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Legendre derivatives direct residual spectral method for solving some types of ordinary differential equations

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Abstract: This paper will focus on determining the approximate numerical solutions for several types of linear and nonlinear Ordinary differential equations. This solution is based on the direct numerical technique, which depends on the Legendre polynomials' derivatives. Then, we will give an approximate solution as a finite sum of the polynomials and unknown coefficients. Finally, some linear and nonlinear differential equations have been solved to show the efficiency and accuracy of the proposed method.

Keywords: Legendre polynomials, Spectral Method, Lane-Emden Equation, Bratu equation.

1 Introduction

Due to the importance of Ordinary differential equations (ODE) [1], many authors are interested in obtaining approximate solutions for several types of ODE. Lane-Emden and Bratu's equations [2,3] are two well-known forms of nonlinear differential equations. Treating these equations is not easy via analytical techniques. So, some researchers use numerical methods for solving these types of equations. Techniques such as finite element, finite difference, and spectral methods are numerical and approximate methods that can resolve many forms of differential equations.

Unlike finite difference and finite element methods, the approximate solution is semi-analytical by applying the spectral method. The basic idea of spectral methods is to approximate the function $y(s)$ by a finite sum of unknown constants and polynomials. Consequently, we can present the solution as follow:

$$y(s) \approx y_n(s) = \sum_{i=0}^{n} a_i p_i(s),$$

where a_i are constants, and $p_i(s)$ is polynomial. After applying spectral methods, the differential equation will convert to a system of equations with unknown constants. Then, one of the numerical methods, such as the Newton or the Gaussian elimination method, will be employed to get the constants. Consequently, this constant is used to obtain the approximate solution.

The spectral methods' importance is in the base function's selection. Several forms of orthogonal polynomials are employed as base functions in the spectral method, Chebyshev polynomials [4,5], Legendre polynomials [6,7,8], and Ultraspherical polynomials [9]. At the same time, the authors in [10,11] used the derivatives of the polynomials. While the authors in [12] applied Bernoulli polynomials.

Also, we have some versions of the spectral method. They are the collocation, Galerkin, and tau methods. The tau and pseudo-Galerkin methods have been applied for solving higher-order ODE in [13,14]. In comparison, the authors choose the collocation method in [15].

The structure of this paper is as follows: The second section presents essential relations of Legendre polynomials. Then, the method used to find the

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approximate solution is discussed in the third section. Additionally, in the fourth section, solving some linear and nonlinear differential equations shows the proposed method's efficiency. The fifth section ended the paper with the concluding remarks.

2 Preliminaries

We will present essential concepts and relationships of the Legendre polynomials (LPs), $L_j(s)$, of degree j, where $s \in [-1,1]$
The LPs are solutions of the differential equation [16]:
$(1-s^2)y''(s) - 2sy'(s) + \lambda y(s) = 0$,
with $\lambda = j(j+1)$.
LPs can be determined via the recurrence relations[16]:

$$(j+1)L_{j+1}(s) = (2j+1)sL_j(s) - jL_{j-1}(s), \tag{1}$$

$$(1-s^2)L'_j(s) = \frac{j(j+1)}{2j+1}(L_{j-1}(s) - L_{j+1}(s)), \tag{2}$$

$$(2j+1)L_j(s) = L'_{j+1}(s) - L'_{j-1}(s), \tag{3}$$

where $L_0(s) = 1$, $L_1(s) = s.$, and $j = 0,1,2,\cdots$
LPs and their derivatives satisfy the following:

$$|L_j(s)| \le 1. \tag{4}$$

$$L'_j(s) = \sum_{\substack{k=0 \\ (j+k)odd}}^{j-1} (2k+1)L_k(s), \tag{5}$$

$$L''_j(s) = \sum_{\substack{k=0 \\ (j+k)even}}^{j-2} (k+\frac{1}{2})[j(j+1) - k(k+1)]L_k(s). \tag{6}$$

The boundary values of LPs and their derivatives:

$$L_j(\pm 1) = (\pm 1)^j, \tag{7}$$

$$L'_j(\pm 1) = \frac{(\pm 1)^{j-1}}{2} j(j+1), \tag{8}$$

$$L''_j(\pm 1) = \frac{(\pm 1)^j}{8}(j-1)j(j+1)(j+2). \tag{9}$$

The LPs in terms of the power of s [7]:

$$L_j(s) = \sum_{k=0}^{[j/2]} \frac{(-1)^k(2j-2k)!}{2^k(j-k)!(n-2k)!k!} s^{j-2k}, \tag{10}$$

where

$$[j/2] = \begin{cases} j/2 & if\ j\ is\ even, \\ (j-1)/2 & if\ j\ is\ odd. \end{cases} \tag{11}$$

In the next section, we will explain the presented method to find approximate solutions for different forms of differential equations.

3 The Proposed Method and Problem Formulation

Consider the ODE:

$$f\left(\beta_r(s)y^{(r)}(s), \beta_{r-1}(s)y^{(r-1)}(s)\right., \cdots, \beta_0(s)y(s), \beta(s)) = 0, \tag{12}$$

for $-1 \le s \le 1$, with the following initial and boundary conditions:

$$\begin{aligned} &y(-1) = \zeta_0, \quad y(1) = \lambda_0, \\ &y'(-1) = \zeta_1, \quad y'(1) = \lambda_1, \\ &\vdots \qquad\qquad \vdots \\ &y^{(n)}(-1) = \zeta_n,\ y^{(m)}(1) = \lambda_m. \end{aligned} \tag{13}$$

The number of constants $\{\zeta_j\}_{j=0}^n$ and $\{\lambda_j\}_{j=0}^m$ equal the order of ODE.

The approximate solution of the given ODE (12,13) can be expanded as follow:

$$\begin{aligned} &y(s) \approx y_n(s) = \sum_{i=0}^n a_i L''_{i+2}(s), \\ &y'(s) \approx y'_n(s) = \sum_{i=0}^n a_i L'''_{i+2}(s), \\ &\vdots \\ &y^{(r)}(s) \approx y_n^{(r)}(s) = \sum_{i=0}^n a_i L_{i+2}^{(r+2)}(s), \end{aligned} \tag{14}$$

where a_i are constant.

By applying equation (14) in equation (12) and conditions (13) we get:

$$f\left(\beta_r(s)\sum_{i=0}^n a_i L_{i+2}^{(r+2)}(s)(s), \beta_{r-1}(s)\sum_{i=0}^n a_i L_{i+2}^{(r+1)}(s), \cdots, \beta_0(s)\sum_{i=0}^n a_i L''_{i+2}(s), \beta(s)\right) = 0, \tag{15}$$

where condition,

$$\begin{aligned} &L''_{i+2}(-1) = \zeta_0, \quad L''_{i+2}(1) = \lambda_0, \\ &L'''_{i+2}(-1) = \zeta_1, \quad L'''_{i+2}(1) = \lambda_1, \\ &\vdots \qquad\qquad \vdots \\ &L_{i+2}^{(n+2)}(-1) = \zeta_n,\ L_{i+2}^{(m+2)}(1) = \lambda_m. \end{aligned} \tag{16}$$

Equations (15) and (16) produce a system of equations with unknown coefficients a_i. Then, we will find the coefficients a_i using one of the numerical methods. Consequently, the approximate solution will be based on derivatives of Legendre polynomials.
The steps of the solution will present in algorithm 1:

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Algorithm 1 Steps for solving ODE via $L''_{n+2}(s)$

Step 1: Enter $n \in \mathbb{N}$;
Step 2: Shift the variable from a defined range to [-1,1];
Step 3: Choose the point $\{s_i\}_{i=0}^{n}$;
Step 5: Expansion the ODE using equation (14);
Step 6: Use the system from step 5 to find a_i;
Step 7: Use the a_i from the previous step to get the approximate solution.

4 Numerical Examples

During this section, we will solve four examples to explain the efficiency and accuracy of the proposed method. Those examples include the Lane-Emden equations, the Bratu equation, and the fourth order differential equation.

*Example 1.*Consider the Non-homogeneous Lane-Emden equation [19]:

$$y''(s)+\frac{8}{s}y'(s)+sy(s)=s^5-s^4+44s^2-30s,\ \ s\in(0,1).(17)$$

$y(0)=0$ *and* $y'(0)=0$, the exact solution $y(s)=s^4-s^3$. By applying our method to solve example (1) and shifting s from (0,1) to (-1,1), using equation (14): $y(s)\approx y_4(s)=\sum_{n=0}^{4}a_nL''_{n+2}=a_0L''_2+a_1L''_3+a_2L''_4+a_3L''_5+a_4L''_6$, then we get the system::

$$3a_0-15a_1+45a_2-105a_3+210a_4=0, \quad (18)$$
$$15a_1-105a_2+420a_3-1260a4=0, \quad (19)$$
$$61440a_0+12659712a_1+35696256a_2+ \quad (20)$$
$$11479776a_3-95734758a_4=-53075, \quad (21)$$
$$2304a_0+333440a_1+1581280a_2+ \quad (22)$$
$$3363920a_3+2942310a_4=2223, \quad (23)$$
$$200704a_0+21364160a_1+143687040a_2+ \quad (24)$$
$$516760160a_3+1277203270a_4=563059, \quad (25)$$

we find: $a_0=-5/252$, $a_1=-1/180$, $a_2=1/1540$, $a_3=1/1260$ and $a_4=1/6930$.
Consequently:
$y_4(s) = \frac{-5}{252}(3) - \frac{1}{180}(15s) + \frac{1}{1540}(\frac{105}{2}s^2 - \frac{15}{2}) + \frac{1}{1260}(\frac{315}{2}s^3 - \frac{105}{2}s) + \frac{1}{6930}(\frac{3465}{8}s^4 - \frac{945}{4}s^2 + 210) = \frac{-1}{16} - \frac{s}{8} + \frac{s^3}{8} + \frac{s^4}{16}$, which equal the exact solution at n=4 for $s\in(-1,1)$.
Comparing with [17] max error = e-07 at n=30, [18] max error=e-11 at n=8 and [19] max error=$e-11$ at n=–, our method is in good agreement with other methods, where the approximate solution equals the exact solution at a small value of n.

*Example 2.*Consider the Non-linear Lane–Emden equation [20,21,22]:

$$y''(s)+\frac{1}{s}y'(s)+e^{y(s)}=0, \quad s\in(0,1). \quad (26)$$

Table 1: The point wise absolute error for Example 2

s	Method	[20]	[21]		[22]
	N=14	N=14	N=14	N=28	N=14
0	2.44e-13	5.79e-12	-	-	6.72e-08
0.1	1.56e-13	3.60e-12	3.14e-10	1.13e-11	6.69e-08
0.2	1.18e-13	2.61e-12	3.07e-10	1.06e-11	7.87e-09
0.3	9.44e-14	2.01e-12	2.99e-10	9.72e-12	6.92e-09
0.4	7.64e-14	1.57e-12	2.88e-10	8.58e-12	2.87e-08
0.5	6.16e-14	1.21e-12	2.82e-10	7.28e-12	7.40e-10
0.6	4.92e-14	8.93e-13	2.14e-10	5.79e-12	6.32e-08
0.7	3.76e-14	6.22e-13	1.51e-10	4.31e-12	6.95e-08
0.8	2.75e-14	3.77e-13	9.45e-11	2.82e-12	3.38e-09
0.9	1.82e-14	1.62e-13	.35e-11	1.34e-12	7.85e-08
1	1.14e-17	8.69e-17	-	-	6.63e-08

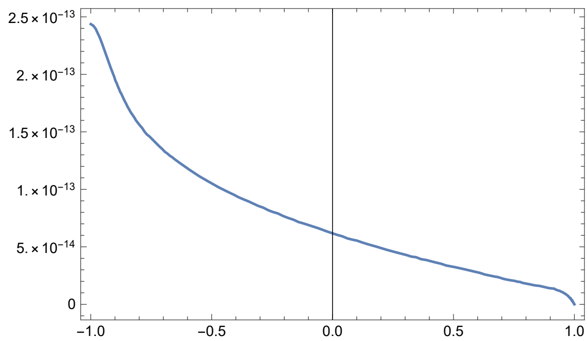

Fig. 1: The point absolute error for Example 2 n=14.

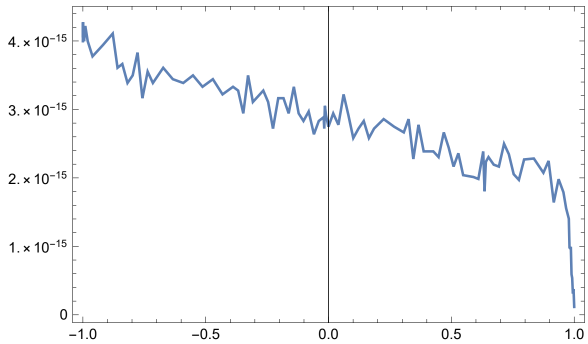

Fig. 2: The point wise absolute error for Example 2 n=16.

$y'(0)=0$ and y(1)=0. The exact solution $y(s)=2\ln(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})s^2+1})$. The point wise Absolute Error (the point wise AE) presented in table (1). Fig (1,2) depict the point wise AE for different value of n, fig (3) comparison of approximate with the exact solution for Example 2 at n=16.

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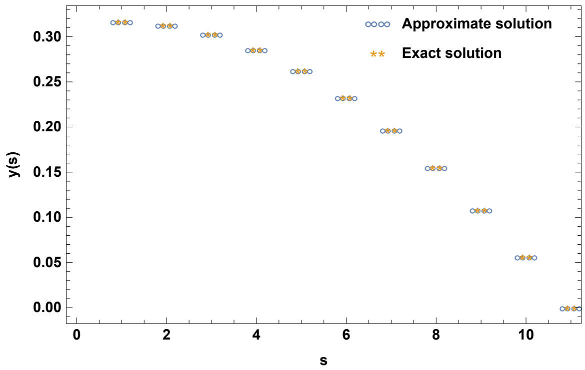

Fig. 3: Comparison between the approximate and the exact solutions for Example 2 n=16.

Table 2: The point wise absolute error for Example 3

s	[23]	Method	
	n=20	n=12	n=14
-1	0	0	0
-0.6	2.17e-14	9.21e-14	2.22e-16
-0.2	3.46e-14	4.00e-14	0
0.2	5.28e-14	6.62e-14	1.11e-16
0.6	1.23e-14	1.09e-13	5.55e-17
1	0	0	0

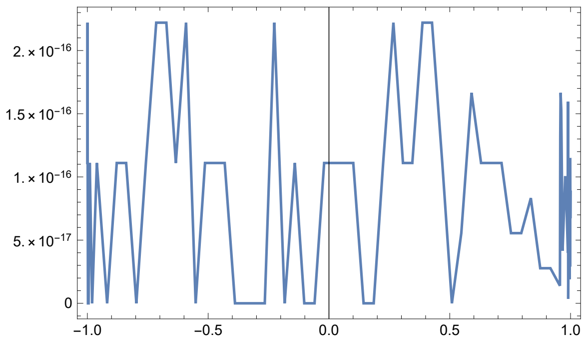

Fig. 4: The point wise absolute error for Example 3 at n=14.

*Example 3.*Consider the one-dimensional fourth-order equation [23]:

$$32y^{(4)}(s) - 8y^{(2)}(s) - 2y(s) = (s-5)e^{\frac{(s+1)}{2}}, \quad (27)$$

for $s \in [-1,1]$, $y(-1) = 1$, $y'(-1) = 0$, $y(1) = 0$, $y'(1) = -\frac{e}{2}$ and the exact solution: $y(s) = \frac{(1-s)}{2}e^{\frac{(1+s)}{2}}$. Table (2) shows the point wise absolute error of example (3) . Fig (4) explains the effect of the proposed method.

*Example 4.*Consider Bratu equation [3]:

$$y''(s) - 2e^{y(s)} = 0, \quad 0 \le s \le 1, \quad (28)$$

Table 3: The point wise absolute error for Example 4

s	Method	[3]	[24]	[25]
0	9.27e-17	-	-	-
0.1	1.87e-08	4.20e-08	1.78e-07	2.99e-04
0.2	4.06e-08	1.72e-07	4.51e-07	0
0.3	6.33e-08	4.05e-07	7.19e-07	1.69e-04
0.4	8.74e-08	7.65e-07	1.01e-06	1.11e-04
0.5	1.14e-07	1.34e-07	1.32e-06	0
0.6	1.43e-07	2.07e-06	1.67e-06	0
0.7	1.76e-07	3.20e-06	2.06e-06	7.77e-05
0.8	2.16e-07	4.88e-06	2.06e-06	0
0.9	2.64e-07	7.36e-06	3.12e-06	3.47e-03
1	3.19e-07	-	-	-

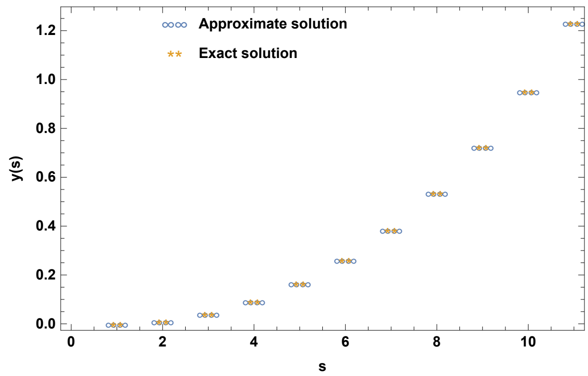

Fig. 5: Comparison between the approximate and the exact solutions for Example 4 n=16.

the exact solution $y(s) = -2ln(cos(s))$, with initial conditions $y(0) = y'(0) = 0$, table (3) shows the point wise absolute error of example (4). Fig (5) report a comparison between the approximate and the exact solutions for Example (4) at n=16.

5 Concluding Remarks

This paper studies a new trail function for solving linear and nonlinear ODE via spectral expansion method. Consequently, some essential relations of the new base function have been presented. Then, the proposed method hae been discussed. Additionally, examples include the Lane-Emden equation, the Bratu equation, and the fourth-order differential equation are solved to show the efficiency of the proposed method.

Availability of data and material

The authors did not use any scientific data during this research.

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Conflict of interest

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All authors contributed equally work.

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