Simplicial Relative Cohomology Rings of Digital Images

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Abstract: The first goal of this paper is to show that the relative cohomology groups of digital images are determined algebraically by the relative homology groups of digital images. Then we state simplicial cup product for digital images and use it to establish ring structure of digital cohomology. Furthermore we give a method for computing the cohomology ring of digital images and give some examples concerning cohomology ring.

Keywords: Digital simplicial relative cohomology group, cup product, cohomology ring.

1 Introduction

In general calculating homology is not enough for determining differences between topological spaces. The cup product in cohomology is finer invariant. The cup product makes the cohomology group of a space into a ring. The ring structure from the cup product is an important advantage of cohomology theory over homology. While the homology groups of a space are equal to the cohomology groups, the ring structure on the cohomologies of the space is different. Then cup product can be used to distinguish the spaces.

Cohomology groups are determined algebraically by the homology groups. We will define the relative cohomology groups of digital images and show that these satisfy basic properties very much like those for the relative homology of digital images.

Although basic properties of cohomology theory are similar to homology theory, there are some differences between them. One of the differences is that cohomology group is contravariant functor while homology group is covariant. Contravariance leads to additional structures in cohomology. These new structures are finer invariants of homotopy type and enable us to distinguish between topological spaces what are called cup products and cohomology operations.

Many researchers(Rosenfeld \cite{24}, Kopperman \cite{20}, Kong \cite{19}, Malgouyres \cite{21}, Boxer \cite{4,5,6,7,9}, Han \cite{11,12}, Karaca \cite{1,10}) have contributed to digital topology with their studies. They wish to characterize the properties of digital images with tools from Algebraic Topology. Their results play an important role in our study.

Arslan, Karaca and Oztel \cite{1}, define simplicial homology group of a digital image and give examples of simplicial homology groups of certain digital images. They also compute simplicial homology groups of $\text{MSS}_{18}$.

Gonzalez-Diaz and Real \cite{15} have their 14-adjacency algorithm to compute cup products on the simplicial complex. The advantage of this method is tried via a small program visualizing the several steps. Gonzalez-Diaz, Jimenez and Medrano \cite{16} introduce a method for computing cup products on cubical approximations. Their cup products are computed directly from the cubical complex. Gonzalez-Diaz, Lamar and Umble \cite{17} present how to simplify the combinatorial structure of cubical complex and obtain a homeomorphic cellular complex with fewer cells. They introduce formulas for a diagonal approximation on a general polygon and use it to compute cup products on the cohomology. The algorithm offered their work can be applied to compute cup products on any polyhedral complex.
approximation of an object embedded in 3-space.

Kaczynski and Mrozek [18] improve a process to compute cup product on cubical complexes for generating a cohomology ring algorithm. This method is useful to the topological analysis of high-dimensional data. Their theory is to construct a cohomology ring algorithm speeding up the algebraic calculations.

Ege and Karaca [10] propose a mathematical framework that can be used for defining cohomology of digital images. They state the Eilenberg-Steenrod axioms for cohomology theory doesn’t be hold in digital images. Moreover they state the cup product for digital images and prove its basic properties.

In Section 2, we review necessary backgrounds on digital topology. In next section we give definitions and theorems that are related to relative cohomology groups of digital images. In the last section we define the simplicial cup product and its general properties. Moreover we give examples about computing the cohomology ring of minimal simple surfaces $\text{MSS}_{18}$ and $\text{MSS}_{18}$.

### 2 Preliminaries

Let $\mathbb{Z}^n$ be the set of lattice points in the $n$-dimensional Euclidean space where $\mathbb{Z}$ is the set of integer. A (binary) digital image is a pair $(X, \kappa)$ where $X \subset \mathbb{Z}^n$ for some positive integer $n$ and $\kappa$ represents certain adjacency relation for the members of $X$. We use a variety of adjacency relations in the study of digital images.

**Definition 21** [5] Let $l, n$ be positive integers, $1 \leq l \leq n$ and distinct two points

$$p = (p_1, p_2, \ldots, p_n), q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$$

$p$ and $q$ are $k_i$-adjacent if there are at most $l$ indices $i$ such that $|p_i - q_i| = 1$ and for all other indices $j$ such that $|p_i - q_i| \neq 1$, $p_j = q_j$.

From Definition 2.1, we have the following:

- Two points $p$ and $q$ in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$.
- Two points $p$ and $q$ in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.
- Two points $p$ and $q$ in $\mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate.
- Two points $p$ and $q$ in $\mathbb{Z}^3$ are 26-adjacent if they are distinct and differ by at most 1 in each coordinate.
- Two points $p$ and $q$ in $\mathbb{Z}^3$ are 18-adjacent if they are 26-adjacent and differ in at most two coordinate.
- Two points $p$ and $q$ in $\mathbb{Z}^3$ are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate.

Let $\kappa \in \{2, 4, 8, 6, 18, 26\}$. A $\kappa$-neighbor of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is $\kappa$-adjacent to $p$ [5]. The $\kappa$-neighborhood of $p$ is defined to be

$$N_\kappa(p) = \{q | q \text{ is } \kappa \text{-adjacent to } p\}.$$ Let $a, b \in \mathbb{Z}$ with $a < b$. A set of the form $[a, b]_\mathbb{Z} = \{z \in \mathbb{Z} | a \leq z \leq b\}$ is called a digital interval [4].

**Definition 22** [14] Let $\kappa$ be an adjacency relation defined on $\mathbb{Z}^n$. A digital image $X \subset \mathbb{Z}^n$ is $\kappa$-connected if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image such that $x = x_0$, $y = x_r$ and $x_i$ and $x_{i+1}$ are $\kappa$-neighbors where $i = 0, 1, \ldots, r - 1$. A $\kappa$-component of a digital image $X$ is a maximal $\kappa$-connected subset of $X$.

**Definition 23** [5] Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with $\kappa_0$-adjacency and $\kappa_1$-adjacency respectively. Then the function $f : X \rightarrow Y$ is said to be $(\kappa_0, \kappa_1)$-continuous if for every $\kappa_0$-connected subset $U$ of $X$, $f(U)$ is a $\kappa_1$-connected subset of $Y$. We say that such a function is digitally continuous.
Proposition 24[5] Let \( X \subseteq \mathbb{Z}^{n_0} \) and \( Y \subseteq \mathbb{Z}^{n_1} \) be digital images with \( \kappa_0 \)-adjacency and \( \kappa_1 \)-adjacency respectively. Then the function \( f : X \rightarrow Y \) is said to be \((\kappa_0, \kappa_1)\)-continuous if and only if for every pair of \( \kappa_0 \)-adjacent points \( \{x_0, x_1\} \) of \( X \), either \( f(x_0) = f(x_1) \) or \( f(x_0) \) and \( f(x_1) \) are \( \kappa_1 \)-adjacent in \( Y \).

A \((2, \kappa)\)-function \( f : [0, m]_\mathbb{Z} \rightarrow X \) such that \( f(0) = x \) and \( f(m) = y \) is called a digital \( \kappa \)-path from \( x \) to \( y \) in a digital image \( X \) [5]. A digital image \( X \) is digital \( \kappa \)-path connected if, for every \( x, y \in X \), there exist a \( \kappa \)-path in \( X \) from \( x \) to \( y \).

A simple closed \( \kappa \)-curve of \( m \geq 4 \) points in a digital image \( X \) is a sequence

\[
\{f(0), f(1), \ldots, f(m-1)\}
\]

of images of the \( \kappa \)-path \( f : [0, m-1]_\mathbb{Z} \rightarrow X \) such that \( f(i) \) and \( f(j) \) are \( \kappa \)-adjacent if and only if \( j = i + 1 \mod m \) [8].

Let \((X, \kappa_0) \subseteq \mathbb{Z}^{n_0}\) and \((Y, \kappa_1) \subseteq \mathbb{Z}^{n_1}\) be digital images. A function \( f : X \rightarrow Y \) is \((\kappa_0, \kappa_1)\)-isomorphism if \( f \) is \((\kappa_0, \kappa_1)\)-continuous and bijective and further \( f^{-1} : Y \rightarrow X \) is \((\kappa_1, \kappa_0)\)-continuous, in which case we denote \( X \cong (\kappa_0, \kappa_1) Y \) [7].

A point \( x \) in \( X \) is called a \( \kappa \)-corner if \( x \) is \( \kappa \)-adjacent to two and only two points \( y, z \in X \) such that \( y \) and \( z \) are \( \kappa \)-adjacent to each other [3]. The \( \kappa \)-corner \( x \) is called simple if \( y \) and \( z \) are not \( \kappa \)-corners and if \( x \) is the only point \( \kappa \)-adjacent to both \( y \) and \( z \) [2]. \( X \) is called a generalized simple closed \( \kappa \)-curve if what is obtained by removing all simple \( \kappa \)-corners of \( X \) is a simple closed \( \kappa \)-curve [21]. For a \( \kappa \)-connected digital image \((X, \kappa)\) in \( \mathbb{Z}^n \), we can state following

\[ |X| = N_{\kappa_0-1}(x) \cap X. \]

Definition 25[12] Let \((X, \kappa) \subseteq \mathbb{Z}^n\) be a digital image, \( n \geq 3 \), and \( X = \mathbb{Z}^n - X \). Then \( X \) is called a closed \( \kappa \)-surface if it satisfies the following:

1. In that case \((\kappa, \overline{\kappa}) \in \{(\kappa, 2\kappa), (2\kappa, 3\kappa - 1)\} \) and \( \kappa \neq 3\kappa - 2\kappa - 1 \), then:

   - for each point \( x \in X \), \( |X|^\kappa \) has exactly one \( \kappa \)-component \( \kappa \)-adjacent to \( x \).
   - \( |X|^\kappa \) has exactly two \( \overline{\kappa} \)-components \( \overline{\kappa} \)-adjacent to \( x \).

2. \( |X|^\kappa \) is a pointed \( \kappa \)-surface.

Furthermore, if a closed \( \kappa \)-surface \( X \) does not have a simple \( \kappa \)-point, then \( X \) is called simple.

Example 26\text{MSS}_{18} \text{ and } \text{MSS}_{19} \text{ are minimal simple closed } 18\text{-surfaces.}

Definition 27[5] Let \((X, \kappa_0) \subseteq \mathbb{Z}^{n_0}\) and \((Y, \kappa_1) \subseteq \mathbb{Z}^{n_1}\) be digital images. Two \((\kappa_0, \kappa_1)\)-continuous functions \( f, g : X \rightarrow Y \) are said to be digitally \((\kappa_0, \kappa_1)\)-homotopic in \( Y \) if there is a positive integer \( m \) and a function \( H : X \times [0, m]_\mathbb{Z} \rightarrow Y \) such that for all \( x \in X \), \( H(x, 0) = f(x) \) and \( H(x, m) = g(x) \); for all \( x \in X \), the induced function \( H_x : [0, m]_\mathbb{Z} \rightarrow Y \) defined by

\[ H_x(t) = H(x, t) \]

is \((2, \kappa_1)\)-continuous; and for all \( t \in [0, m]_\mathbb{Z} \), the induced function \( H_t : X \rightarrow Y \) defined by

\[ H_t(x) = H(x, t) \]

is \((\kappa_0, \kappa_1)\)-continuous. The function \( H \) is called a digital \((\kappa_0, \kappa_1)\)-homotopy [2] between \( f \) and \( g \). A digital image \((X, \kappa)\) is said to be \( \kappa \)-contractible if its identity map \((\kappa, \kappa)\)-homotopic to a constant function \( \tilde{c} \) for some \( c \in X \) where the constant function \( \tilde{c} : X \rightarrow Y \) is defined by \( \tilde{c}(x) = c \) for all \( x \in X \).

For a digital image \((X, \kappa)\) and its subset \((A, \kappa)\), we call \((X, A)\) a digital image pair with \( \kappa \)-adjacency. Moreover, if \( A \) is a singleton set \( x_0 \), then \((X, x_0)\) is called a pointed digital image.

Definition 28[26] Let \( S \) be a set of nonempty subset of a digital image \((X, \kappa)\). Then the members of \( S \) are called simplices of \((X, \kappa)\) if the following hold:

- If \( p \) and \( q \) are distinct points of \( s \in S \), then \( p \) and \( q \) are \( \kappa \)-adjacent.
- If \( s \in S \) and \( \overline{\kappa} \neq t \subset s \), then \( t \in S \).

A \( m \)-simplex is a simplex \( S \) such that \( |S| = m + 1 \). Let \( P \) be a digital \( m \)-simplex. If \( P^0 \) is a nonempty proper subset of \( P \), then \( P^0 \) is called a face of \( P \). We write \( Vert(P) \) to denote the vertex set of \( P \), namely, the set of all digital 0-simplexes in \( P \). A digital subcomplex \( A \) of a digital simplicial complex \( X \) with \( \kappa \)-adjacency is a digital simplicial complex contained in \( X \) with \( Vert(A) \subseteq Vert(X) \).

Let \((X, \kappa)\) be a finite collection of digital \( m \)-simplices, \( 0 \leq m \leq d \) for some non-negative integer \( d \), \((X, \kappa)\) is called a finite digital simplicial complex [1] if the following statements hold:

- If \( P \) belongs to \( X \), then every face of \( P \) also belongs to \( X \).
Let \( \phi \) be a homomorphism. If \( \sigma = [v_0, \ldots, v_q] \) is an oriented simplex with \( 0 < q \leq m \), we define

\[
\partial_q \sigma = \partial_q[v_0, \ldots, v_q] = \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]
\]

where the symbol \( \hat{v}_i \) means that the vertex \( v_i \) is to be deleted from the array. Since \( C_q^\kappa(X) \) is the trivial group for \( q < 0 \), \( m < q \) the operator \( \partial_q \) is the trivial homomorphism for \( q \leq 0 \), \( m < q \).

**Proposition 211** If \( m \geq q \), we have \( \partial_{q-1} \circ \partial_q = 0 \).

**Definition 212** A simplicial complex with \( m \)-dimension. A homomorphism

\[
\partial_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)
\]

is called the boundary operator. If \( \sigma = [v_0, \ldots, v_q] \) is an oriented simplex with \( 0 < q \leq m \), we define

\[
\partial_q \sigma = \partial_q[v_0, \ldots, v_q] = \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]
\]

where the symbol \( \hat{v}_i \) means that the vertex \( v_i \) is to be deleted from the array. Since \( C_q^\kappa(X) \) is the trivial group for \( q < 0 \), \( m < q \) the operator \( \partial_q \) is the trivial homomorphism for \( q \leq 0 \), \( m < q \).

**Theorem 213** If \( (X, \kappa) \subset \mathbb{Z}^n \) is a digital \( \kappa \)-path connected space then \( H_q^\kappa(X) \cong \mathbb{Z} \).

**Example 217** Let’s compute the 0-cocycles of \( \text{MSC}_8^\kappa \).

![Fig. 5: MSC_8](image)

Let \( \text{MSC}_8^\kappa = \{p_0 = (1,2), p_1 = (2,1), p_2 = (3,2), p_3 = (2,3) \} \subset \mathbb{Z}^2 \) and \( p_0 < p_1 < p_2 < p_3 \).

0-simplexes are \( \{p_0\}, \{p_1\}, \{p_2\}, \{p_3\} \) and 1-simplexes are \( e_0 = \{p_0p_1\}, e_1 = \{p_3p_2\}, e_2 = \{p_1p_2\}, e_3 = \{p_0p_3\} \).

We first find the 0-cochains. Since

\[
\partial_1(e_0) = p_3 - p_0 \\
\partial_1(e_1) = p_2 - p_3 \\
\partial_1(e_2) = p_2 - p_1 \\
\partial_1(e_3) = p_1 - p_0
\]

we get 0-cochains,

\[
\partial_0^0 p_0^0 = -e_0 + e_3 \\
\partial_0^0 p_1^0 = -e_2 + e_3 \\
\partial_0^0 p_2^0 = e_1 + e_2 \\
\partial_0^0 p_3^0 = e_0 - e_1
\]

Therefore we have that \( p_0^0 + p_1^0 + p_2^0 + p_3^0 \) is a 0-cocycle.
Definition 218[22] The cohomology group of a digital image \((X, \kappa)\) with coefficients in \(G\) is the group

\[H^0(X; G) = \mathbb{Z}^0(X; G) / B^0(X; G).\]

Theorem 219[10] If \((X, \kappa)\) is a single vertex, then

\[H^0(X) = \{ Z = 0, q = 0 \quad 0, q \neq 0.\]

Theorem 220 If \((X, \kappa) \subset \mathbb{Z}^n\) is a digital \(\kappa\)-path connected space then \(H^0(X) \cong \mathbb{Z}\).

Proof. Assume that \(0\)-simplexes of \(X\) are \(\langle p_0 \rangle, \langle p_1 \rangle, \ldots, \langle p_n \rangle\). We get the following sequence

\[0 \xrightarrow{\delta^{-1}} C^0(X) \xrightarrow{\delta^0} C^1(X).\]

As the image of \(\delta^{-1}\) is zero, \(B^0(X) = \{ 0 \}\). Let us find \(Z^0(X) = \text{Ker} \delta^0\). Let

\[A = \{ \sum_{i=0}^{n} k_ip_i \mid k_i = k, i = 0, 1, \ldots, n \}.\]

We claim that \(Z^0(X) = A\). If this claim is true, it is clear that \(Z^0(X) = \mathbb{Z}\) and we find \(H^0(X) \cong \mathbb{Z}\).

Let us prove the claim. Choose two points \(p_r, p_s \in X\). Since \(X\) is \(\kappa\)-path connected, there is a path \(\sigma_i\) in \(X\) from \(p_r\) to \(p_s\) for each \(i\). \(\sigma_i\) is the set of digital 1-simplexes that \(\kappa\)-path in \(X\) from \(p_r\) to \(p_s\).

\[\sigma_i = \{ \langle p_{r_i}, p_{r+1} \rangle, \langle p_{r+1}, p_{r+2} \rangle, \ldots, \langle p_{s-1}, p_s \rangle \}.\]

Let \(e_k = \langle p_{k_i}, p_{k+1} \rangle\), for \(k = r, r+1, \ldots, s\). We get

\[\sigma_i = \{ e_{r_i}, e_{r+1}, \ldots, e_{s-1} \}.\]

It is clear that

\[\partial_1(e_{r_i}) = p_{r_i+1} - p_{r_i}, \quad \partial_1(e_{r+1}) = p_{r+2} - p_{r+1}, \quad \partial_1(e_{s-2}) = p_{s-1} - p_{s-2}, \quad \partial_1(e_{s-1}) = p_s - p_{s-1}.\]

Hence we have

\[\delta^0(p_{r+1}) = e_{r_i} - e_{r+1}, \quad \delta^0(p_{r+2}) = e_{r+1} - e_{r+2}, \quad \delta^0(p_{s-1}) = e_{s-2} - e_{s-1},\]

Let \(\gamma_i\) is the set of digital 0-simplexes on the path \(\sigma_i\). For \(\omega = \sum k_i \gamma_i \in \mathbb{Z}\), we get

\[\delta^0(\omega) = \delta^0(\sum k_i \gamma_i) = k \sum \delta^0(\gamma_i) = k \sum (e_{r_i} - e_{r+1} - e_{s-1})\]

So we get \(\omega \in Z^0(X)\).

Conversely, if \(\theta \in Z^0(X)\), then \(\theta = \sum_{i=0}^{n} k_ip_i \in C^0(X)\) and

\[\delta^0(\theta) = \delta^0(\sum_{i=0}^{n} k_ip_i) = \sum_{i=0}^{n} k_i \delta^0(p_i) = 0.\]

We have

\[\sum_{i=0}^{n} k_i e_{r_i} - \sum_{i=0}^{n} k_i e_{s-1} - 0.\]

Since \(\sum_{i=0}^{n} k_i e_i = \sum_{i=0}^{n} k_i e_{s-1}, k_i = k, i = 0, \ldots, n\). Therefore we have that \(\theta \in A\). Thus we find that \(Z^0(X) = A \cong \mathbb{Z}\) and

\[H^0(X) = \mathbb{Z}/B^0(X) \cong \mathbb{Z}.\]

Example 221 Let’s compute the cohomology of \(\text{MSS}'_{18}\).

![Fig. 6: MSS'_{18}](image)

Let

\[\text{MSS}'_{18} = \{ p_0 = (1, 1, 0), p_1 = (0, 2, 0), p_2 = (-1, 1, 0), p_3 = (0, 0, 0), p_4 = (0, 1, -1), p_5 = (0, 1, 1) \} \subset \mathbb{Z}^3,\]

where \(p_2 < p_3 < p_4 < p_5 < p_1 < p_0\).

\[C_0(\text{MSS}'_{18}), C_1(\text{MSS}'_{18})\text{ and } C_2(\text{MSS}'_{18})\text{ are free abelian groups with bases, respectively.}\]

0-simplexes

\[\langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_3 \rangle, \langle p_4 \rangle, \langle p_5 \rangle\]

1-simplexes

\[e_0 = \langle p_2p_1 \rangle, e_1 = \langle p_2p_3 \rangle, e_2 = \langle p_2p_4 \rangle, e_3 = \langle p_2p_5 \rangle,\]

\[e_4 = \langle p_4p_1 \rangle, e_5 = \langle p_3p_4 \rangle, e_6 = \langle p_4p_0 \rangle, e_7 = \langle p_3p_1 \rangle,\]

\[e_8 = \langle p_3p_5 \rangle, e_9 = \langle p_5p_0 \rangle, e_{10} = \langle p_1p_0 \rangle, e_{11} = \langle p_3p_0 \rangle\]

and

2-simplexes

\[\sigma_0 = \langle p_2p_4p_1 \rangle, \sigma_1 = \langle p_4p_1p_0 \rangle, \sigma_2 = \langle p_3p_4p_0 \rangle,\]

\[\sigma_3 = \langle p_2p_3p_4 \rangle, \sigma_4 = \langle p_2p_5p_1 \rangle, \sigma_5 = \langle p_2p_3p_5 \rangle,\]

\[\sigma_6 = \langle p_5p_1p_0 \rangle, \sigma_7 = \langle p_3p_5p_0 \rangle\]

Since \(C_m(\text{MSS}'_{18}) \cong \{ 0 \}\) for \(m \geq 3\), we get the following short sequence:
So we can find 0-cochains,
\[ \delta^0 p_0^0 = e_6 + e_9 + e_{10} + e_{11}, \]
\[ \delta^0 p_1^0 = e_0 + e_4 + e_7 - e_{10}. \]
\[ \delta^0 p_2^0 = -e_0 - e_1 - e_2 - e_3. \]
\[ \delta^0 p_3^0 = e_1 - e_4 - e_8 - e_{11}. \]
\[ \delta^0 p_4^0 = -e_2 - e_4 + e_5 - e_6. \]
\[ \delta^0 p_5^0 = e_3 - e_7 + e_8 - e_9. \]

From the definition of a homomorphism \( \partial \), it is easy to see that
\[ \partial_1 (\sigma_0) = \delta_2 ((p_2 p_1 p_1)) = (p_4 p_1) - (p_2 p_1) + (p_2 p_4) = e_4 - e_6 + e_2, \]
\[ \partial_2 (\sigma_1) = \delta_2 ((p_4 p_1 p_0)) = (p_1 p_0) - (p_4 p_0) + (p_4 p_1) = e_{10} - e_6 + e_4, \]
\[ \partial_3 (\sigma_2) = \delta_2 ((p_3 p_1 p_0)) = (p_4 p_0) - (p_3 p_0) + (p_3 p_4) = e_6 - e_1 + e_11, \]
\[ \partial_4 (\sigma_3) = \delta_2 ((p_3 p_2 p_1)) = (p_3 p_1) - (p_2 p_1) + (p_2 p_3) = e_1 - e_2 + e_3, \]
\[ \partial_5 (\sigma_4) = \delta_2 ((p_2 p_2 p_1)) = (p_5 p_1) - (p_2 p_1) + (p_2 p_5) = e_7 - e_0 + e_3, \]
\[ \partial_6 (\sigma_5) = \delta_2 ((p_5 p_1 p_0)) = (p_5 p_0) - (p_3 p_0) + (p_3 p_1) = e_{10} - e_5 + e_6, \]
\[ \partial_7 (\sigma_6) = \delta_2 ((p_3 p_3 p_0)) = (p_5 p_0) - (p_5 p_0) + (p_3 p_1) = e_9 - e_11 + e_6. \]
Thus we can get 1-cochains,
\[ \delta^1 e_0^0 = -\sigma_0 + \sigma_4, \]
\[ \delta^1 e_1^0 = -\sigma_1 + \sigma_5, \]
\[ \delta^1 e_2^0 = -\sigma_0 - \sigma_1 + \sigma_5, \]
\[ \delta^1 e_3^0 = -\sigma_1 + \sigma_5 + \sigma_7, \]
\[ \delta^1 e_4^0 = -\sigma_1 + \sigma_5 + \sigma_7 + \sigma_9, \]
\[ \delta^1 e_5^0 = -\sigma_2 - \sigma_7. \]

We first find the kernel of \( \delta^0 \). We have
\[ \delta^0 (\sum_{i=0}^{5} p_i^0) = n_0 (e_6 + e_9 + e_{10} + e_{11}) + n_1 (e_0 + e_4 + e_7 - e_{10}) + n_2 (e_1 - e_2 - e_3) + n_3 (e_1 - e_4 + e_7 + e_{11}) + n_4 (e_2 + e_4 + e_5 - e_6) + n_5 (e_3 - e_7 + e_8 - e_9). \]

Solving the equation
\[ e_0 (n_1 - n_2) + e_1 (-n_2 + n_3) + e_2 (-n_2 + n_4) + e_3 (-n_2 + n_3) + e_4 (n_1 - n_4) + e_5 (-n_1 + n_4) + e_6 (n_4 - n_5) + e_7 (n_1 - n_5) + e_8 (n_1 + n_5) + e_9 (n_1 - n_2) + e_{10} (n_0 - n_1) + e_{11} (n_0 - n_3) = 0, \]
we must have
\[ n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = n. \]
Hence, we get
\[ Z_{0,18} (\text{MSS}_{18}^-) = \{ n(p_0 + p_1 + p_2 + p_3 + p_4 + p_5) | n \in \mathbb{Z} \} \]
\[ \cong \mathbb{Z}. \]
Since \( B_{0,18} (\text{MSS}_{18}^-) \cong 0 \), the zero dimension cohomology group of \( \text{MSS}_{18}^- \) is isomorphic to the abelian group of integers. Let
\[ \delta^1 (\sum_{i=0}^{11} k_i e_i^0) = k_0 (-\sigma_0 + \sigma_4) + k_1 (\sigma_0 + \sigma_3) + k_2 (\sigma_0 - \sigma_3) + k_3 (\sigma_0 - \sigma_3) + k_4 (\sigma_0 + \sigma_3) + k_1 (\sigma_0 + \sigma_3) + k_6 (\sigma_0 - \sigma_3) + k_7 (\sigma_0 + \sigma_3) + k_8 (\sigma_0 + \sigma_3) + k_9 (\sigma_0 - \sigma_3) + k_{10} (\sigma_0 + \sigma_3) + k_{11} (\sigma_0 - \sigma_3). \]
We find the kernel of \( \delta^1 \). We have the following equation
\[ \sigma_0 (-k_0 + k_2 + k_4) + \sigma_1 (k_4 - k_6 + k_{10}) + \sigma_2 (k_5 + k_6 - k_{11}) + \sigma_3 (k_1 - k_2 + k_3) + \sigma_4 (k_1 + k_7 - k_0) + \sigma_5 (k_1 + k_8 - k_3) + \sigma_6 (k_7 - k_9 + k_{10}) + \sigma_7 (k_8 - k_9 - k_{11}) = 0. \]
Solving the equation above, we find
\[ \begin{align*}
  k_0 &= k_1 + k_4 + k_5, \\
  k_2 &= k_1 + k_5, \\
  k_3 &= k_1 + k_5 + k_4 - k_7, \\
  k_4 &= k_1 + k_6 - k_5, \\
  k_5 &= k_1 + k_7 + k_8, \\
  k_6 &= k_1 + k_{10}, \\
  k_7 &= k_1 + k_5 + k_4 - k_7, \\
  k_8 &= k_1 + k_7 + k_8, \\
  k_9 &= k_1 + k_5 + k_4 - k_7, \\
  k_{10} &= k_1 + k_6 - k_5, \\
  k_{11} &= k_1 + k_5 + k_{10}. 
\end{align*} \]
Hence, we get the group of one dimensional cocycle
\[ Z_{1,18} (\text{MSS}_{18}^-) = \{ (k_1 + k_4 + k_5) e_0^0 + (k_1 + k_5) e_2^0 + (k_1 + k_5) e_4^0 + (k_1 + k_5) e_5^0 + (k_1 + k_4 + k_5) e_6^0 + (k_1 + k_4 + k_5) e_7^0 + (k_1 + k_4 + k_5) e_8^0 + (k_1 + k_4 + k_5) e_9^0 + (k_1 + k_4 + k_5) e_{10}^0 + (k_1 + k_4 + k_5) e_{11}^0. \]
Therefore we conclude that
\[ p = (k_4 + k_1) e_5 + k_7 e_7 + (k_4 + k_5 - k_7) e_9 \]
\[ + (k_4 + k_7) e_8 + k_1 e_9 + (k_4 + k_5 + k_7) e_{11} \]
\[ |k_i \in \mathbb{Z}, i = 1, 4, 5, 7, 10) \cong \mathbb{Z}^5. \]

On the other hand, we obtain the group of one dimensional coboundary
\[ B^{1,18}(\text{MSS}'_{18}) = \{ t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9 \} \]
\[ + (t_0 - t_2) e_4 + (t_2 - t_1) e_5 + t_6 e_6 + (t_0 - t_1) e_7 \]
\[ + (t_3 - t_1) e_5 + (t_2 - t_3 + t_4) e_9 \]
\[ + (-t_0 + t_2 + t_4) e_{10} + (-t_1 + t_2 + t_4) e_{11} \]
\[ |t_i \in \mathbb{Z}, i = 0, 1, 2, 3, 4) \cong \mathbb{Z}^5. \]

Since \( B^{1,18}(\text{MSS}'_{18}) = Z^{1,18}(\text{MSS}'_{18}) \), we have that,
\[ H^{1,18}(\text{MSS}'_{18}) \cong \{ 0 \}. \]

We find that
\[ B^{2,18}(\text{MSS}'_{18}) = \{ h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9 \} \]
\[ + (h_0 - h_1 + h_3 - h_2 + h_4 + h_5) e_6 + (h_5 - h_0 + h_2 - h_3 + h_4 + h_9) e_9 \]
\[ |h_i \in \mathbb{Z}, i = 0, 1, 2, 3, 4, 5, 6) \cong \mathbb{Z}^7. \]

Since \( Z^{2,18}(\text{MSS}'_{18}) \cong \mathbb{Z}^8 \), we have that,
\[ H^{2,18}(\text{MSS}'_{18}) \cong \mathbb{Z}. \]

Therefore we conclude that
\[ H^{3,18}(\text{MSS}'_{18}) = \{ Z, q = 0, 2 \}
\[ 0, q \neq 0, 2. \]

\[ \square \]

**Example 222** Let’s compute the cohomology of \( \text{MSS}_{18} \).

\[ \text{Fig. 7: MSS}_{18} \]

Let \( \text{MSS}_{18} = \{ p_0 = (0, 0, 0), p_1 = (1, 1, 1), \]
\[ p_2 = (1, 2, 1), p_3 = (0, 3, 1), p_4 = (-1, 2, 1), \]
\[ p_5 = (-1, 1, 1), p_6 = (0, 1, 0), p_7 = (0, 2, 0), \]
\[ p_8 = (0, 0, 2), p_9 = (0, 1, 2) \} \subset \mathbb{Z}^{3}, \]
where
\[ p_5 < p_4 < p_0 < p_6 < p_9 < p_7 < p_8 < p_3 < p_1 < p_2. \]

\( C^{0}_{18}(\text{MSS}_{18}), C^{1}_{18}(\text{MSS}_{18}) \text{ and } C^{2}_{18}(\text{MSS}_{18}) \) are free
abelian groups with bases, respectively.

**0-simplexes**
\[ \{ p_0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9 \} \]

1-simplexes
\[ e_0 = (p_0 p_1), e_1 = (p_0 p_9), e_2 = (p_5 p_0), e_3 = (p_0 p_6), \]
\[ e_4 = (p_5 p_9), e_5 = (p_5 p_1), e_6 = (p_5 p_6), e_7 = (p_0 p_1), \]
\[ e_8 = (p_1 p_2), e_9 = (p_5 p_6), e_{10} = (p_6 p_7), e_{11} = (p_5 p_4), \]
\[ e_{12} = (p_8 p_2), e_{13} = (p_4 p_8), e_{14} = (p_8 p_3), e_{15} = (p_4 p_3), \]
\[ e_{16} = (p_4 p_7), e_{17} = (p_3 p_2), e_{18} = (p_7 p_2), e_{19} = (p_7 p_3) \]
and
\[ \text{2-simplexes} \]
\[ \sigma_0 = (p_0 p_9 p_1), \sigma_1 = (p_0 p_6 p_1), \sigma_2 = (p_5 p_0 p_6), \]
\[ \sigma_3 = (p_5 p_9 p_0), \sigma_4 = (p_4 p_8 p_3), \sigma_5 = (p_4 p_7 p_3), \]
\[ \sigma_6 = (p_5 p_3 p_2), \sigma_7 = (p_7 p_5 p_2) \]

Since \( C^{m}_{18}(\text{MSS}_{18}) \) is a trivial group for \( m \geq 3 \), we have
\[ 0 \xrightarrow{\partial_1} C^{1}_{2}(\text{MSS}_{18}) \xrightarrow{\partial_2} C^{1}_{1}(\text{MSS}_{18}) \xrightarrow{\partial_3} C^{1}_{0}(\text{MSS}_{18}) \xrightarrow{\partial_0} 0. \]

By the definition of cochain, we obtain
\[ C^{0}_{18}(\text{MSS}_{18}) \cong \text{Hom}(C^{1}_{18}(\text{MSS}_{18}), \mathbb{Z}), \]
\[ C^{1}_{18}(\text{MSS}_{18}) \cong \text{Hom}(C^{2}_{18}(\text{MSS}_{18}), \mathbb{Z}), \]
\[ C^{2}_{18}(\text{MSS}_{18}) \cong \text{Hom}(C^{3}_{18}(\text{MSS}_{18}), \mathbb{Z}). \]

Hence we get
\[ 0 \xrightarrow{\delta^1} C^{0}_{18}(\text{MSS}_{18}) \xrightarrow{\delta^0} C^{1}_{18}(\text{MSS}_{18}) \xrightarrow{\delta^1} C^{2}_{18}(\text{MSS}_{18}) \xrightarrow{\delta^2} 0. \]

It is easy to see that
\[ \delta^0 p_0 = -e_0 - e_1 + e_2 - e_3, \]
\[ \delta^0 p_1 = e_0 + e_5 + e_7 - e_8, \]
\[ \delta^0 p_2 = e_8 + e_12 + e_17 + e_18, \]
\[ \delta^0 p_3 = e_14 + e_15 + e_17 + e_19, \]
\[ \delta^0 p_4 = e_1 - e_5 - e_14 - e_16, \]
\[ \delta^0 p_5 = -e_2 - e_6 - e_9 - e_11, \]
\[ \delta^0 p_6 = -e_5 - e_7 + e_9 + e_{10}, \]
\[ \delta^0 p_7 = e_10 + e_16 - e_18 - e_19, \]
\[ \delta^0 p_8 = e_4 - e_12 + e_13 - e_{14}, \]
\[ \delta^0 p_9 = e_1 - e_4 - e_5 + e_6. \]
From the definition, we can easily obtain
\[
\begin{align*}
\delta_2(\alpha_1) &= \{ \delta p_1 \} - \{ \delta p_1 \} + \{ \delta p_0 \} = e_5 - e_0 + e_1, \\
\delta_2(\alpha_1) &= \{ \delta p_1 \} - \{ \delta p_1 \} + \{ \delta p_0 \} = e_7 - e_0 + e_3, \\
\delta_2(\alpha_1) &= \{ \delta p_0 \} - \{ \delta p_0 \} - \{ \delta p_0 \} = e_3 - e_0 + e_2, \\
\delta_2(\alpha_1) &= \{ \delta p_0 \} - \{ \delta p_0 \} - \{ \delta p_0 \} = e_1 - e_0 + e_2, \\
\delta_2(\alpha_1) &= \{ \delta p_3 \} - \{ \delta p_3 \} + \{ \delta p_3 \} = e_17 - e_18 + e_19.
\end{align*}
\]
Thus, we get 1-cochains,
\[
\begin{align*}
\delta^1(e_0) &= -\alpha_0 - \alpha_1, \\
\delta^1(e_1) &= -\alpha_0 - \alpha_1, \\
\delta^1(e_2) &= -\alpha_0 - \alpha_1, \\
\delta^1(e_3) &= -\alpha_0 - \alpha_1, \\
\delta^1(e_4) &= \{ \alpha_0 \}, \\
\delta^1(e_5) &= \{ \alpha_0 \}, \\
\delta^1(e_6) &= \{ \alpha_0 \}, \\
\delta^1(e_7) &= \{ \alpha_0 \}, \\
\delta^1(e_8) &= \{ \alpha_0 \}, \\
\delta^1(e_9) &= \{ \alpha_0 \}.
\end{align*}
\]

Let's find the kernel of \(\delta^0\). By the definition of \(\delta^0\), we see that
\[
\delta^0(\sum_{i=0}^{9} n_i p_i) = n_0(-e_0 - e_1 + e_2 - e_3)
\]
\[
+ n_1(e_0 + e_5 + e_7 - e_8)
\]
\[
+ n_2(e_8 + e_12 + e_{17} + e_{18})
\]
\[
+ n_3(e_{14} + e_{15} - e_{17} + e_{19})
\]
\[
+ n_4(e_{11} - e_{13} - e_{15} - e_{16})
\]
\[
+ n_5(-e_2 - e_6 - e_9 - e_{11})
\]
\[
+ n_6(e_3 - e_7 + e_9 - e_{10})
\]
\[
+ n_7(e_{10} + e_{16} - e_{18} - e_{19})
\]
\[
+ n_8(e_4 - e_12 - e_{13} - e_{14})
\]
\[
+ n_9(e_1 - e_4 - e_5 + e_6).
\]

Solving the equation
\[
e_0(-n_0 + n_1) + e_1(-n_0 + n_2) + e_2(n_0 - n_5) + e_3(n_0 - n_5)
\]
\[
+ e_4(-n_0 + n_6) + e_5(n_0 - n_5) + e_6(n_0 - n_5) + e_7(n_0 - n_6) + e_8(n_0 - n_5)
\]
\[
+ e_9(n_0 - n_6) + e_{10}(n_0 - n_7) + e_{11}(n_0 - n_7)
\]
\[
+ e_{12}(n_2 - n_8) + e_{13}(n_4 - n_8) + e_{14}(n_3 - n_8)
\]
\[
+ e_{15}(n_3 - n_4) + e_{16}(n_3 - n_7) + e_{17}(n_2 - n_3)
\]
\[
+ e_{18}(n_2 - n_7) + e_{19}(n_3 - n_7) = 0,
\]
we find
\[
n_0 = n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8 = n_9 = n.
\]
Thus, we get the group of zero dimensional cocycles
\[
Z^0(\text{MSS}_18) = \{ n(p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8) \mid n \in \mathbb{Z} \} \cong \mathbb{Z}.
\]
Since \(B^0(\text{MSS}_18) \cong 0\), we obtain
\[
H^0(\text{MSS}_18) \cong \mathbb{Z}.
\]
Let
\[
\delta^1(\sum_{i=0}^{19} k_i e_i) = k_0(-\alpha_0 - \alpha_1) + k_1(\alpha_0 + \alpha_3) + k_2(\alpha_2 + \alpha_3)
\]
\[
+ k_3(\alpha_3 + \alpha_2) + k_4(\{0\}) + k_5(\sigma_0)
\]
\[
+ k_6(-\alpha_1 + k_7(\sigma_1 + \sigma_8(\{0\})
\]
\[
+ k_7(-\alpha_3 + k_9(\{0\}) + k_{11}(\{0\})
\]
\[
+ k_{12}(-\sigma_3 + k_{13}(\sigma_4 + \sigma_8) + k_{14}(\sigma_4 + \sigma_8)
\]
\[
+ k_{15}(-\sigma_3 + \sigma_8) + k_{16}(\sigma_8)
\]
\[
+ k_{17}(\sigma_3 + \sigma_8) + k_{18}(-\sigma_3 + k_{19}(\sigma_5 + \sigma_7).
\]
We find the kernel of \(\delta^1\). We have
\[
\alpha_0(-k_0 + k_1 + k_5) + \alpha_1(-k_0 + k_3 + k_7)
\]
\[
+ \alpha_2(k_2 + k_3 - k_0) + \alpha_3(k_2 + k_3 - k_0)
\]
\[
+ \alpha_4(k_13 + k_14 - k_15) + \alpha_5(-k_15 + k_16 + k_19)
\]
\[
+ \alpha_6(-k_12 + k_14 + k_17) + \alpha_7(k_17 - k_18 + k_19) = 0.
\]
Solving the equation above, we get
\[
k_0 = k_1 + k_3,
\]
\[
k_6 = k_1 + k_3,
\]
\[
k_7 = k_1 + k_3 - k_3.
\]
\[
k_9 = k_2 + k_3 - k_3.
\]
\[
k_{12} = k_14 + k_17.
\]
\[
k_{15} = k_{13} + k_{14}.
\]
\[
k_{18} = k_{13} + k_{14} - k_{16} + k_17.
\]
\[
k_{19} = k_{13} + k_{14} - k_{16}.
\]
Hence, we obtain
\[
Z^1(\text{MSS}_18) = \{ (k_1 + k_3) e_0 + k_1 e_3 + k_2 e_2 + k_3 e_3
\]
\[
+ k_4 e_0 + k_5 e_2 + k_6 e_5 + k_7 e_0 + k_8 e_5
\]
\[
+ k_9 + k_{10} e_0 + k_{11} e_1 + k_{12} + k_{13} e_1 + k_{14} e_1
\]
\[
+ k_{15} + k_{14} e_1 + k_{16} e_1
\]
\[
+ k_{17} e_1 + (k_{13} + k_{14} - k_{16} + k_{17}) e_1
\]
\[
+ (k_{13} + k_{14} - k_{16} - k_{17}) e_1
\]
\[
\mid k_{i} \in \mathbb{Z}, i = 1, 4, 5, 7, 10 \cong \mathbb{Z}^9.
\]
On the other hand, we obtain
\[
B^1(\text{MSS}_18) = \{ (t_0 + t_1 e_1 + t_2 e_1 + t_3 e_3 + t_4 e_4
\]
\[
+ (t_0 - t_1) e_5 + (t_1 + t_2) e_5 + (t_0 - t_2) e_7
\]
\[
+ t_3 e_8 + (t_2 + t_3) e_6 + t_6 e_{10} + t_7 e_{11}
\]
\[
+ (t_0 - t_1 - t_4 + t_5) e_{12} + (t_1 + t_2 + t_4 - t_7) e_{13}
\]
\[
+ t_{14} e_{14} + (t_1 + t_2 + t_4 - t_7) e_{14}
\]
\[
+ (t_2 + t_3 + t_4 - t_7) e_{16} + (t_0 - t_4 + t_5 - t_8) e_{17}
\]
\[
+ (t_0 - t_3 - t_5 - t_6) e_{18} + (t_1 - t_3 + t_4 - t_6 + t_8) e_{19}
\]
\[
\mid t_i \in \mathbb{Z}, i = 0, 1, 2, 3, 4, 5, 6, 7, 8 \cong \mathbb{Z}^9.
\]
We have that,
\[
H^1(\text{MSS}_18) \cong \mathbb{Z}.
\]
We find that
\[ B^{2,18}(\text{MSS}_{18}) = \{ h_0 \sigma_0 + h_1 \sigma_1 + h_2 \sigma_2 + h_3 \sigma_3 + h_4 \sigma_4 + h_5 \sigma_5 + h_6 \sigma_6 + h_7 \sigma_7 \mid h_i \in \mathbb{Z}, i = 0, 1, 2, 3, 4, 5, 6, 7 \} \cong \mathbb{Z}^8. \]
Since \( Z^{2,18}(\text{MSS}_{18}) \cong \mathbb{Z}^5 \), we have that,
\[ H^{2,18}(\text{MSS}_{18}) \cong \{0\}. \]
We get our result
\[ H^{0,18}(\text{MSS}_{18}) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^3 & q = 1 \\ 0 & q \geq 2. \end{cases} \]

3 Digital Relative Cohomology Groups

**Definition 31** [22] Let \((K, \kappa) \subset \mathbb{Z}^n\) be a digital simplicial complex and \((K_0, \kappa_0)\) a digital subcomplex of \((K, \kappa).\) For any abelian group \(G\) and for any \(p \in \mathbb{Z},\) the \(p\)-dimensional digital relative cochain group
\[ C_p^{\partial,\kappa}(K, K_0; G) = \text{Hom}(C_p^\kappa(K), G) \]
The boundary operator
\[ \partial_p : C_p^{\partial,\kappa}(K, K_0; G) \rightarrow C_{p-1}^{\partial,\kappa}(K, K_0; G) \]
is the dual homomorphism of \(\partial_{p+1}.\)

\[ Z_p^{\partial,\kappa}(K, K_0; G) = \text{kernel of this homomorphism}, \]
\[ B_p^{\partial,\kappa}(K, K_0; G) = \text{image of its. These are called the group of digital relative simplicial} \]
\[ p\text{-cocycles and the group of digital relative simplicial} \]
\[ p\text{-coboundaries, respectively. Then the} \]
\[ p\text{th digital relative simplicial cohomology group of} \]
\[ (K, K_0) \text{is the quotient group} \]
\[ H_p^{\partial,\kappa}(K, K_0; G) = Z_p^{\partial,\kappa}(K, K_0; G) / B_p^{\partial,\kappa}(K, K_0; G) \]
The quotient group \( C_{p}^{\partial,\kappa}(K; G) / C_{p}^{\partial,\kappa}(K_0; G) \) is called the group of relative chains of \(K\) modulo \(K_0\) and is denoted by \( C_{p}^{\partial,\kappa}(K; G). \) \( C_{p}^{\partial,\kappa}(K; G) \) is subgroup of \( C_{p}^{\partial,\kappa}(K; G). \)

A digital relative cochain \(C_{p}^{\partial,\kappa}(K_0; G)\) is a homomorphism from \(C_{p}^{\kappa}(K_0)\) to \(G\). The group of such homomorphisms corresponds precisely to the group of all homomorphisms of \(C_{p}^{\kappa}(K)\) into \(G\) that vanish on the subgroup \(C_{p}^{\kappa}(K_0)\). This is just a subgroup of the group of all homomorphisms of \(C_{p}^{\kappa}(K)\) into \(G\). Thus \(C_{p}^{\partial,\kappa}(K_0; G)\) can be naturally considered to be the subgroup of \(C_{p}^{\partial,\kappa}(K; G)\) consisting of those cochains that vanish on every simplex of \(K_0\).

For chains, we had an exact sequence
\[ 0 \rightarrow C_{p}^{\kappa}(K_0) \xrightarrow{i} C_{p}^{\kappa}(K) \xrightarrow{\partial} C_{p}^{\kappa}(K, K_0) \xrightarrow{j} 0 \]
where \(C_{p}^{\kappa}(K_0)\) is a subgroup of \(C_{p}^{\kappa}(K)\), and \(C_{p}^{\kappa}(K, K_0)\) is their quotient. The sequence splits because the relative chain group is free. Therefore, the sequence
\[ 0 \rightarrow C_{p}^{\partial,\kappa}(K_0; G) \xrightarrow{\tilde{i}} C_{p}^{\partial,\kappa}(K; G) \xrightarrow{\tilde{\partial}} C_{p}^{\partial,\kappa}(K, K_0; G) \rightarrow 0 \]
is exact and splits. The dual of the projection map \(j\) is an inclusion map \(\tilde{j}\) and the dual of the inclusion map \(i\) is a restriction map \(\tilde{i}\).

Let us now consider the homomorphism of cohomology induced by a digital simplicial map. Recall that if \(f : (K, K_0) \rightarrow (L, L_0)\) is a simplicial map, then one has a corresponding chain map
\[ f_{\sharp} : C_{p}^{\partial,\kappa}(K, K_0) \rightarrow C_{p}^{\partial,\kappa}(L, L_0) \]
The dual of \(f_{\sharp}\) maps cochains to cochains; we usually denote it by \(f^{\sharp}\). Because \(f_{\sharp}\) commutes with \(\partial\), the map \(f^{\sharp}\) commutes with \(\partial\), since the dual of the equation \(f_{\sharp} \circ \partial = \partial \circ f_{\sharp}\) is the equation \(\tilde{\delta} \circ f^{\sharp} = f^{\sharp} \circ \delta\). Hence \(f^{\sharp}\) carries cocycles to cocycles and coboundaries to coboundaries. \(f^{\sharp}\) is called a cochain map; it induces a homomorphism of digital cohomology groups,
\[ H_{p}^{\partial,\kappa}(K, K_0; G) \rightarrow H_{p}^{\partial,\kappa}(L, L_0; G) \]
Functoriality holds, even on the cochain level. For if \(i\) is the identity, then \(i_{\sharp}\) is the identity and so is \(i^{\sharp}\). Similarly, the equation \((g \circ f)_{\sharp} = g_{\sharp} \circ f_{\sharp}\) gives, when dualized, the equation \((g \circ f)^{\sharp} = f^{\sharp} \circ g^{\sharp}\).

Just as in the case of homology, one has a long exact sequence in cohomology involving the relative groups. But again, there are a few differences.

**Theorem 32** [22] Let \(K_0\) be a subcomplex of a digital complex \(K\). There exists a long exact sequence
\[ \ldots \rightarrow H_{q}^{\partial,\kappa}(K_0; G) \rightarrow H_{q}^{\partial,\kappa}(K; G) \rightarrow H_{q+1}^{\partial,\kappa}(K_0; G) \rightarrow \ldots \]

**Proof.** If we apply the Zig-zag Lemma to the diagram
\[ \begin{array}{ccc} 0 & \rightarrow & C_{q+1}^{\partial,\kappa}(K_0; G) \xrightarrow{\tilde{i}} C_{q+1}^{\partial,\kappa}(K; G) \xrightarrow{\tilde{j}} C_{q+1}^{\partial,\kappa}(K, K_0; G) \rightarrow 0 \\ \downarrow{\tilde{\delta}} & & \downarrow{\delta} \\ 0 & \rightarrow & C_{q}^{\partial,\kappa}(K_0; G) \xrightarrow{\tilde{i}} C_{q}^{\partial,\kappa}(K; G) \xrightarrow{\tilde{j}} C_{q}^{\partial,\kappa}(K, K_0; G) \rightarrow 0 \end{array} \]
this theorem follows.

**Example 33** We consider \(A = \{(0, 1, 1)\}\) as subspace of \(MSS_{18} = \{p_0 = (1, 1, 0), p_1 = (0, 2, 0), p_2 = (-1, 1, 0),\} \).
Let us compute $H^{q,18}(MSS_{18}^r)$ with Theorem 32. Since $A$ is a single vertex, from Theorem 2.19 we conclude that

$$H^{q,18}(A) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}, & q \neq 0. \end{cases}$$

From Example 221, we get

$$H^{q,18}(MSS_{18}^r) = \begin{cases} \mathbb{Z}, & q = 0, 2 \\ \mathbb{Z}, & q \neq 0, 2. \end{cases}$$

By Theorem 32, we have the exact sequence

$$\cdots \to H^{q,18}(MSS_{18}^r) \to H^{q+1,18}(MSS_{18}^r) \to H^{q+1,18}(A) \to \cdots$$

We get $H^{0,18}(MSS_{18}^r) \cong \mathbb{Z}$, from Theorem 2.19.

$$\delta^0 : \mathbb{Z}^j \to \mathbb{Z}^i \xrightarrow{i^*} \mathbb{Z}^j \to \delta^1 : H^{1,18}(MSS_{18}^r) \xrightarrow{\kappa} 0$$

Applying to the First Isomorphism Theorem, we have

$$H^{1,18}(MSS_{18}^r)/\text{Ker } k^* \cong \text{Im } k^*.$$

Since the sequence is exact, we have $\text{Im } \delta^1 = \text{Ker } k^*$ and $\text{Im } i^* = \text{Ker } \delta^1$. As $i^*$ is isomorphism, we conclude that $\text{Ker } i^* = 0$ and $\text{Im } i^* = \mathbb{Z}$. We find $\text{Ker } \delta^1 = \mathbb{Z}$. Again applying the First Isomorphism Theorem, since

$$\mathbb{Z}/\text{Ker } \delta^1 \cong \text{Im } \delta^1,$$

we have $\text{Im } \delta^1 = 0 = \text{Ker } k^*$ and

$$H^{1,18}(MSS_{18}^r) = 0.$$

Therefore we get

$$H^{q,18}(MSS_{18}^r) = \begin{cases} \mathbb{Z}, & q \neq 0, 2 \\ \mathbb{Z}, & q \neq 0. \end{cases}$$

### 4 Cup Product For Digital Images

**Definition 41** Let $(X, \kappa)$ be a digital simplicial complex. Suppose that the coefficient group $\Gamma$ is the additive group of a commutative ring with identity. The digital simplicial cup product

$$\cup : C^p,\kappa(X, G) \times C^q,\kappa(X, G) \to C^{p+q},\kappa(X, G)$$

of cochains $c^p$ and $c^q$ is defined by the formula

$$c^p \cup c^q = c^p \left( \left[ \cdots, v_{p+q} \right] \right) \left( \left[ \cdots, v_{p+q} \right] \right) \cdots \left( \left[ \cdots, v_p \right] \right)$$

where $v_0 < \cdots < v_{p+q}$ in the given ordering and $'$ is the product in $\Gamma$.

**Theorem 42** Let $\alpha, \alpha_1, \alpha_2 \in H^{p,\kappa}(X, G_1)$ and $\beta, \beta_1, \beta_2 \in H^{q,\kappa}(X, G_2)$, we get

$$(\alpha_1 + \alpha_2) \cup \beta = \alpha_1 \cup \beta + \alpha_2 \cup \beta$$

and

$$\alpha \cup (\beta_1 + \beta_2) = \alpha \cup \beta_1 + \alpha \cup \beta_2.$$
Example 49 Consider MSS$_{18}$.

$$H^{3,18} (MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^2, & q = 1 \\ 0, & q \geq 2 \end{cases}$$

By example 222, we obtain 1-cocycles of simplicial complex:

$$x = e_0^1 + e_1^1 + e_2^1 + e_3^1, \quad a = e_{13}^1 + e_{15}^1 + e_{18}^1 + e_{19}^1,$$

$$y = e_0^2 + e_3^2 + e_9^2, \quad b = e_{16}^2 + e_{17}^1 + e_{19}^1 + e_{12}^1,$$

$$z = e_2^1 - e_3^1 + e_6^1 - e_7^1, \quad c = e_{14}^1 + e_{15} + e_{16} + e_{12}^1,$$

$$r = e_1^1 - e_5^1 + e_9^1 + e_6^1, \quad d = e_{14}^1 - e_{17} + e_{19} - e_{16}^1,$$

$$w = e_0^1 + e_1^1 + e_4^1 - e_2^1, \quad e = e_{14}^1 - e_{15} - e_{17} + e_{19},$$

$$k = e_1^1 - e_5^1 + e_6^1, \quad f = e_{14}^1 - e_{13} + e_{12},$$

$$l = e_5^1 - e_2^1 + e_9^1, \quad g = e_{15} + e_{11}^1 + e_{16},$$

$$u = e_0^1 + e_5^1 + e_7^1, \quad h = e_{17}^1 + e_{12} + e_{18},$$

$$v = e_2^1 + e_0^1 + e_9^1, \quad i = e_{19}^1 - e_{16} + e_{18}^1.$$

We are ready to compute the cup product of 1-cocycles $x, y, z, r, l, u, v, a, b, c, d, e, f, g, h, i$.

$$\langle x \smile z, \sigma_5 \rangle = \langle x \smile [p_5 p_6 p_7] \rangle$$

$$= \langle x, [p_5] \rangle \cdot \langle x, [p_6] \rangle = 0.1 = 1,$$

$$\langle x \smile y, \sigma_6 \rangle = \langle x \smile [p_6 p_9] \rangle$$

$$= \langle x, [p_6] \rangle \cdot \langle x, [p_9] \rangle = 1.1 = 1,$$

$$\langle x \smile z, \sigma_6 \rangle = \langle x \smile [p_6 p_9] \rangle$$

$$= \langle x, [p_6] \rangle \cdot \langle x, [p_9] \rangle = 0.1 = 0.$$
The aim of this paper is to show that some properties from algebraic topology are hold in digital topology. At first relative cohomology groups of digital images are defined and we show that cohomology groups of a digital image are determined the simpler way. Secondly we present that ring structure be existed on the digital simplicial cohomology groups with the cup product. Also we give some examples relevant to computing the cohomology ring of digital images. We expect that these topics will be useful to research digital cohomology operations.

After calculating cap product of other cocycles, we obtain the following table.

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5 Conclusion

The aim of this paper is to show that some properties from algebraic topology are hold in digital topology. At first relative cohomology groups of digital images are defined and we show that cohomology groups of a digital image are determined the simpler way. Secondly we present that ring structure be existed on the digital simplicial cohomology groups with the cup product. Also we give some examples relevant to computing the cohomology ring of digital images. We expect that these topics will be useful to research digital cohomology operations.

References


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