionable and their numerical examples are incorrect. Consequently, this article will adopt the calculus approach not only to overcome shortcomings [40], but also to develop the complete solution procedures for it.

Firstly, we must investigate and discuss whether or not the time \(t_w\) in which inventory level reduces to \(W\) is greater than \(M\), so the inequality \(M \geq t_w\) holds true if and only if

\[
\frac{1}{\alpha} \ln \left( \frac{De^{\alpha T} - \alpha W}{D} \right) \leq T.
\]

For notational convenience, let

\[
M^* = \frac{1}{\alpha} \ln \left( \frac{De^{\alpha M} - \alpha W}{D} \right).
\]

Then the inequality \(M \geq t_w\) holds true if and only if \(M^* \geq T\). Secondly, in the case when the inequality \(T_a \leq t_w\) holds true if and only if

\[
\frac{1}{\alpha} \ln \left( \frac{De^{\alpha T_a} - \alpha W}{D} \right) \leq T,
\]

we let

\[
T_a^* = \frac{1}{\alpha} \ln \left( \frac{De^{\alpha T_a} - \alpha W}{D} \right).
\]

Then the inequality \(T_a \leq t_w\) holds true if and only if \(T_a^* \leq T\). Afterwards, the total average cost can be divided
into the following three cases:

**Case (1):** \( T^*_a < M < M^* \);

**Case (2):** \( M < T^*_a < M^* \);

and

**Case (3):** \( M < M^* < T^*_a \).

We remark in passing that the circumstances of Cases (2) and (3) were not discussed by Liang and Zhou [40].

**Case 1.** Suppose that \( T^*_a < M < M^* \). Under this circumstance, Liang and Zhou [40] revealed that the total relevant cost \( TC(T) \) is given by

\[
TC(T) = \begin{cases} 
TC_1(T) & \text{if } M^* < T \\
TC_2(T) & \text{if } M < T \leq M^* \\
TC_3(T) & \text{if } T^*_a < T \leq M, 
\end{cases}
\]

where

\[
TC_1(T) = \frac{A}{T} + \frac{D}{T^2} \left( \phi^{b_0} - \beta \nu - 1 \right) + \frac{W}{T} \left( \phi^{b_0} - \beta \nu - 1 \right) - \frac{p_D c M^2}{2T^2} + \frac{D}{a^2 T} \left( \phi^{b_0} - \beta \nu - 1 \right),
\]

and

\[
TC_2(T) = \frac{A}{T} + \frac{D}{T^2} \left( \phi^{b_0} - \beta \nu - 1 \right) + \frac{W}{T} \left( \phi^{b_0} - \beta \nu - 1 \right) - \frac{p_D c M^2}{2T^2} + \frac{D}{a^2 T} \left( \phi^{b_0} - \beta \nu - 1 \right).
\]

Substituting Eq. (1) into Eqs. (4), (6) and (7), we obtain the total relevant cost function as follows:

\[
TC_1(T) = \frac{A}{T} + \frac{(h_0 + c \alpha)}{a^2 T} \left( W - D(T - \tau) \right) + \frac{D}{T^2} \left( \phi^{b_0} - \beta \nu - 1 \right) - \frac{p_D c M^2}{2T^2} + \frac{D}{a^2 T} \left( \phi^{b_0} - \beta \nu - 1 \right) + \frac{1}{a} \left( W - a \phi^{b_0} - D(T - \tau) \right) \tag{8}
\]

\[
TC_2(T) = \frac{A}{T} + \frac{(h_0 + c \alpha)}{a^2 T} \left( W - D(T - \tau) \right) + \frac{D}{T^2} \left( \phi^{b_0} - \beta \nu - 1 \right) - \frac{p_D c M^2}{2T^2} + \frac{D}{a^2 T} \left( \phi^{b_0} - \beta \nu - 1 \right) \tag{9}
\]

and

\[
TC_3(T) = \frac{A}{T} + \frac{(h_0 + c \alpha)}{a^2 T} \left( W - D(T - \tau) \right) + \frac{D}{T^2} \left( \phi^{b_0} - \beta \nu - 1 \right) - \frac{p_D c M^2}{2T^2} + \frac{D}{a^2 T} \left( \phi^{b_0} - \beta \nu - 1 \right) \tag{10}
\]

For convenience, we treat all \( TC_i(T) \) \( (i = 1, 2, 3) \) defined on \( T > 0 \). Eqs. (8), (9) and (10) yield

\[
TC_1(M^*) = TC_2(M^*) \quad \text{and} \quad TC_2(M) = TC_3(M).
\]

So, clearly, the function \( TC(T) \) is continuous and well-defined on \( T \geq T^*_a \).

**Case 2.** Suppose that \( M < T^*_a < M^* \). Under this circumstance, we see that the total relevant cost \( TC(T) \) is given by:

\[
TC(T) = \begin{cases} 
TC_1(T) & \text{if } M^* < T \\
TC_2(T) & \text{if } T^*_a < T \leq M^*. 
\end{cases}
\]

Since \( TC_1(M^*) = TC_2(M^*) \), \( TC(T) \) is continuous and well-defined on \( T \geq T^*_a \).

**Case 3.** Suppose that \( M < M^* < T^*_a \). Under this circumstance, we find that the total relevant cost \( TC(T) \) is given by

\[
TC(T) = TC_1(T). \tag{12}
\]

Clearly, therefore, the function \( TC(T) \) is continuous and well-defined on \( T \geq T^*_a \) as well.

3 **Theoretical Results and Optimal Solutions**

First of all, Eqs. (8) to (10) yield

\[
TC_1(T) = \frac{1}{T^2} \left\{ -A + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} - W \right) + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} + \beta \nu + \beta_\nu + 1 \right) \right. \\
+ \frac{1}{a} \left( W - a \phi^{b_0} - D_{\nu_0} - DT \frac{dt_0}{dT} \right) \left. + \frac{p_D c M^2}{2a^2 T^2} \right\} \tag{13}
\]

and

\[
TC_2(T) = \frac{1}{T^2} \left\{ -A + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} - W \right) + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} + \beta \nu + \beta_\nu + 1 \right) \right. \\
+ \frac{1}{a} \left( W - a \phi^{b_0} - D_{\nu_0} - DT \frac{dt_0}{dT} \right) \left. + \frac{p_D c M^2}{2a^2 T^2} \right\} \tag{14}
\]

and

\[
TC_3(T) = \frac{1}{T^2} \left\{ -A + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} - W \right) + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} + \beta \nu + \beta_\nu + 1 \right) \right. \\
+ \frac{1}{a} \left( W - a \phi^{b_0} - D_{\nu_0} - DT \frac{dt_0}{dT} \right) \left. + \frac{p_D c M^2}{2a^2 T^2} \right\} \tag{15}
\]

Next, we let

\[
f_i(T) = -A + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} - W \right) + \frac{(h_0 + c \alpha)}{a^2 T} \left( DT \frac{dt_0}{dT} - D_{\nu_0} + \beta \nu + \beta_\nu + 1 \right) \\
+ \frac{1}{a} \left( W - a \phi^{b_0} - D_{\nu_0} - DT \frac{dt_0}{dT} \right) + \frac{p_D c M^2}{2a^2 T^2} \tag{16}
\]
$f_5(T) = -A + \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

and

$\alpha_{3} = \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

Thus, clearly, we have

$\alpha_{1} = \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

and

$\alpha_{2} = \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

Thus, we have

$\alpha_{1} = \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

and

$\alpha_{2} = \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

Thus, clearly, we have

$\alpha_{1} = \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

and

$\alpha_{2} = \frac{(b_0 + c x_{i})}{a} \left( DT \frac{d x_{i}}{d T} - D x_{i} - W \right) + \frac{(b_0 + c \beta + D)}{\beta} \left( DT \frac{d x_{i}}{d T} - D x_{i} - e^{b_0 + \beta} + 1 \right) + c T \frac{D}{x_{i}} \left( \alpha T e^{\alpha (T - M)} - e^{\alpha (T - M)} - \alpha M + 1 \right) + \frac{p e_D M^2}{2}$  

We also have

$\beta \left( \frac{d x_{i}}{d T} \right)^2 + \left( \frac{d^2 x_{i}}{d T^2} \right)^2 > 0$  

Therefore, we find that $f'_i(T) > 0$ and $f_i(T)$ (i = 1, 2, 3) is increasing on $T \geq 0$, respectively.

Finally, we consider the case when

$f_1(T) = 0$,  

$f_2(T) = 0$  

and

$f_3(T) = 0$.  

Let $T_1^*$, $T_2^*$ and $T_3^*$ denote the roots of Eqs. (23), (24) and (25), respectively, if their roots exist. Furthermore, Eqs. (16) to (18) reveal the fact that

$\Delta_1 = f_1(M^*)$,  

$\Delta_2 = f_2(M^*)$,  

$\Delta_3 = f_3(T_a^*)$.  

Most strikingly, when $T = M^*$, we obtain $t_w = M$, so we have

$f_1(M^*) = f_2(M^*)$,  

that is,

$\Delta_1 = \Delta_2 = \Delta_3$.  

Eqs. (26) to (29) imply that

$\Delta_1 > \Delta_2 > \Delta_3$.  

We recall that $T_{C_i}^*(T)$ (i = 1, 2, 3) is increasing on $T \geq 0$. From

$\lim_{T \to \infty} f_i(T) = \infty$  

(i = 1, 2, 3)

and the Intermediate Value Theorem (see, for example, [41] and [42]), we have the following results.

**Theorem 1.** Suppose that $T_a^* < M < M^*$. Then

1. If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$, then $T^* = T_a^*$ and $T(C(T^*)) = T(C(T_a^*))$.
2. If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 \leq 0$, then $T^* = T_3^*$ and $T(C(T^*)) = T(C(T_3^*))$.
3. If $\Delta_1 > 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$, then $T^* = T_2^*$ and $T(C(T^*)) = T(C(T_2^*))$.
4. If $\Delta_1 \leq 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$, then $T^* = T_1^*$ and $T(C(T^*)) = T(C(T_1^*))$.

**Proof.** We consider the following cases:

1. If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$, then

$\frac{d T C_1(M^*)}{d T} > 0$, \quad \frac{d T C_2(M^*)}{d T} > 0$  

and

$\frac{d T C_3(M^*)}{d T} > 0$.

Hence, we get $M^* > T_1^*$, $M > T_2^*$ and $T_a^* > T_1^*$. Additionally, we have

(a) $T C_1(T)$ is increasing on $M^*, \infty$.
(b) $T C_2(T)$ is increasing on $M, M^*$.
(c) $T C_3(T)$ is increasing on $T_a^*, M$.

Combining (a) to (c), we conclude that $T(C(T)$ is increasing on $T_a^*, \infty$). So, we have

$T^* = T_a^*$ and $T(C(T^*)) = T(C(T_a^*))$.  

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(2) If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 \leq 0$, then

\[
\frac{dT C_1 (M^*)}{d T} = \frac{dT C_2 (M^*)}{d T} > 0,
\]

\[
\frac{dT C_2 (M)}{d T} = \frac{dT C_3 (M)}{d T} > 0
\]

and

\[
\frac{dT C_3 (T^*_a)}{d T} \leq 0.
\]

Hence, we see that $M^* > T^*_1$, $M > T^*_2$ and $T^*_3 \leq T^*_3 < M$. Additionally, we have

(a) $T C_1 (T)$ is increasing on $[M^*, \infty)$. 
(b) $T C_2 (T)$ is increasing on $[M, M^*]$. 
(c) $T C_3 (T)$ is decreasing on $[T^*_a, T^*_3]$ and increasing on $[T^*_3, M]$. 

Combining (a) to (c), we conclude that $T C (T)$ is decreasing on $[T^*_3, T^*_3]$ and increasing on $[T^*_3, \infty)$. So, we have

\[
T^*_3 = T^*_3 \quad \text{and} \quad T C (T^*_3) = T C (T^*_3).
\]

(3) If $\Delta_1 > 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$, then

\[
\frac{dT C_1 (M^*)}{d T} = \frac{dT C_2 (M^*)}{d T} > 0,
\]

\[
\frac{dT C_2 (M)}{d T} = \frac{dT C_3 (M)}{d T} \leq 0
\]

and

\[
\frac{dT C_3 (T^*_a)}{d T} \leq 0.
\]

Hence, we observe that $M^* > T^*_1$, $M \leq T^*_2 \leq M^*$ and $T^*_3 > M$. Additionally, we have

(a) $T C_1 (T)$ is increasing on $[M^*, \infty)$. 
(b) $T C_2 (T)$ is decreasing on $[M, M^*]$ and increasing on $[T^*_2, M^*]$. 
(c) $T C_3 (T)$ is decreasing on $[T^*_a, M]$. 

Combining (a) to (c), we conclude that $T C (T)$ is decreasing on $[T^*_1, T^*_2]$ and increasing on $[T^*_2, \infty)$. So, we have

\[
T^*_3 = T^*_2 \quad \text{and} \quad T C (T^*_3) = T C (T^*_2).
\]

(4) If $\Delta_1 \leq 0$, $\Delta_2 \leq 0$ and $\Delta_3 \leq 0$, then

\[
\frac{dT C_1 (M^*)}{d T} = \frac{dT C_2 (M^*)}{d T} \leq 0,
\]

\[
\frac{dT C_2 (M)}{d T} = \frac{dT C_3 (M)}{d T} \leq 0
\]

and

\[
\frac{dT C_3 (T^*_a)}{d T} \leq 0.
\]

Hence, we see that $M^* \leq T^*_1$, $M \leq T^*_2$ and $T^*_3 > M$. Additionally, we have

(a) $T C_1 (T)$ is decreasing on $[M^*, T^*_1]$ and increasing on $[T^*_1, \infty)$. 
(b) $T C_2 (T)$ is decreasing on $[M^*, M^*]$. 
(c) $T C_3 (T)$ is decreasing on $[T^*_a, M]$. 

Combining (a) to (c), we conclude that $T C (T)$ is decreasing on $[T^*_1, T^*_1]$ and increasing on $[T^*_1, \infty)$. So, $T^*_3 = T^*_1$ and $T C (T^*_3) = T C (T^*_1)$. 

We thus have completed the proof of Theorem 1.

The above arguments are to reveal the exploration of functional behaviors of $T C_1 (T)$, $T C_2 (T)$ and $T C_3 (T)$ on intervals $[M^*, \infty)$, $[M, M^*]$ and $[T^*_a, M]$, respectively, to jointly decide whether or not $T^*_1$, $T^*_2$, $T^*_3$ or $T^*_3$ is the optimal solution $T^*$ of $T C (T)$ on the whole domain $T \geq T^*_a$.

### 4 Further Theoretical Results and Optimal Solutions

The inventory models of the following cases:

**Case (2):** $M < T^*_a < M^*$

**Case (3):** $M < M^* < T^*_a$

were ignored by Liang and Zhou [40], so we will present complete solution procedures (which are missing in [40]) in this section.

In the case when $M < T^*_a < M^*$, Eq. (14) yields

\[
T C^*_2 (T^*_a) = \frac{1}{T^*_a^2} \cdot f_2 (T^*_a).
\] (30)

Similarly, if we let

\[
\Delta_4 = f_2 (T^*_a),
\] (31)

then we are led to the fact that $\Delta_1 > \Delta_4$. As implied above, we have the following results.

**Theorem 2.** Suppose that $M < T^*_a < M^*$. Then

(1) If $\Delta_1 > 0$, $\Delta_4 > 0$, then $T^*_3 = T^*_a$ and $T C (T^*_3) = T C (T^*_a)$.

(2) If $\Delta_1 > 0$, $\Delta_4 \leq 0$, then $T^*_3 = T^*_2$ and $T C (T^*_3) = T C (T^*_2)$.

(3) If $\Delta_1 \leq 0$, $\Delta_4 \leq 0$, then $T^*_3 = T^*_1$ and $T C (T^*_3) = T C (T^*_1)$.

**Proof.** We consider the following cases:

(1) If $\Delta_1 > 0$ and $\Delta_4 > 0$, then

\[
\frac{dT C_1 (M^*)}{d T} = \frac{dT C_2 (M^*)}{d T} > 0
\]

and

\[
\frac{dT C_2 (T^*_a)}{d T} > 0.
\]

Hence, we see that $M^* > T^*_1$ and $T^*_a > T^*_1$. Furthermore, we have

(a) $T C_1 (T)$ is increasing on $[M^*, \infty)$. 
(b) We have completed the proof of Theorem 1. 
(c) $T C_3 (T)$ is decreasing on $[M, M^*]$.

We thus have completed the proof of Theorem 1.
(b) $TC_2(T)$ is increasing on $[T^*_a, M^*]$. Combining (a) and (b), we conclude that $TC(T)$ is increasing on $[T^*_a, \infty)$. So, we have

$$
T^* = T^*_a \quad \text{and} \quad TC(T^*) = TC(T^*_a).
$$

(2) If $\Delta_1 > 0$ and $\Delta_4 \leq 0$, then

$$
\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} > 0
$$

and

$$
\frac{dTC_2(T^*_a)}{dT} \leq 0.
$$

Hence, we see that $M^* > T^*_a$ and $T^*_a \leq T^*_a \leq M^*$. Furthermore, we have

(a) $TC_1(T)$ is increasing on $[M^*, \infty)$.

(b) $TC_2(T)$ is decreasing on $[T^*_a, T^*_a]$ and increasing on $[T^*_a, M^*]$. Combining (a) and (b), we conclude that $TC(T)$ is decreasing on $[T^*_a, T^*_a]$ and increasing on $[T^*_a, \infty)$. So, $T^* = T^*_a$ and $TC(T^*) = TC(T^*_a)$.

(3) If $\Delta_1 \leq 0$ and $\Delta_4 \leq 0$, then

$$
\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} \leq 0
$$

and

$$
\frac{dTC_2(T^*_a)}{dT} \leq 0.
$$

Hence, we see that $M^* \geq T^*_a$ and $T^*_a \leq T^*_a \leq M^*$. Furthermore, we have

(a) $TC_1(T)$ is decreasing on $[M^*, T^*_a]$ and increasing on $[T^*_a, \infty)$.

(b) $TC_2(T)$ is decreasing on $[T^*_a, M^*]$. Combining (a) and (b), we conclude that $TC(T)$ is decreasing on $[T^*_a, T^*_a]$ and increasing on $[T^*_a, \infty)$.

We have thus completed the proof of Theorem 2.

Finally, in the case when $M < M^* < T^*_a$, we find that

$$
TC(T) = TC_1(T). \quad \text{Eq. (13) yields}
$$

$$
TC_1(T^*_a) = \frac{1}{T^*_a^2} f_1(T^*_a)
$$

(32)

Similarly, if we let $\Delta_5 = f_1(T^*_a)$, then we have the following results.

**Theorem 3.** Suppose now that $M < M^* < T^*_a$. Then

(1) If $\Delta_5 > 0$, then $T^* = T^*_a$ and $TC(T^*) = TC(T^*_a)$.

(2) If $\Delta_5 \leq 0$, then $T^* = T^*_a$ and $TC(T^*) = TC(T^*_a)$.

**Proof.** We consider the following cases:

(1) If $\Delta_5 > 0$, then

$$
\frac{dTC_1(T^*_a)}{dT} > 0.
$$

Hence, we observe that $T^*_a > T^*_a$ and $TC_1(T)$ is increasing on $[T^*_a, \infty)$. So, we conclude that

$$
T^* = T^*_a \quad \text{and} \quad TC(T^*) = TC(T^*_a).
$$

(2) If $\Delta_5 \leq 0$, then

$$
\frac{dTC_1(T^*_a)}{dT} \leq 0.
$$

Hence, we see that $T^*_a \leq T^*_a$ and $TC_1(T)$ is decreasing on $[T^*_a, T^*_a]$ and increasing on $[T^*_a, \infty)$. We conclude that $T^* = T^*_a$ and $TC(T^*) = TC(T^*_a)$.

We have completed the proof of Theorem 3.

### 5 Illustrative Numerical Examples and Tables

In this section, we will illustrate all of the theoretical results by numerical examples and tables. For example, our main results (Theorems 1, 2 and 3 of the Sections 3 and 4) are illustrated by Tables 1, 2 and 3, respectively. In addition, due to the uncertainties in any decision-making situation, sensitivity analysis will provide significant assistance in the decision-making process, so sensitivity analysis of the optimal solution with respect to the parameters of the system is also carried out. We will first explore all dimensions of the parameters in the following examples are the same as those in Example 1 of [40] (see, for details, Examples 1, 2 and 3 below).

**Example 1.** Let $A = $1500/order, $D = 2000$/units/year, $M = \frac{250}{350}$, $\alpha = 0.1$, $\beta = 0.06$, $h_o = $1/unit/year, $h_i = \frac{\$3}{unit/year}$, $c = $10/unit/year, $p = $15/unit/year, $I_p = $0.15/S/year, $I_e = $0.12/S/year and $W = 100$/units. According to Theorem 1-4), the exact optimal solution is given by

$$
T^* = 0.0995 < M^* = 0.2986, \quad T^*_a = 0.5336 \quad \text{and} \quad TC(T^*) = 4624.
$$

**Table 1:** The results of Theorem 1

<table>
<thead>
<tr>
<th>A</th>
<th>D</th>
<th>W</th>
<th>$M$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$c$</th>
<th>$p$</th>
<th>$I_p$</th>
<th>$I_e$</th>
<th>$W$</th>
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<tbody>
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<td>10</td>
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<td>100</td>
<td>0.5</td>
<td>1-(1)</td>
<td>0.1</td>
<td>0.06</td>
<td>1</td>
<td>$\frac{$3}{unit/year}$</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>1000</td>
<td>0.5</td>
<td>1-(2)</td>
<td>0.1</td>
<td>0.06</td>
<td>1</td>
<td>$\frac{$3}{unit/year}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>400</td>
<td>0.5</td>
<td>1-(3)</td>
<td>0.1</td>
<td>0.06</td>
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<td></td>
</tr>
<tr>
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<td>0.5</td>
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<td>0.1</td>
<td>0.06</td>
<td>1</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 2.** Let $h_o = $1/unit/year, $h_r = $3/unit/year, $\alpha = 0.1$, $\beta = 0.06$, $c = $10/unit/year, $p = $15/unit/year, $I_p = $0.15/S/year, $I_e = $0.12/S/year and $T_o = 0.2469$. Then we have $T^*_a = 0.4879$ and $M^* = 0.7350$.

**Table 2:** The results of Theorem 2

<table>
<thead>
<tr>
<th>A</th>
<th>D</th>
<th>W</th>
<th>$M$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$c$</th>
<th>$p$</th>
<th>$I_p$</th>
<th>$I_e$</th>
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<tr>
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<td>1</td>
<td>$\frac{$3}{unit/year}$</td>
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</table>

**Example 3.** Let $h_o = $1/unit/year, $h_r = $3/unit/year, $\alpha = 0.1$, $\beta = 0.06$, $c = $10/unit/year, $p = $15/unit/year, $I_p = $0.15/S/year, $I_e = $0.12/S/year and $T_o = 0.2469$. Then we have $T^*_a = 0.4879$ and $M^* = 0.5397$.

**Table 3:** The results of Theorem 3

<table>
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<tr>
<th>A</th>
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<th>W</th>
<th>$M$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
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<td>100</td>
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<td>3-(2)</td>
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</tr>
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Table 4: Sensitivity Analysis

<table>
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<th>$W$</th>
<th>$A$</th>
<th>$D$</th>
<th>$T^*_A$</th>
<th>$M^*$</th>
<th>Theorem</th>
<th>$\Delta_1$</th>
<th>$\Delta_2$</th>
<th>$\Delta_3$</th>
<th>$T^*$</th>
<th>$TC(T^*)$</th>
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<td>2000</td>
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<td>$\leq 0$</td>
<td>$\leq 0$</td>
<td>0</td>
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</tr>
<tr>
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<td>$\leq 0$</td>
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<td>2000</td>
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<td>$\leq 0$</td>
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<td>0.2986</td>
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<td>$T^*_1 = 0.4664$</td>
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<td>2000</td>
<td>0.3922</td>
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<td>$\geq 0$</td>
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<td>$T^*_3 = 0.3339$</td>
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<td>$T^*_2 = 0.4307$</td>
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</tr>
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</table>

Example 4. Let $h_o = \$1/unit/year$, $h_r = \$3/unit/year$, $c = \$10/unit/year$, $p = \$15/unit/year$, $\alpha = 0.1$, $\beta = 0.06$, $I_p = \$0.15/$year$, $I_r = \$0.12/$year$ and $T_o = 0.2469$. Then we have $T^*_A = 0.4879$ and $M^* = 0.4421$.

Finally, we will explore the sensitivity analysis of the optimal solution with respect to the parameters $W$, $A$ and $D$, which are the same as in [40], in order to obtain the exact results. The results are shown in Table 4.

Based on the computational results, we obtain the exact solutions and the following results:

1. An increase in the value of $D$ will result in an increase in $TC(T^*)$, but a decrease in $T^*$. That is, a higher value of $D$ causes a higher value of $TC(T^*)$, but a lower value of $T^*$.

2. As the value of $A$ increases, the optimal replenishment cycle time $T^*$ and the minimum total average cost $TC(T^*)$ will be increased. It implies that, if the ordering cost is higher, it is reasonable that the retailer orders more quantity to lower its ordering cost when the ordering cost increases.

3. As the storage capacity of OW is increased, the replenishment cycle time $T^*$ increases, but the cost goes down. That is, a higher value of $W$ causes a higher value of $T^*$, but lower value of $TC(T^*)$.

6 Conclusion

This study has discussed the optimal ordering decisions in [40]. It focused on the condition that $Q \geq W$, that is, the retailer must hire the rented warehouse for holding inventory. However, the object of the paper [40] was seen to find $(T^*, T^*_o)$ such that $TC(T^*, T^*_o)$ is the minimum value, which is questionable. For this, we explored the same problems as that in [40], but we have presented an exact and mathematically valid solution procedure based upon the functional behavior of the total average cost under the conditions of each of the following three cases:

Case (1): $T^*_o < M < M^*$;
Case (2): $M < T^*_o < M^*$;
Case (3): $M < M^* < T^*_o$.

We remark in passing that Case (2) and Case (3) in our article were not discussed by Liang and Zhou [40]. Furthermore, we have established several theoretical results which are given as Theorems 1, 2 and 3 in order to determine the optimal solutions under various circumstances. Finally, numerical examples and sensitivity analysis of the optimal solution with respect to the parameters have also been included in order to illustrate the theoretical results and validate the exact solution procedures presented here.
References


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