Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100309

Some Mathematical Analytic Arguments for Determining Valid Optimal Lot Size for Deteriorating Items with Limited Storage Capacity under Permissible Delay in Payments

Jui-Jung Liao [1], *Kuo-Nan Huang* [2], *Kun-Jen Chung* [3,4,5], *Pin-Shou Ting* [5], *Shy-Der Lin* [6] *and H. M. Srivastava* [7,8,*].

[1] Department of Business Administration, Chihlee University of Technology, Taipei 22050, Taiwan, Republic of China.
[2] Department of Industrial Engineering and Management, St. John's University, Tamsui 25135, Taiwan, Republic of China.
[3] Chung Yuan Christian University, Chung-Li 32023, Taiwan, Republic of China.
[4] National Taiwan University of Science and Technology, Taipei 10607, Taiwan, Republic of China.
[5] Department of International Business Management, Shih Chien University, Taipei 10462, Taiwan, Republic of China.
[6] Departments of Applied Mathematics and Business Administration, Chung Yuan Christian University, Chung-Li 32023, Taiwan, Republic of China.
[7] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada.
[8] China Medical University, Taichung 40402, Taiwan, Republic of China.

Received: 9 Oct. 2015, Revised: 9 Jan. 2016, Accepted: 10 Jan. 2016
Published online: 1 May 2016

Abstract: In the year 2011, Y. Liang and F. Zhou presented an inventory model with two levels of storages, in which one has finite dimension and the other has infinite dimension, and with conditionally permissible delay in payments. In essence, it concentrated on the establishment of the inventory model, but did not concentrate on the validity of the processes of finding the optimal solution from the viewpoint of logic. In addition, it ignored whether the case of the trade credit period, M, is greater than the time interval and whether the order quantity is greater than W units or not, so the discussion of the optimal solution is questionable. The main purpose of this paper is to characterize the optimal solutions in accordance with the functional behavior of the total average cost under different circumstances, not only to overcome the shortcomings in the aforementioned work of Y. Liang and F. Zhou, but also to obtain accurate and reliable solution procedures. Finally, numerical examples are given to illustrate the theoretical results and the sensitivity analysis of the optimal solution with respect to the parameters of the system is carried out to reveal the exact results.

Keywords: Inventory modelling; Economic order quantity; Deteriorating items; Permissible delay in payments; Limited storage capacity; Mathematical solution procedures; Optimal solution; Sensitivity analysis.

1 Introduction

In modern business transactions, allowing a grace period for settling the amount owed is becoming ubiquitous. Usually, there is no charge if the outstanding amount is settled within the permitted fixed settlement period; beyond the the permitted fixed settlement period, interest is charged. Conversely, the retailer can sell items, accumulate revenues and finally earn interest during the permissible delay period. Furthermore, the main purpose of the permissible delay period is to encourage the retailer to buy more, to increase market share or to deplete inventories of certain items. Recently, a lot of articles consider inventory models for deteriorating items with permissible delay in payments. For example, [1] developed an EOQ model for deteriorating items under supplier credits linked to ordering quantity. [2], [3], [4], [5] provided lot-sizing decisions under permissible delay in payments depending on the ordering quantity in different circumstances. [6] established inventory ordering policies of delayed deteriorating items under permissible delay in payments. [7] established an EPQ

* Corresponding author e-mail: harimsri@math.uvic.ca

model for deteriorating items with up-stream full trade credit and down-stream partial trade credit. [8] revealed the retailers optimal ordering policy for deteriorating items with maximum lifetime under suppliers trade credit financing. Many additional related developments can be found in [9,10,11,12,13,14,15,16,17,18,19,20,21,22] and [23], and in the references cited therein.

On the other hand, in many practical situations, when an attractive price discount for bulk purchase is available or the items are seasonal or the cost of procuring items is higher than other inventory related cost or demand of the items is very high, these items cannot be accommodated in the existing store facility located at busy market places. In this regard, for storing the excess items, one additional storage facility is hired on rental basis, which may be located a little away from it. Furthermore, a rented warehouse (RW) is used to store the excess units over the fixed capacity on the own warehouse (OW) and the retailer is served first RW, then from OW. [24] incorporated the concepts of the basic two-warehouse inventory model and the conditions of permissible delay in payments in order to generalize the earlier work [25]. Recently, [26] and [27] explored an inventory model for deteriorating items with two warehouses in which the deterioration rates of items in OW and RW are different. [28] studied lot-sizing decisions for deteriorating items with capacity constraints under an order-size-dependent trade credit. [29] studied a two-warehouse inventory model for deteriorating items and stock dependent demand under conditionally permissible delay in payment in imprecise environment. [30] studied a two-warehouse inventory model for deteriorating items with linear increasing demand under conditionally permissible delay in payment. [31] explored the optimal strategy of deterioration items with capacity constraints under two-levels of trade credit policy. Many additional related investigations can be found in [32,33,34,35,36,37,38] and [39], and also in the references cited therein.

This paper explores an inventory model under the situation considered in the work of Liang and Zhou [40], who explored a two-warehouse inventory model for deterioration items under conditionally permissible delay in payment and its object was seen to find (T^*, t_w^*) such that $TC(T^*, t_w^*)$ is the minimum value. However, it is found that t_w is a function of T which results in observation that the total average cost TC is established by a decision variable, T, not by both T and t_w. In addition, Liang and Zhou [40] ignored whether M is greater than the time interval and that the order quantity is greater than W units or not, so this paper characterizes the validity of the optimal solutions in accordance with the functional behavior of the total average cost under different circumstances in order to overcome the shortcomings in the work of Liang and Zhou [40]. It also presents simple and easy-to-understand solution procedures. Finally, numerical examples to illustrate the theoretical results are presented and the sensitivity analysis with respect to the parameters of the system is performed.

2 Mathematical Formulation

The notation and assumptions in this paper are the same as that of [40] except for the last item. In addition, Liang and Zhou [40] focused on the assumption that the order quantity is always greater than the capacity of the owned warehouse, so this article denote by T_a the time interval in which the order quantity is greater than W units so that the inequality $Q \geqq W$ holds true if and only if $T \geqq T_a$.

On the other hand, because of the continuity of $I_0(t)$ at time t_w, Liang and Zhou [40] revealed that

$$W \cdot e^{-\alpha t_w} = \frac{D}{\alpha}\left(e^{\alpha(T-t_w)} - 1\right). \tag{1}$$

From Eq. (1), it is obvious that

$$t_w = \frac{1}{\alpha}\ln\left(\frac{De^{\alpha T} - \alpha W}{D}\right). \tag{2}$$

Thus, it can be easily seen that t_w is a function of T, and that the total average cost, TC, is a function of T, not T and t_w. However, Liang and Zhou [40] supposed that the total average cost was a function of T and t_w, so their solution procedures are questionable and their numerical examples are incorrect. Consequently, this article will adopt the calculus approach not only to overcome shortcomings [40], but also to develop the complete solution procedures for it.

Firstly, we must investigate and discuss whether or not the time t_w in which inventory level reduces to W is greater than M, so the inequality $M \geqq t_w$ holds true if and only if

$$\frac{1}{\alpha}\ln\left(\frac{De^{\alpha T} - \alpha W}{D}\right) \geqq T.$$

For notational convenience, let

$$M^* = \frac{1}{\alpha}\ln\left(\frac{De^{\alpha M} - \alpha W}{D}\right).$$

Then the inequality $M \geqq t_w$ holds true if and only if $M^* \geqq T$. Secondly, in the case when the inequality $T_a \leqq t_w$ holds true if and only if

$$\frac{1}{\alpha}\ln\left(\frac{De^{\alpha T} - \alpha W}{D}\right) \leqq T,$$

we let

$$T_a^* = \frac{1}{\alpha}\ln\left(\frac{De^{\alpha T_a} - \alpha W}{D}\right).$$

Then the inequality $T_a \leqq t_w$ holds true if and only if $T_a^* \leqq T$. Afterwards, the total average cost can be divided into the following three cases:

Case (1): $T_a^* < M < M^*$;

Case (2): $M < T_a^* < M^*$;

and

Case (3): $M < M^* < T_a^*$.

We remark in passing that the circumstances of Cases (2) and (3) were not discussed by Liang and Zhou [40].

Case 1. Suppose that $T_a^* < M < M^*$. Under this circumstance, Liang and Zhou [40] revealed that the total relevant cost $TC(T)$ is given by

$$TC(T) = \begin{cases} TC_1(T) & \text{if } M^* < T & \text{(3a)} \\ TC_2(T) & \text{if } M < T \leqq M^* & \text{(3b)} \\ TC_3(T) & \text{if } T_a^* < T \leqq M, & \text{(3c)} \end{cases}$$

where

$$\begin{aligned} TC_1(T) = {} & \frac{A}{T} + \frac{D}{\beta^2 T}\left[(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right)\right. \qquad (4) \\ & \left. + cI_p\left(e^{\beta(t_w - M)} - \beta(t_w - M) - 1\right)\right] \\ & - \frac{pI_eDM^2}{2T} + \frac{W}{\alpha T}\left[(h_o + c\alpha)\left(1 - e^{-\alpha t_w}\right) + cI_p\left(e^{-\alpha M} - e^{-\alpha t_w}\right)\right] \\ & + \frac{D}{\alpha^2 T}(h_o + c\alpha + cI_p)\left(e^{\alpha(T - t_w)} - \alpha(T - t_w) - 1\right), \qquad (5) \end{aligned}$$

$$\begin{aligned} TC_2(T) = {} & \frac{A}{T} + \frac{D}{\beta^2 T}\left[(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right)\right] + \frac{W}{\alpha T}\left[(h_o + c\alpha)\left(1 - e^{-\alpha t_w}\right)\right] \\ & - \frac{pI_eDM^2}{2T} + \frac{D}{\alpha^2 T}(h_o + c\alpha)\left(e^{\alpha(T - t_w)} - \alpha(T - t_w) - 1\right) \\ & + \frac{cI_pD}{\alpha^2 T}\left(e^{\alpha(T - M)} - \alpha(T - M) - 1\right) \qquad (6) \end{aligned}$$

and

$$\begin{aligned} TC_3(T) = {} & \frac{A}{T} + \frac{D}{\beta^2 T}\left[(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right)\right] + \frac{W}{\alpha T}\left[(h_o + c\alpha)\left(1 - e^{-\alpha t_w}\right)\right] \\ & + \frac{D}{\alpha^2 T}(h_o + c\alpha)\left(e^{\alpha(T - t_w)} - \alpha(T - t_w) - 1\right) + \frac{pI_eD}{2}(2M - T). \qquad (7) \end{aligned}$$

Substituting Eq. (1) into Eqs. (4), (6) and (7), we obtain the total relevant cost function as follows:

$$\begin{aligned} TC_1(T) = {} & \frac{A}{T} + \frac{(h_o + c\alpha)}{\alpha T}(W - D(T - t_w)) + \frac{D(h_r + c\beta)}{\beta^2 T} + \left(e^{\beta t_w} - \beta t_w - 1\right) \\ & - \frac{pI_eDM^2}{2T} + \frac{cI_p}{T}\left[\frac{D}{\beta^2}\left(e^{\beta(t_w - M)} - \beta(t_w - M) - 1\right)\right. \\ & \left. + \frac{1}{\alpha}\left(We^{-\alpha M} - D(T - t_w)\right)\right] \qquad (8) \end{aligned}$$

$$\begin{aligned} TC_2(T) = {} & \frac{A}{T} + \frac{(h_o + c\alpha)}{\alpha T}(W - D(T - t_w)) + \frac{D(h_r + c\beta)}{\beta^2 T}\left(e^{\beta t_w} - \beta t_w - 1\right) \\ & - \frac{pI_eDM^2}{2T} + \frac{cI_pD}{\alpha^2 T}\left(e^{\alpha(T - M)} - \alpha(T - M) - 1\right) \qquad (9) \end{aligned}$$

and

$$\begin{aligned} TC_3(T) = {} & \frac{A}{T} + \frac{(h_o + c\alpha)}{\alpha T}(W - D(T - t_w)) + \frac{D(h_r + c\beta)}{\beta^2 T}\left(e^{\beta t_w} - \beta t_w - 1\right) \\ & - \frac{pI_eD(2M - T)}{2} \qquad (10) \end{aligned}$$

For convenience, we treat all $TC_i(T)$ $(i = 1, 2, 3)$ defined on $T > 0$. Eqs. (8), (9) and (10) yield

$$TC_1(M^*) = TC_2(M^*) \qquad \text{and} \qquad TC_2(M) = TC_3(M).$$

So, clearly, the function $TC(T)$ is continuous and well-defined on $T \geqq T_a^*$.

Case 2. Suppose that $M < T_a^* < M^*$. Under this circumstance, we see that the total relevant cost $TC(T)$ is given by:

$$TC(T) = \begin{cases} TC_1(T) & \text{if } M^* < T & \text{(11a)} \\ TC_2(T) & \text{if } T_a^* < T \leqq M^*. & \text{(11b)} \end{cases}$$

Since $TC_1(M^*) = TC_2(M^*)$, $TC(T)$ is continuous and well-defined on $T \geqq T_a^*$.

Case 3. Suppose that $M < M^* < T_a^*$. Under this circumstance, we find that the total relevant cost $TC(T)$ is given by

$$TC(T) = TC_1(T). \qquad (12)$$

Clearly, therefore, the function $TC(T)$ is continuous and well-defined on $T \geqq T_a^*$ as well.

3 Theoretical Results and Optimal Solutions

First of all, Eqs. (8) to (10) yield

$$\begin{aligned} TC_1'(T) = {} & \frac{1}{T^2}\left\{-A + \frac{(h_o + c\alpha)}{\alpha}\left(DT\frac{dt_w}{dT} - Dt_w - W\right)\right. \\ & + \frac{(h_r + c\beta)D}{\beta^2}\left(\beta Te^{\beta t_w}\frac{dt_w}{dT} - \beta T\frac{dt_w}{dT} - e^{\beta t_w} + \beta t_w + 1\right) \\ & + \frac{cI_pDT}{\beta}\left(e^{\beta(t_w - M)} - 1\right)\frac{dt_w}{dT} - \frac{D}{\beta^2}\left(e^{\beta(t_w - M)} - \beta(t_w - M) - 1\right) \\ & \left. - \frac{1}{\alpha}\left(We^{-\alpha M} + Dt_w - DT\frac{dt_w}{dT}\right) + \frac{pI_eDM^2}{2}\right\}, \qquad (13) \end{aligned}$$

$$\begin{aligned} TC_2'(T) = {} & \frac{1}{T^2}\left\{-A + \frac{(h_o + c\alpha)}{\alpha}\left(DT\frac{dt_w}{dT} - Dt_w - W\right)\right. \\ & + \frac{(h_r + c\beta)D}{\beta^2}\left(\beta Te^{\beta t_w}\frac{dt_w}{dT} - \beta T\frac{dt_w}{dT} - e^{\beta t_w} + \beta t_w + 1\right) \\ & \left. + \frac{cI_pD}{\alpha^2}\left(\alpha Te^{\alpha(T - M)} - e^{\alpha(T - M)} - \alpha M + 1\right) + \frac{pI_eDM^2}{2}\right\} \qquad (14) \end{aligned}$$

and

$$\begin{aligned} TC_3'(T) = {} & \frac{1}{T^2}\left\{-A + \frac{(h_o + c\alpha)}{\alpha}\left(DT\frac{dt_w}{dT} - Dt_w - W\right)\right. \\ & + \frac{(h_r + c\beta)D}{\beta^2}\left(\beta Te^{\beta t_w}\frac{dt_w}{dT} - \beta T\frac{dt_w}{dT} - e^{\beta t_w} + \beta t_w + 1\right) \\ & \left. + \frac{pI_eDT^2}{2}\right\}. \qquad (15) \end{aligned}$$

Next, we let

$$\begin{aligned} f_1(T) = {} & -A + \frac{(h_o + c\alpha)}{\alpha}\left(DT\frac{dt_w}{dT} - Dt_w - W\right) \\ & + \frac{(h_r + c\beta)D}{\beta^2}\left(\beta Te^{\beta t_w}\frac{dt_w}{dT} - \beta T\frac{dt_w}{dT} - e^{\beta t_w} + \beta t_w + 1\right) \\ & + \frac{cI_pDT}{\beta}\left(e^{\beta(t_w - M)} - 1\right)\frac{dt_w}{dT} - \frac{D}{\beta^2}\left(e^{\beta(t_w - M)} - \beta(t_w - M) - 1\right) \\ & - \frac{1}{\alpha}\left(We^{-\alpha M} + Dt_w - DT\frac{dt_w}{dT}\right) + \frac{pI_eDM^2}{2}, \qquad (16) \end{aligned}$$

$$f_2(T) = -A + \frac{(h_o + c\alpha)}{\alpha}\left(DT\frac{dt_w}{dT} - Dt_w - W\right) + \frac{(h_r + c\beta)D}{\beta^2}\left(\beta T e^{\beta t_w}\frac{dt_w}{dT} - \beta T\frac{dt_w}{dT} - e^{\beta t_w} + \beta t_w + 1\right) + \frac{cI_pD}{\alpha^2}\left(\alpha T e^{\alpha(T-M)} - e^{\alpha(T-M)} - \alpha M + 1\right) + \frac{pI_eDM^2}{2} \tag{17}$$

and

$$f_3(T) = -A + \frac{(h_o + c\alpha)}{\alpha}\left(DT\frac{dt_w}{dT} - Dt_w - W\right) + \frac{(h_r + c\beta)D}{\beta^2}\left(\beta T e^{\beta t_w}\frac{dt_w}{dT} - \beta T\frac{dt_w}{dT} - e^{\beta t_w} + \beta t_w + 1\right) + \frac{pI_eDT^2}{2}. \tag{18}$$

Thus, clearly, we have

$$f_1'(T) = \frac{(h_o\beta - h_r\alpha)DT}{\alpha\beta}\left(\frac{d^2t_w}{dT^2}\right) + \frac{(h_r + c\beta)DT}{\beta^2}e^{\beta t_w}\left[\beta\left(\frac{dt_w}{dT}\right)^2 + \left(\frac{d^2t_w}{dT^2}\right)\right] + cI_pDT\left\{\frac{e^{\beta(t_w - M)}}{\beta}\left[\beta\left(\frac{dt_w}{dT}\right)^2 + \left(\frac{d^2t_w}{dT^2}\right)\right] + \left(\frac{\beta-\alpha}{\alpha}\right)\left(\frac{d^2t_w}{dT^2}\right)\right\}, \tag{19}$$

$$f_2'(T) = \frac{(h_o\beta - h_r\alpha)DT}{\alpha\beta}\left(\frac{d^2t_w}{dT^2}\right) + \frac{(h_r + c\beta)DT}{\beta^2}e^{\beta t_w}\left[\beta\left(\frac{dt_w}{dT}\right)^2 + \left(\frac{d^2t_w}{dT^2}\right)\right] + cI_pDTe^{\alpha(T-M)} \tag{20}$$

and

$$f_3'(T) = \frac{(h_o\beta - h_r\alpha)DT}{\alpha\beta}\left(\frac{d^2t_w}{dT^2}\right) + \frac{(h_r + c\beta)DT}{\beta^2}e^{\beta t_w}\left[\beta\left(\frac{dt_w}{dT}\right)^2 + \left(\frac{d^2t_w}{dT^2}\right)\right] + pI_eDT. \tag{21}$$

We also have

$$\beta\left(\frac{dt_w}{dT}\right)^2 + \frac{d^2t_w}{dT^2} = \frac{De^{\alpha T}\left(\beta De^{\alpha T} - \alpha^2 W\right)}{\left(De^{\alpha T} - \alpha W\right)^2} > 0 \text{ if } \beta D > \alpha^2 W \tag{22}$$

Therefore, we find that $f_i'(T) > 0$ and $f_i(T)$ $(i = 1,2,3)$ is increasing on $T \geqq 0$, respectively.

Finally, we consider the case when

$$f_1(T) = 0, \tag{23}$$

$$f_2(T) = 0 \tag{24}$$

and

$$f_3(T) = 0. \tag{25}$$

Let T_1^*, T_2^* and T_3^* denote the roots of Eqs. (23), (24) and (25), respectively, if their roots exist. Furthermore, Eqs. (16) to (18) reveal the fact that

$$\Delta_1 = f_1(M^*), \tag{26}$$

$$\overline{\Delta_1} = f_2(M^*), \tag{27}$$

$$\Delta_2 = f_2(M) = f_3(M) \tag{28}$$

and

$$\Delta_3 = f_3(T_a^*). \tag{29}$$

Most strikingly, when $T = M^*$, we obtain $t_w = M$, so we have

$$f_1(M^*) = f_2(M^*),$$

that is,

$$\Delta_1 = \overline{\Delta_1}.$$

Eqs. (26) to (29) imply that

$$\Delta_1 > \Delta_2 > \Delta_3.$$

We recall that $TC_i'(T)$ $(i = 1,2,3)$ is increasing on $T \geqq 0$. From

$$\lim_{T\to\infty} f_i(T) = \infty \qquad (i = 1,2,3)$$

and the *Intermediate Value Theorem* (see, for example, [41] and [42]), we have the following results.

Theorem 1. *Suppose that $T_a^* < M < M^*$. Then*

(1) *If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$, then $T^* = T_a^*$ and $TC(T^*) = TC(T_a^*)$.*
(2) *If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 \leqq 0$, then $T^* = T_3^*$ and $TC(T^*) = TC(T_3^*)$.*
(3) *If $\Delta_1 > 0$, $\Delta_2 \leqq 0$ and $\Delta_3 \leqq 0$, then $T^* = T_2^*$ and $TC(T^*) = TC(T_2^*)$.*
(4) *If $\Delta_1 \leqq 0$, $\Delta_2 \leqq 0$ and $\Delta_3 \leqq 0$, then $T^* = T_1^*$ and $TC(T^*) = TC(T_1^*)$.*

Proof. We consider the following cases:

(1) If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 > 0$, then

$$\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} > 0,$$

$$\frac{dTC_2(M)}{dT} = \frac{dTC_3(M)}{dT} > 0$$

and

$$\frac{dTC_1(T_a^*)}{dT} > 0.$$

Hence, we get $M^* > T_1^*$, $M > T_2^*$ and $T_a^* > T_1^*$. Additionally, we have
(a) $TC_1(T)$ is increasing on $[M^*, \infty)$.
(b) $TC_2(T)$ is increasing on $[M, M^*]$.
(c) $TC_3(T)$ is increasing on $[T_a^*, M]$.
Combining (a) to (c), we conclude that $TC(T)$ is increasing on $[T_a^*, \infty)$. So, we have

$$T^* = T_a^* \qquad \text{and} \qquad TC(T^*) = TC(T_a^*).$$

(2) If $\Delta_1 > 0$, $\Delta_2 > 0$ and $\Delta_3 \leqq 0$, then

$$\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} > 0,$$

$$\frac{dTC_2(M)}{dT} = \frac{dTC_3(M)}{dT} > 0$$

and

$$\frac{dTC_1(T_a^*)}{dT} \leqq 0.$$

Hence, we see that $M^* > T_1^*$, $M > T_2^*$ and $T_a^* \leqq T_3^* < M$. Additionally, we have
(a) $TC_1(T)$ is increasing on $[M^*, \infty)$.
(b) $TC_2(T)$ is increasing on $[M, M^*]$.
(c) $TC_3(T)$ is decreasing on $[T_a^*, T_3^*]$ and increasing on $[T_3^*, M]$.
Combining (a) to (c), we conclude that $TC(T)$ is decreasing on $[T_a^*, T_3^*]$ and increasing on $[T_3^*, \infty)$. So, we have

$$T^* = T_3^* \qquad \text{and} \qquad TC(T^*) = TC(T_3^*).$$

(3) If $\Delta_1 > 0$, $\Delta_2 \leqq 0$ and $\Delta_3 \leqq 0$, then

$$\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} > 0,$$

$$\frac{dTC_2(M)}{dT} = \frac{dTC_3(M)}{dT} \leqq 0$$

and

$$\frac{dTC_3(T_a^*)}{dT} \leqq 0.$$

Hence, we observe that $M^* > T_1^*$, $M \leqq T_2^* \leqq M^*$ and $T_3^* > M$. Additionally, we have
(a) $TC_1(T)$ is increasing on $[M^*, \infty)$.
(b) $TC_2(T)$ is decreasing on $[M, T_2^*]$ and increasing on $[T_2^*, M^*]$.
(c) $TC_3(T)$ is decreasing on $[T_a^*, M]$.
Combining (a) to (c), we conclude that $TC(T)$ is decreasing on $[T_a^*, T_2^*]$ and increasing on $[T_2^*, \infty)$. So, we have

$$T^* = T_2^* \qquad \text{and} \qquad TC(T^*) = TC(T_2^*).$$

(4) If $\Delta_1 \leqq 0$, $\Delta_2 \leqq 0$ and $\Delta_3 \leqq 0$, then

$$\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} \leqq 0,$$

$$\frac{dTC_2(M)}{dT} = \frac{dTC_3(M)}{dT} \leqq 0$$

and

$$\frac{dTC_3(T_a^*)}{dT} \leqq 0.$$

Hence, we see that $M^* \leqq T_1^*$, $M \leqq T_2^*$ and $T_3^* > M$. Additionally, we have
(a) $TC_1(T)$ is decreasing on $[M^*, T_1^*]$ and increasing on $[T_1^*, \infty)$.
(b) $TC_2(T)$ is decreasing on $[M, M^*]$.
(c) $TC_3(T)$ is decreasing on $[T_a^*, M]$.
Combining (a) to (c), we conclude that $TC(T)$ is decreasing on $[T_a^*, T_1^*]$ and increasing on $[T_1^*, \infty)$. So, $T^* = T_1^*$ and $TC(T^*) = TC(T_1^*)$.

We thus have completed the proof of Theorem 1.

The above arguments are to reveal the exploration of functional behaviors of $TC_1(T)$, $TC_2(T)$ and $TC_3(T)$ on intervals $[M^*, \infty)$, $[M, M^*]$ and $[T_a^*, M]$, respectively, to jointly decide whether or not T_1^*, T_2^*, T_3^* or T_a^* is the optimal solution T^* of $TC(T)$ on the whole domain $T \geqq T_a^*$.

4 Further Theoretical Results and Optimal Solutions

The inventory models of the following cases:
Case (2): $M < T_a^* < M^*$
Case (3): $M < M^* < T_a^*$
were ignored by Liang and Zhou [40], so we will present complete solution procedures (which are missing in [40]) in this section.

In the case when $M < T_a^* < M^*$, Eq. (14) yields

$$TC_2'(T_a^*) = \frac{1}{T_a^{*2}} \cdot f_2(T_a^*). \tag{30}$$

Similarly, if we let

$$\Delta_4 = f_2(T_a^*), \tag{31}$$

then we are led to the fact that $\Delta_1 > \Delta_4$. As implied above, we have the following results.

Theorem 2. *Suppose that $M < T_a^* < M^*$. Then*

(1) *If $\Delta_1 > 0$, $\Delta_4 > 0$, then $T^* = T_a^*$ and $TC(T^*) = TC(T_a^*)$.*
(2) *If $\Delta_1 > 0$, $\Delta_4 \leqq 0$, then $T^* = T_2^*$ and $TC(T^*) = TC(T_2^*)$.*
(3) *If $\Delta_1 \leqq 0$, $\Delta_4 \leqq 0$, then $T^* = T_1^*$ and $TC(T^*) = TC(T_1^*)$.*

Proof. We consider the following cases:

(1) If $\Delta_1 > 0$ and $\Delta_4 > 0$, then

$$\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} > 0$$

and

$$\frac{dTC_2(T_a^*)}{dT} > 0.$$

Hence, we see that $M^* > T_1^*$ and $T_a^* > T_2^*$. Furthermore, we have
(a) $TC_1(T)$ is increasing on $[M^*, \infty)$.

(b) $TC_2(T)$ is increasing on $[T_a^*, M^*]$.
Combining (a) and (b), we conclude that $TC(T)$ is increasing on $[T_a^*, \infty)$. So, we have

$$T^* = T_a^* \quad \text{and} \quad TC(T^*) = TC(T_a^*).$$

(2) If $\Delta_1 > 0$ and $\Delta_4 \leqq 0$, then

$$\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} > 0$$

and

$$\frac{dTC_2(T_a^*)}{dT} \leqq 0.$$

Hence, we see that $M^* > T_1^*$ and $T_a^* \leqq T_2^* \leqq M^*$. Furthermore, we have
(a) $TC_1(T)$ is increasing on $[M^*, \infty)$.
(b) $TC_2(T)$ is decreasing on $[T_a^*, T_2^*]$ and increasing on $[T_2^*, M^*]$.
Combining (a) and (b), we conclude that $TC(T)$ is decreasing on $[T_a^*, T_2^*]$ and increasing on $[T_2^*, \infty)$. So, $T^* = T_2^*$ and $TC(T^*) = TC(T_2^*)$.

(3) If $\Delta_1 \leqq 0$ and $\Delta_4 \leqq 0$, then $\frac{dTC_1(M^*)}{dT} = \frac{dTC_2(M^*)}{dT} \leqq 0$ and $\frac{dTC_2(T_a^*)}{dT} \leqq 0$. Hence, $M^* \leqq T_1^*$ and $M^* \leqq T_2^*$. Furthermore, we have
(a) $TC_1(T)$ is decreasing on $[M^*, T_1^*]$ and increasing on $[T_1^*, \infty)$.
(b) $TC_2(T)$ is decreasing on $[T_a^*, M^*]$.
Combining (a) and (b), we conclude that $TC(T)$ is decreasing on $[T_a^*, T_1^*]$ and increasing on $[T_1^*, \infty)$. So, $T^* = T_1^*$ and $TC(T^*) = TC(T_1^*)$.

We have thus completed the proof of Theorem 2.

Finally, in the case when $M < M^* < T_a^*$, we find that $TC(T) = TC_1(T)$. Eq. (13) yields

$$TC_1'(T_a^*) = \frac{1}{T_a^{*2}} f_1(T_a^*) \tag{32}$$

Similarly, if we let $\Delta_5 = f_1(T_a^*)$, then we have the following results.

Theorem 3. *Suppose now that $M < M^* < T_a^*$. Then*
(1) *If $\Delta_5 > 0$, then $T^* = T_a^*$ and $TC(T^*) = TC(T_a^*)$.*
(2) *If $\Delta_5 \leqq 0$, then $T^* = T_1^*$ and $TC(T^*) = TC(T_1^*)$.*

Proof. We consider the following cases:

(1) If $\Delta_5 > 0$, then

$$\frac{dTC_1(T_a^*)}{dT} > 0.$$

Hence, we observe that $T_a^* > T_1^*$ and $TC_1(T)$ is increasing on $[T_a^*, \infty)$. So, we conclude that

$$T^* = T_a^* \quad \text{and} \quad TC(T^*) = TC(T_a^*).$$

(2) If $\Delta_5 \leqq 0$, then $\frac{dTC_1(T_a^*)}{dT} \leqq 0$. Hence, we see that $T_a^* \leqq T_1^*$ and $TC_1(T)$ is decreasing on $[T_a^*, T_1^*]$ and increasing on $[T_1^*, \infty)$. We conclude that $T^* = T_1^*$ and $TC(T^*) = TC(T_1^*)$.

We have completed the proof of Theorem 3.

5 Illustrative Numerical Examples and Tables

In this section, we will illustrate all of the theoretical results by numerical examples and tables. For example, our main results (Theorems 1, 2 and 3 of the Sections 3 and 4) are illustrated by Tables 1, 2 and 3, respectively. In addition, due to the uncertainties in any decision-making situation, sensitivity analysis will provide significant assistance in the decision-making process, so sensitivity analysis of the optimal solution with respect to the parameters of the system is also carried out. We will first explore all dimensions of the parameters in the following examples are the same as those in Example 1 of [40] (see, for details, Examples 1, 2 and 3 below).

Example 1. Let $A = \$1500/\text{order}$, $D = 2000\text{units/year}$, $M = \frac{3}{12} = 0.25\text{year}$, $\alpha = 0.1$, $\beta = 0.06$, $h_o = \$1/\text{unit/year}$, $h_r = \$3/\text{unit/year}$, $c = \$10/\text{unit/year}$, $p = \$15/\text{unit/year}$, $I_p = \$0.15/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$ and $W = 100\textit{units}$, According to Theorem 1-(4), the exact optimal solution is given by

$$T_a^* = 0.0995 < M^* = 0.2986, \quad T^* = T_1^* = 0.5336 \quad \text{and} \quad TC(T^*) = 4624.$$

Table 1: The results of Theorem 1

A	D	W	M	Theorem	Δ_1	Δ_2	Δ_3	T^*	$TC(T^*)$
10	400	100	0.5	1-(1)	>0	>0	>0	$T_a^* = 0.4879$	74.5469
400	1000	100	0.5	1-(2)	>0	>0	$\leqq 0$	$T_3^* = 0.3857$	1048.30
300	400	100	0.5	1-(3)	>0	$\leqq 0$	$\leqq 0$	$T_2^* = 0.5391$	663.2074
600	400	100	0.5	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.7560$	1126.300

Example 2. Let $h_o = \$1/\text{unit/year}$, $h_r = \$3/\text{unit/year}$, $\alpha = 0.1$, $\beta = 0.06$, $c = \$10/\text{unit/year}$, $p = \$15/\text{unit/year}$, $I_p = \$0.15/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$ and $T_a = 0.2469$. Then we have $T_a^* = 0.4879$ and $M^* = 0.7350$.

Table 2: The results of Theorem 2

A	D	W	M	Theorem	Δ_1	Δ_4	T^*	$TC(T^*)$
10	400	100	0.3	2-(1)	>0	>0	$T_a^* = 0.4879$	398.7944
250	400	100	0.3	2-(2)	>0	$\leqq 0$	$T_2^* = 0.5020$	705.8018
500	400	100	0.3	2-(3)	$\leqq 0$	$\leqq 0$	$T_1^* = 0.6967$	1122.70

Example 3. Let $h_o = \$1/\text{unit/year}$, $h_r = \$3/\text{unit/year}$, $c = \$10/\text{unit/year}$, $p = \$15/\text{unit/year}$, $\alpha = 0.1$, $\beta = 0.06$, $I_p = \$0.15/\$/\text{year}$, $I_e = \$0.12/\$/\text{year}$ and $T_a = 0.2469$. Then we have $T_a^* = 0.4879$ and $M^* = 0.5397$.

Table 3: The results of Theorem 3

A	D	W	M	Theorem	Δ_5	T^*	$TC(T^*)$
150	400	100	0.2	3-(1)	>0	$T_a^* = 0.484$	567.7781
250	400	100	0.3	3-(2)	>0	$T_3^* = 0.5039$	772.1533

Table 4: Sensitivity Analysis

M	W	A	D	T_a^*	M^*	Theorem	Δ_1	Δ_2	Δ_3	T^*	$TC(T^*)$
0.25	100	1500	2000	0.0995	0.2986	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.5336$	4624.0
			3500	0.0570	0.2778	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.4028$	5797.2
			5000	0.0399	0.2695	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.3355$	6603.8
		2000	2000	0.0995	0.2986	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.6166$	5493.0
			3500	0.0570	0.2778	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.4658$	6948.2
			5000	0.0399	0.2695	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.3895$	7983.2
		2500	2000	0.0995	0.2986	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.6886$	6259.1
			3500	0.0570	0.2778	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.5208$	7961.3
			5000	0.0399	0.2695	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.4355$	9195.9
	250	1500	2000	0.2469	0.3712	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.5362$	4430.3
			3500	0.1418	0.3194	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.4044$	5593.0
			5000	0.0995	0.2986	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.3366$	6394.1
		2000	2000	0.2469	0.3712	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.6182$	5296.1
			3500	0.1418	0.3194	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.4664$	6741.5
			5000	0.0995	0.2986	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.3896$	7771.4
		2500	2000	0.2469	0.3712	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.6902$	6060.3
			3500	0.1418	0.3194	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.5214$	7753.3
			5000	0.0995	0.2986	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.4356$	8983.0
0.4	400	1500	2000	0.3922	0.5903	1-(3)	$\geqq 0$	$\leqq 0$	$\leqq 0$	$T_2^* = 0.5379$	3759.5
			3500	0.2260	0.5092	1-(2)	$\geqq 0$	$\geqq 0$	$\leqq 0$	$T_3^* = 0.3999$	4496.4
			5000	0.1587	0.4766	1-(2)	$\geqq 0$	$\geqq 0$	$\leqq 0$	$T_3^* = 0.3339$	4870.3
		2000	2000	0.3922	0.5903	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.6193$	4623.6
			3500	0.2260	0.5092	1-(3)	$\geqq 0$	$\leqq 0$	$\leqq 0$	$T_2^* = 0.4631$	5655.1
			5000	0.1587	0.4766	1-(2)	$\geqq 0$	$\geqq 0$	$\leqq 0$	$T_3^* = 0.3845$	6262.0
		2500	2000	0.3922	0.5903	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.6903$	5386.9
			3500	0.2260	0.5092	1-(4)	$\leqq 0$	$\leqq 0$	$\leqq 0$	$T_1^* = 0.5182$	6673.7
			5000	0.1587	0.4766	1-(3)	$\geqq 0$	$\leqq 0$	$\leqq 0$	$T_2^* = 0.4307$	7489.5

Example 4. Let $h_o = \$1$/unit/year, $h_r = \$3$/unit/year, $c = \$10$/unit/year, $p = \$15$/unit/year, $\alpha = 0.1$, $\beta = 0.06$, $I_p = \$0.15$/\$/year, $I_e = \$0.12$/\$/year and $T_a = 0.2469$. Then we have $T_a^* = 0.4879$ and $M^* = 0.4421$.

Finally, we will explore the sensitivity analysis of the optimal solution with respect to the parameters W, A and D, which are the same as in [40], in order to obtain the exact results. The results are shown in Table 4.

Based on the computational results, we obtain the exact solutions and the following results:

1. An increase in the value of D will result in an increase in $TC(T^*)$, but a decrease in T^*. That is, a higher value of D causes a higher value of $TC(T^*)$, but a lower value of T^*.
2. As the value of A increases, the optimal replenishment cycle time T^* and the minimum total average cost $TC(T^*)$ will be increased. It implies that, if the ordering cost is higher, it is reasonable that the retailer orders more quantity to lower its ordering cost when the ordering cost increases.
3. As the storage capacity of OW is increased, the replenishment cycle time T^* increases, but the cost goes down. That is, a higher value of W causes a higher value of T^*, but lower value of $TC(T^*)$.

6 Conclusion

This study has discussed the optimal ordering decisions in [40]. It focused on the condition that $Q \geqq W$, that is, the retailer must hire the rented warehouse for holding inventory. However, the object of the paper [40] was seen to find (T^*, t_w^*) such that $TC(T^*, t_w^*)$ is the minimum value, which is questionable. For this, we explored the same problems as that in [40], but we have presented an exact and mathematically valid solution procedure based upon the functional behavior of the total average cost under the conditions of each of the following three cases:

Case (1): $T_a^* < M < M^*$;
Case (2): $M < T_a^* < M^*$;
Case (3): $M < M^* < T_a^*$.

We remark in passing that Case (2) and Case (3) in our article were not discussed by Liang and Zhou [40]. Furthermore, we have established several theoretical results which are given as Theorems 1, 2 and 3 in order to determine the optimal solutions under various circumstances. Finally, numerical examples and sensitivity analysis of the optimal solution with respect to the parameters have also been included in order to illustrate the theoretical results and validate the exact solution procedures presented here.

References

[1] V. B. Kreng, S.-J. Tan, The optimal replenishment decisions under two levels of trade credit policy depending on the order quantity, Expert Systems with Applications 37 (2010) 5514–5522. doi:10.1016/j.eswa.2009.12.014.

[2] K.-J. Chung, S.-D. Lin, H. M. Srivastava, The complete solution procedures for the mathematical analysis of some families of optimal inventory models with order-size dependent trade credit and deterministic and constant demand, Applied Mathematics and Computation 219 (2012) 142—157. doi:10.1016/j.amc.2012.06.001.

[3] K.-J. Chung, L. E. Cardenas-Barron, The simplified solution procedure for deteriorating items under stock-dependent demand and two-level trade credit in the supply chain management, Applied Mathematical Modelling 37 (2013) 4653—4660. doi:0.1016/j.apm.2012.10.018.

[4] L.-Y. Ouyang, C.-T. Yang, Y.-L. Chan, L. E. Cardenas-Barron, A comprehensive extension of the optimal replenishment decisions under two levels of trade credit policy depending on the order quantity, Applied Mathematics and Computation 224 (2013) 268–277. doi:10.1016/j.amc.2013.08.062.

[5] J.-T. Teng, H.-L. Yang, M.-S. Chern, An inventory model for increasing demand under two levels of trade credit linked to order quantity, Applied Mathematical Modelling 37 (2013) 7624–7632. doi:10.1016/j.apm.2013.02.009.

[6] A. Musa, B. Sani, Inventory ordering policies of delayed deteriorating items under permissible delay in payments, International Journal of Production Economics 136 (2012) 75–83. doi:10.1016/j.ijpe.2011.09.013.

[7] S.-C. Chen, J.-T. Teng, K. Skouri, Economic production quantity models for deteriorating items with up-stream full trade credit and down-stream partial trade credit, International Journal of Production Economics 155 (2014) 302–309. doi:10.1016/j.ijpe.2013.07.024.

[8] S.-C. Chen, J.-T. Teng, Retailer's optimal ordering policy for deteriorating items with maximum lifetime under supplier's trade credit financing, Applied Mathematical Modelling 38 (2014) 4049—4061. doi:10.1016/j.apm.2013.11.056.

[9] K. V. Geetha, R. Uthayakumar, Economic design of an inventory policy for non-instantaneous deteriorating items under permissible delay in payments, Journal of Computational and Applied Mathematics 233 (2010) 2492—2505. doi:10.1016/j.cam.2009.10.031.

[10] J. Min, Y.-W. Zhou, J. Zhao, An inventory model for deteriorating items under stock-dependent demand and two-level trade credit, Applied Mathematical Modelling 34 (2010) 3273–3285. doi:10.1016/j.apm.2010.02.019.

[11] K.-N. Huang, J.-J. Liao, An inventory model for deteriorating items with two levels of trade credit taking account of time discounting, Acta Applicandae Mathematicae 110 (2010) 313–326. doi:10.1007/s10440-008-9411-3.

[12] S. Khanra, S. K. Ghosh, K. S. Chaudhuri, An eoq model for a deteriorating item with time dependent quadratic demand under permissible delay in payment, Applied Mathematics and Computation 218 (2011) 1–9. doi:10.9790/5728-022303.

[13] K.-J. Chung, J.-J. Liao, The simplified solution algorithm for an integrated supplier-buyer inventory model with two-part trade credit in a supply chain system, European Journal of Operational Research 213 (2011) 156—165. doi:10.1016/j.ejor.2011.03.018.

[14] K. Mohini, T. P. M. Pakkla, Inventory policy under trade credit when time payment is uncertain, International Journal of Mathematical Archive 3 (2012) 2669–2679.

[15] A. Thangam, Optimal price discounting and lot-sizing policies for perishable items in a supply chain under advance payment scheme and two-echelon trade credits, International Journal of Production Economics 139 (2012) 459–472. doi:10.1016/j.ijpe.2012.03.030.

[16] B. Sarkar, An eoq model with delay in payments and time varying deterioration rate, Mathematical and Computer Modelling 55 (2012) 367–377. doi:10.1016/j.mcm.2011.08.009.

[17] T. L. Urban, An extension of inventory models incorporating financing agreements with both suppliers and customers, Applied Mathematical Modelling 36 (2012) 6323–6330. doi:10.1016/j.apm.2012.01.050.

[18] K.-J. Chung, S.-D. Lin, H. M. Srivastava, The inventory models under conditional trade credit in a supply chain system, Applied Mathematical Modelling 37 (2013) 10036–10052.

[19] K.-R. Lou, W.-C. Wang, Optimal trade credit and order quantity when trade credit impacts on both demand rate and default risk, Journal of the Operational Research Society 64 (2013) 1551–1556. doi:10.1057/jors.2012.134.

[20] P.-S. Ting, K.-J. Chung, Some formulations for optimal solutions with delays in payment and price-discount offers in the supply chain system, Applied Mathematics and Computation 230 (2014) 180–192. doi:10.1016/j.amc.2013.12.069.

[21] J. Wu, L.-Y. Ouyang, L. E. Cardenas-Barron, S. K. Goyal, Optimal credit period and lot size for deteriorating items with expiration dates under two-level trade credit financing, European Journal of Operational Research 237 (2014) 898–908. doi:10.1016/j.ejor.2014.03.009.

[22] E. Bazan, M. Y. Jaber, S. Zanoni, A review of mathematical inventory models for reverse logistics and the future of its modeling: An environmental perspective, Applied Mathematical Modelling 40 (2016) 4151–4178. doi:10.1016/j.apm.2015.11.027.

[23] K.-J. Chung, J.-J. Liao, P.-S. Ting, S.-D. Lin, H. M. Srivastava, The algorithm for the optimal cycle time and pricing decisions for an integrated inventory system with order-size dependent trade credit in supply chain management, Applied Mathematics and Computation 268 (2015) 322–333. doi:10.1016/j.amc.2015.06.039.

[24] K.-J. Chung, T.-S. Huang, The optimal retailer's ordering policies for deteriorating items with limited storage capacity under trade credit financing, International Journal of Production Economics 106 (2007) 127–145. doi:10.1016/j.ijpe.2006.05.008.

[25] S. P. Aggarwal, C. K. Jaggi, Ordering policies of deteriorating items under permissible delay in payments, Journal of the Operational Research Society 46 (1995) 658–662. doi:10.2307/2584538.

[26] J.-J. Liao, K.-N. Huang, Deterministic inventory model for deteriorating items with trade credit financing and capacity constraints, Computers & Industrial Engineering 59 (2010) 611—618. doi:10.1016/j.cie.2010.07.006.

[27] J.-J. Liao, K.-J. Chung, K.-N. Huang, A deterministic inventory model for deteriorating items with two warehouses and trade credit in a supply chain system, International Journal of Production Economics 146 (2013) 557–565. doi:10.1016/j.ijpe.2013.08.001.

[28] J.-J. Liao, K.-N. Huang, K.-J. Chung, Lot-sizing decisions for deteriorating items with two warehouses under an order-size-dependent trade credit, International Journal of Production Economics 137 (2012) 102–115. doi:10.1016/j.ijpe.2012.01.020.

[29] D. K. Jana, K. Maity, T. K. Roy, A two-warehouse eoq model for deteriorating items and stock dependent demand under conditionally permissible delay in payment in imprecise environment, Advanced Modeling and Optimization 15 (2013) 173–193.

[30] T. Singh, H. Pattnayak, A two-warehouse inventory model for deteriorating items with linear demand under conditionally permissible delay in payment, International Journal of Management Science and Engineering Management 9 (2014) 104–113. doi:10.1080/17509653.2013.862931.

[31] J.-J. Liao, K.-N. Huang, P.-S. Ting, Optimal strategy of deteriorating items with capacity constraints under two-levels of trade credit policy, Applied Mathematics and Computation 233 (2014) 647–658. doi:10.1016/j.amc.2014.01.077.

[32] S. B. Kumar, B. Sarkar, A. Goswami, A two-warehouse inventory model with increasing demand and time varying deterioration, Scientia Iranica 19 (2012) 1969–1977. doi:10.1016/j.scient.2012.10.040.

[33] G. Yen, K. Chung, T. Chen, The optimal retailer's ordering policies with trade credit financing and limited storage capacity in the supply chain system, International Journal of System Science 1 (2011) 1–16. doi:10.1080/00207721.2011.565133.

[34] H.-L. Yang, C.-T. Chang, A two-warehouse partial backlogging inventory model for deteriorating items with permissible delay in payment under inflation, Applied Mathematical Modelling 37 (2013) 2717–2726. doi:10.1016/j.apm.2012.05.008.

[35] Y.-G. Zhong, Y.-W. Zhou, Improving the supply chain's performance through trade credit under inventory-dependent demand and limited storage capacity, International Journal of Production Economics 143 (2013) 364–370. doi:10.1016/j.ijpe.2012.07.013.

[36] H. N. Soni, Optimal replenishment policies for deteriorating items with stock sensitive demand under two-level trade credit and limited capacity, Applied Mathematical Modelling 37 (2013) 5887–5895. doi:10.1016/j.apm.2012.11.006.

[37] J.-J. Liao, K.-N. Huang, K.-J. Chung, Optimal pricing and ordering policy for perishable items with limited storage capacity and partial trade credit, IMA Journal of Management Mathematics 24 (2013) 45–61. doi:10.1093/imaman/dps003.

[38] K.-J. Chung, J.-J. Liao, P.-S. Ting, S.-D. Lin, H. M. Srivastava, The algorithm for the optimal cycle time and pricing decisions for an integrated inventory system with order-size dependent trade credit in supply chain managemen, Applied Mathematics and Computation 268 322–333. doi:10.1016/j.amc.2015.06.039.

[39] A. K. Bhunia, C. K. Jaggi, A. Sharma, R. Sharma, A two-warehouse inventory model for deteriorating items under permissible delay in payment with partial backlogging, Applied Mathematics and Computation 232 (2014) 1125–1137. doi:10.1016/j.amc.2014.01.115.

[40] Y. Liang, F. Zhou, A two-warehouse inventory model for deteriorating items under conditionally permissible delay in payment, Applied Mathematical Modelling 35 (2011) 2221–2231. doi:10.1016/j.apm.2010.11.014.

[41] H. M. Srivastava, B. R. K. Kashyap, Special Functions in Queuing Theory and Related Stochastic Processes, Academic Press, New York and London, 1982.

[42] G. B. Thomas, R. L. Finney, Calculus with Analytic Geometry, Addison-Wesley Publishing Company, Boston, 1996.

Jui-Jung Liao is a Professor in the Department of Business Administration at Chihlee University of Technology in New Taipei City in Taiwan (Republic of China). She earned her Ph.D. degree from the Department of Industrial Management at the National Taiwan University of Science and Technology in Taipei, an M.S. degree in Mathematics from Tamkang University in Tamsui, and a B.S. degree in Statistics from Tamkang University in Tamsui. Her research interests are in the field of Inventory Controls and Probability and Statistics. She has published articles in such journals as (for example) Computers and Operations Research, European Journal of Operational Research, International Journal of Production Economics, Journal of the Operations Research Society of Japan, Journal of Statistics and Management Systems, International Journal of Information and Optimization Sciences, International Journal of Information and Management Sciences, Computers and Industrial Engineering, Mathematical and Computer Modelling, Applied Mathematical Modelling, Computers and Mathematics with Applications, Acta Applicandae Mathematicate, IMA Journal of Management Mathematics, Applied Mathematics and Computation, and so on

Kuo-Nan Huang is an Associate Professor in the Department of Industrial Engineering and Management at St. Johns University in Tamsui in Taiwan (Republic of China). He received his Ph.D. degree in Industrial Engineering and Management from Yuan Ze University in Taoyuan City in Taiwan (Republic of China). His research interests are in the field of Inventory Control and Response Surface Methodology. He has publications in such journals as (for example) Computers and Industrial Engineering, Journal of Statistics and Management Systems, International Journal of Information and Optimization Sciences, International Journal of Information and Management Sciences, Mathematical and Computer Modelling, Computers and Mathematics with Applications, Acta Applicandae Mathematicate, IMA Journal of Management Mathematics, Journal of Statistical Computation and Simulation, Journal of Chemometrics, Statistics and Probability Letters, Journal of Quality, and so on.

Kun-Jen Chung is the Chair Professor in the Department of International Business Management at Shih Chien University in Taipei in Taiwan (Republic of China). He was awarded his Ph.D. degree in Industrial Management from the Georgia Institute of Technology in Atlanta (Georgia) in U.S.A. His interests include Markov Decision Processes, Economic Designs of Control Charts, and Inventory Control and Reliability. He has authored over 196 articles which are published or accepted for publication in scientific research journals such as Operations Research, International Journal of Production Research, Optimization, Journal of the Operations Research Society, European Journal of Operational Research, Operations Research Letters, Computers and Operations Research, Engineering Optimization, Applied Mathematical Modelling, SIAM Journal on Control and Optimization, Computers and Industrial Engineering, International Journal of Production Economics, IIE Transactions, IEEE Transactions on Reliability, Production Planning and Control, International Journal of Operations and Production Management, The Engineering Economist, Microelectronics and Reliability, Asia-Pacific Journal of Operational Research, TOP, International Journal of Systems Science, Journal of Industrial and Management Optimization, Omega, Applied Mathematics and Computation, Applied Mathematics and Information Sciences, and others.

Pin-Shou Ting is a Professor in the Department of International Management at Shih Chien University in Taipei in Taiwan (Republic of China). He has also served as the Senior Vice President of Shih Chien University since 2000. He joined the faculty at Shih Chien University in 1993 after receiving his Ph.D. degree in Industrial Management from the National Taiwan University of Science and Technology in Taipei in Taiwan (Republic of China). His areas of expertise include Quantitative Methodology, Supply Chain Management, and Logistical Systems. His publications have appeared in such journals as (for example) JORS, EJOR, Applied Mathematics and Computation, and so on.

Shy-Der Lin is a Full Professor in the Department of Applied Mathematics and the Department of Business Administration at Chung Yuan Christian University in Chung-Li in Taiwan (Republic of China). He holds a Ph.D. degree in Technology Management from the National Taiwan University of Science and Technology in Taipei City in Taiwan (Republic of China). His interests include Inventory Management, Financial Engineering, Financial Mathematics, Fractional Calculus, Special Functions and Differential Equations. His work has been published in journals such as Journal of Fractional Calculus, Hiroshima Mathematical Journal, Indian Journal of Pure and Applied Mathematics, International Journal of Quality and Reliability Management, Journal of Operations Research Society, Computer and Industrial Engineering, Journal of Information and Optimization Sciences, Journal of Statistics and Management Systems, Applied Mathematics and Computation, Computers and Mathematics with Applications, Journal of Mathematical Analysis and Applications, Applied Mathematics Letters, Applied Mathematical Modelling, Integral Transforms and Special Functions, Acta Applicandae Mathematicae, Revista de la Academia Canaria de Ciencias, Rendiconti del Seminario Matematico dellUniversita e Politecnico di Torino, Russian Journal of Mathematical Physics, Taiwanese Journal of Mathematics, Applied Mathematics, Czechoslovak Mathematical Journal, African Journal of Business Management, Boundary Value Problems, Filomat, Applied Mathematics and Information Sciences, and other international scientific research journals.

H. M. Srivastava For the author's biographical and other professional details (including the lists of his most recent publications such as Journal Articles, Books, Monographs and Edited Volumes, Book Chapters, Encyclopedia Chapters, Papers in Conference Proceedings, Forewords to Books and Journals, *et cetera*), the interested reader should look into the following Web Site: http://www.math.uvic.ca/faculty/harimsri