Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/100514

On Hermite-Hadamard Type Integral Inequalities for n-times Differentiable s-Logarithmically Convex Functions With Applications

Muhammad Amer Latif[1,*] *and Sever Silvestru Dragomir*[2,3]

[1] School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
[2] School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC8001, Australia
[3] School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

Received: 11 Apr. 2016, Revised: 10 Jun. 2016, Accepted: 11 Jun. 2016
Published online: 1 Sep. 2016

Abstract: In this paper, we establish Hermite-Hadamard type inequalities for functions whose nth derivatives are s-logarithmically convex functions. From our results, several results for classical trapezoidal and classical midpoint inequalities are obtained in terms second derivatives that are s-logarithmically convex functions as special cases. Finally, applications to special means of the obtained results are given.

Keywords: Hermite-Hadamard's inequality, s-logarithmically convex function, Hölder inequality

1 Introduction

The classical convexity is defined as follows.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1}$$

for all $x, y \in I$ and $\lambda \in [0,1]$. The inequality (1) holds in reverse direction if f is a concave function.

The following double inequality holds

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{2}$$

for convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and is know as the Hermite-Hadamard inequality. The inequality (2) holds in reverse direction if f is a concave function.

The inequality (2) has been subject of extensive research and has been refined and generalized by a number of mathematicians for over one hundred years see for instance [1]-[8], [11]-[15], [18]-[22], [24]-[27] and the references therein.

Many mathematicians are trying to generalize the classical convexity in a number of ways and one of them is so called logarithmically convexity defined as follows.

Definition 2. *[26] If a function $f : I \subseteq \mathbb{R} \to (0,\infty)$ satisfies*

$$f(\lambda x + (1-\lambda)y) \le [f(x)]^{\lambda}[f(y)]^{1-\lambda}, \tag{3}$$

for all $x, y \in I$, $\lambda \in [0,1]$, the function f is called logarithmically convex on I. If the inequality (3) reverses, the function f is called logarithmically concave on I.

The notion of logarithmically convex functions was generalized by Xi el al. in [26].

Definition 3. *[26] For some $s \in (0,1]$, a positive function $f : I \subseteq \mathbb{R} \to (0,\infty)$ is said to be s-logarithmically convex on I if and only if*

$$f(\lambda x + (1-\lambda)y) \le [f(x)]^{\lambda^s}[f(y)]^{(1-\lambda)^s}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

* Corresponding author e-mail: m_amer_latif@hotmail.com

It is obvious that when $s=1$ in Definition 3, the s-logarithmically convex function becomes usual logarithmically convex.

Xi et al. [26] obtained the following Hermite-Hadamard type inequalities for s-logarithmically convex functions.

Theorem 1.[26] *Let $f: I\subseteq[0,\infty)\to(0,\infty)$ be a differentiable function on I°, a, $b\in I^{\circ}$ with $a<b$ and $f^{'}\in L([a,b])$. If $|f(x)|^{q}$ for $q\geq 1$ is s-logarithmically convex on $[a,b]$ for some given $s\in(0,1]$, then*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{3^{(q-1)/q}\left[L_1(\mu,q)\right]^{1/q} + \left[L_2(\mu,q,b)\right]^{1/q}\right\}, \tag{4}$$

where

$$L_1(\mu,q) \leq \begin{cases} \left|f^{'}(a)f^{'}(b)\right|^{sq/2}F_1(\mu_1), & 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1,\\ \left|f^{'}(a)f^{'}(b)\right|^{q/(2s)}F_1(\mu_2), & 1\leq\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|,\\ \left|f^{'}(a)f^{'}(b)\right|^{sq/2}F_1(\mu_3), & 0<\left|f^{(n)}(a)\right|\leq 1<\left|f^{(n)}(b)\right|,\\ \left|f^{'}(a)f^{'}(b)\right|^{q/(2s)}F_1(\mu_4), & 0<\left|f^{(n)}(b)\right|\leq 1<\left|f^{(n)}(a)\right|,\end{cases}$$

$$L_2(\mu,q,u) \leq \begin{cases} \left|f^{'}(u)\right|^{sq/2}F_1(\mu_1), & 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1,\\ \left|f^{'}(u)\right|^{q/(2s)}F_1(\mu_2), & 1\leq\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|,\\ \left|f^{'}(u)\right|^{sq/2}F_1(\mu_3), & 0<\left|f^{(n)}(a)\right|\leq 1<\left|f^{(n)}(b)\right|,\\ \left|f^{'}(u)\right|^{q/(2s)}F_1(\mu_4), & 0<\left|f^{(n)}(b)\right|\leq 1<\left|f^{(n)}(a)\right|,\end{cases}$$

$$F_1(\nu)=\begin{cases}\frac{1}{\ln\nu}\left(2\nu-1-\frac{\nu-1}{\ln\nu}\right) & \nu\neq 1,\\ \frac{3}{2} & \nu=1,\end{cases}$$

$$F_2(\nu)=\begin{cases}\frac{1}{\ln\nu}\left(\nu-\frac{\nu-1}{\ln\nu}\right) & \nu\neq 1,\\ \frac{1}{2} & \nu=1,\end{cases}$$

and

$$\mu_1=\left|\frac{f^{'}(a)}{f^{'}(b)}\right|^{sq/2},\ \mu_2=\left|\frac{f^{'}(a)}{f^{'}(b)}\right|^{q/(2s)},\ \mu_3=\frac{\left|f^{'}(a)\right|^{sq/2}}{\left|f^{'}(b)\right|^{q/(2s)}},\ \mu_4=\frac{\left|f^{'}(a)\right|^{q/(2s)}}{\left|f^{'}(b)\right|^{qs/2}}.$$

Theorem 2.[26] *Under the conditions of Theorem 1, we have*

$$\left|f(b)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[L_2(\mu,q,a)\right]^{1/q} + 3^{(q-1)/q}\left[L_1\left(\mu^{-1},q\right)\right]^{1/q}\right\}, \tag{5}$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ and μ_i for $i=1$, 2, 3, 4 are defined as in Theorem 1.

Theorem 3.[26] *Under the conditions of Theorem 1, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[L_2(\mu,q,b)\right]^{1/q} + \left[L_1\left(\mu^{-1},q,a\right)\right]^{1/q}\right\}, \tag{6}$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ and μ_i for $i=1$, 2, 3, 4 are defined as in Theorem 1.

Applications to special means of positive numbers of the above results are also given in [26].

Motivated by the above definitions and the results, the main purpose of the present paper is to establish new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are s-logarithmically convex. These results not only generalize the results from [26] but many other interesting results can be obtained for functions whose second derivatives in absolute value are s-logarithmically convex which may be better than those from [26].

2 Main Results

First we quote some useful lemmas to prove our mains results.

Lemma 1.[11] *Suppose $f: I\subseteq\mathbb{R}\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n\in\mathbb{N}$, $n\geq 1$. If $f^{(n)}$ is integrable* *on $[a,b]$, for $a,b\in I$ with $a<b$, the equality holds*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\\&-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\\&=\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)f^{(n)}(ta+(1-t)b)dt,\end{aligned}\tag{7}$$

where the sum above takes 0 when $n=1$ and $n=2$.

Lemma 2.[16] *Suppose $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n\in\mathbb{N}$, $n\geq 1$. If $f^{(n)}$ is integrable on $[a,b]$, for $a,b\in I$ with $a<b$, the equality holds*

$$\begin{aligned}&\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\\&=\frac{(-1)(b-a)^n}{n!}\int_0^1 K_n(t)f^{(n)}(ta+(1-t)b)dt,\end{aligned}\tag{8}$$

where

$$K_n(t):=\begin{cases}t^n, & t\in\left[0,\frac{1}{2}\right],\\ (t-1)^n, & t\in\left(\frac{1}{2},1\right].\end{cases}$$

The following useful result will also help us establishing our results.

Lemma 3.[16] *If $\mu>0$ and $\mu\neq 1$, then*

$$\begin{aligned}&\int_0^1 t^n\mu^t dt\\&=\frac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}}+n!\mu\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!(\ln\mu)^{k+1}}.\end{aligned}\tag{9}$$

Lemma 4.[16] *If $\mu>0$ and $\mu\neq 1$, then*

$$\begin{aligned}&\int_0^{\frac{1}{2}} t^n\mu^t dt\\&=\frac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}}+n!\mu^{1/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}(n-k)!(\ln\mu)^{k+1}}.\end{aligned}\tag{10}$$

*Proof.*It follows from Lemma 3 by making use of the substitution $t=\frac{u}{2}$.

Lemma 5.[16] *If $\mu>0$ and $\mu\neq 1$, then*

$$\begin{aligned}&\int_{\frac{1}{2}}^1(1-t)^n\mu^t dt\\&=\frac{n!\mu}{(\ln\mu)^{n+1}}-n!\mu^{1/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}(n-k)!(\ln\mu)^{k+1}}.\end{aligned}\tag{11}$$

*Proof.*It follows from Lemma 4 by making the substitution $1-t=u$.

Lemma 6.[23] *For $\alpha>0$ and $\mu>0$, we have*

$$I(\alpha,\mu):=\int_0^1 t^{\alpha-1}\mu^t dt=\mu\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(\alpha)_k}<\infty,$$

where

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+k-1).$$

Moreover, it holds

$$\begin{aligned}&\left|I(\alpha,\mu)-\mu\sum_{k=1}^{m}(-1)^{k-1}\frac{(\ln\mu)^{k-1}}{(\alpha)_k}\right|\\&\leq\frac{|\ln\mu|}{\alpha\sqrt{2\pi(m-1)}}\left(\frac{|\ln\mu|e}{m-1}\right)^{m-1}.\end{aligned}$$

We are now ready to set off our first result.

Theorem 4.*Let $I\subseteq[0,\infty)$ be an open real interval and let $f:I\to(0,\infty)$ be a function such that $f^{(n)}$ exists on I, a, $b\in I$ with $a<b$ and $f^{(n)}$ is integrable on $[a,b]$ for $n\in\mathbb{N}$, $n\geq 2$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for $q\in[1,\infty)$, $s\in(0,1]$, we have the inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.\\&\left.-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\&\leq\frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\\&\times\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}[F_1(\mu,n)]^{1/q},\end{aligned}\tag{12}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$,

$$F_1(\mu,n)=\begin{cases}\frac{(-1)^n n![\ln\mu+2]}{(\ln\mu)^{n+1}}-\frac{2\mu}{\ln\mu}-n!\mu\sum_{k=1}^n\frac{(-1)^k[\ln\mu+2]}{(n-k)!(\ln\mu)^{k+1}}, & \mu\neq 1,\\ \frac{n-1}{n+1}, & \mu=1,\end{cases}$$

and

$$(\delta,\theta)=\begin{cases}(0,s), & \text{if } 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1,\\ (1-s,1), & \text{if } 1\leq\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|,\\ (0,1), & \text{if } 0<\left|f^{(n)}(a)\right|\leq 1<\left|f^{(n)}(b)\right|,\\ (1-s,s), & \text{if } 0<\left|f^{(n)}(b)\right|\leq 1<\left|f^{(n)}(a)\right|.\end{cases}$$

Proof. Suppose $n \geq 2$. By s-logarithmically convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, Lemma 1 and Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-1/q} \times\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{1/q}. \tag{13}$$

Let $0<\xi\leq 1\leq\eta$, $0\leq\lambda\leq 1$ and $0<s\leq 1$. Then

$$\xi^{\lambda^s}\leq\xi^{s\lambda} \text{ and } \eta^{\lambda^s}\leq\eta^{s\lambda+1-s}. \tag{14}$$

For $0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1$, from (14) and Lemma 3, we have

$$\begin{aligned}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt &\leq \int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qst}\left|f^{(n)}(b)\right|^{sq(1-t)}dt\\ &= \left|f^{(n)}(b)\right|^{sq}\int_0^1 t^{n-1}(n-2t)\mu^t dt\\ &= \left|f^{(n)}(b)\right|^{sq}F_1(\mu,n),\end{aligned} \tag{15}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$.

For $1\leq\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|$, from (14) and by using Lemma 3, we have

$$\begin{aligned}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt &\leq \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{q}\int_0^1 t^{n-1}(n-2t)\mu^t dt\\ &= \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{q}F_1(\mu,n).\end{aligned} \tag{16}$$

For $0<\left|f^{(n)}(a)\right|\leq 1\leq\left|f^{(n)}(b)\right|$, from (14) and by Lemma 3, we obtain

$$\begin{aligned}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt &\leq \left|f^{(n)}(b)\right|^{q}\int_0^1 t^{n-1}(n-2t)\mu^t dt\\ &= \left|f^{(n)}(b)\right|^{q}F_1(\mu,n).\end{aligned} \tag{17}$$

Lastly for $0<\left|f^{(n)}(b)\right|\leq 1\leq\left|f^{(n)}(a)\right|$ from (14) and Lemma 3, we get that

$$\begin{aligned}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt &\leq \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{sq}\int_0^1 t^{n-1}(n-2t)\mu^t dt\\ &= \left|f^{(n)}(b)\right|^{sq}\left|f^{(n)}(a)\right|^{q(1-s)}F_1(\mu,n).\end{aligned} \tag{18}$$

Combining (15), (16), (17) and (18), we get the required result. This completes the proof of the theorem.

Corollary 1. *Suppose the assumptions of Theorem 4 are satisfied and if $q=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n}{2n!}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}F_1(\mu,n), \tag{19}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{s}$, $F_1(\mu,n)$ and (δ,θ) are defined as in Theorem 4.

Corollary 2. *Under the assumptions of Theorem 4, if $n=2$, we have the inequalities*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}\left(\frac{1}{3}\right)^{1-1/q} \times\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left[F_1(\mu,2)\right]^{1/q}, \tag{20}$$

where $\mu=\left|\frac{f''(a)}{f''(b)}\right|^{sq}$,

$$F_1(\mu,2)=\begin{cases}\frac{2(1+\ln\mu)\ln\mu+4(1-\mu)}{(\ln\mu)^3}, & \mu\neq 1,\\ \frac{1}{3}, & \mu=1,\end{cases}$$

and

$$(\delta,\theta)=\begin{cases}(0,s), & \text{if } 0<\left|f''(a)\right|,\left|f''(b)\right|\leq 1,\\ (1-s,1), & \text{if } 1\leq\left|f''(a)\right|,\left|f''(b)\right|,\\ (0,1), & \text{if } 0<\left|f''(a)\right|\leq 1\leq\left|f''(b)\right|,\\ (1-s,s), & \text{if } 0<\left|f''(b)\right|\leq 1\leq\left|f''(a)\right|.\end{cases}$$

Remark. For $s=1$, one can get very interesting inequalities from (12), (19) and (20) for log-convex functions.

Theorem 5. *Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I, a, $b \in I$ with $a < b$ and $f^{(n)}$ is integrable on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for $q \in (1,\infty)$, $s \in (0,1]$, we have the inequality*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right. \\ &\left. - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right| \\ &\leq \frac{(b-a)^n \left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q} n!} \\ &\times \left(\frac{q-1}{2q-1}\right)^{1-1/q} \left|f^{(n)}(a)\right|^{\delta} \left|f^{(n)}(b)\right|^{\theta} [F_2(\mu,n)]^{1/q}, \end{aligned} \tag{21}$$

where $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$,

$$F_2(\mu,n) = \begin{cases} \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(nq-q+1)_k} < \infty, & \mu \neq 1, \\ \frac{1}{nq-q+1}, & \mu = 1, \end{cases}$$

$(nq-q+1)_k = (nq-q+1)(nq-q+2)\cdots(nq-q+k)$ and (δ,θ) are defined as in Theorem 4.

Proof. Since $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for $q \in (1,\infty)$, $s \in (0,1]$, hence from Lemma 1 and the Hölder inequality, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right. \\ &\left. - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right| \\ &\leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (n-2t)^{q/(q-1)}\,dt\right)^{1-1/q} \\ &\times \left(\int_0^1 t^{q(n-1)} \left|f^{(n)}(ta+(1-t)b\right|^q dt\right)^{1/q} \\ &\leq \frac{(b-a)^n}{2^{2-1/q} n!} \left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q} \\ &\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ &\times \left(\int_0^1 t^{q(n-1)} \left|f^{(n)}(a)\right|^{qt^s} \left|f^{(n)}(b)\right|^{q(1-t)^s} dt\right)^{1/q}. \end{aligned} \tag{22}$$

From (14), Lemma 6 and by using similar arguments as in proving Theorem 4, we have the inequality (21). This completes the proof of the theorem.

Corollary 3. *Suppose the assumptions of Theorem 5 are satisfied and $n = 2$. Then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ &\times \left|f''(a)\right|^{\delta} \left|f''(b)\right|^{\theta} [F_2(\mu,2)]^{1/q}, \end{aligned} \tag{23}$$

where $\mu = \left|\frac{f''(a)}{f''(b)}\right|^{sq}$,

$$F_2(\mu,2) = \begin{cases} \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(q+1)_k} < \infty, & \mu \neq 1, \\ \frac{1}{q+1}, & \mu = 1, \end{cases}$$

$(q+1)_k = (q+1)(q+2)\cdots(q+k)$ and (δ,θ) is as defined in Corollary 2.

Corollary 4. *Suppose the assumptions of Theorem 5 are satisfied and $n = 2$, $s = 1$. Then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \left|f''(b)\right| [F_2(\mu,2)]^{1/q}, \end{aligned} \tag{24}$$

where $\mu = \left|\frac{f''(a)}{f''(b)}\right|^{q}$,

$$F_2(\mu,2) = \begin{cases} \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(q+1)_k} < \infty, & \mu \neq 1, \\ \frac{1}{q+1}, & \mu = 1, \end{cases}$$

and $(q+1)_k = (q+1)(q+2)\cdots(q+k)$.

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable s-logarithmically convex functions.

Theorem 6. *Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I, a, $b \in I$ with $a < b$ and $f^{(n)}$ is integrable on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 1$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for $q \in [1,\infty)$, $s \in (0,1]$, we have the inequality*

$$\begin{aligned} &\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \right. \\ &\left. - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq \frac{(b-a)^n \left|f^{(n)}(a)\right|^{\delta} \left|f^{(n)}(b)\right|^{\theta}}{n!\,2^{(n+1)(q-1)/q}(n+1)^{1-1/q}} \\ &\times \left\{[F_3(\mu,n)]^{1/q} + [F_4(\mu,n)]^{1/q}\right\}, \end{aligned} \tag{25}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$,

$$F_3(\mu,n)=\begin{cases}\frac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}}+n!\mu^{1/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}(n-k)!(\ln\mu)^{k+1}}, & \mu\neq 1,\\ \frac{1}{2^{n+1}(n+1)}, & \mu=1,\end{cases}$$

$$F_4(\mu,n)=\begin{cases}\frac{n!\mu}{(\ln\mu)^{n+1}}-n!\mu^{1/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}(n-k)!(\ln\mu)^{k+1}}, & \mu\neq 1,\\ \frac{1}{2^{n+1}(n+1)}, & \mu=1,\end{cases}$$

and (δ,θ) *are defined as in Theorem 4.*

*Proof.*Suppose $n\geq 1$. By using Lemma 2, the s-logarithmically convexity of $\left|f^{(n)}\right|$ and the Hölder inequality, we have

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)^n}{n!}\left[\left(\int_{\frac12}^1(1-t)^n dt\right)^{1-1/q}\right.\\&\times\left(\int_{\frac12}^1(1-t)^n\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{1/q}\\&+\left(\int_0^{\frac12}t^n dt\right)^{1-1/q}\\&\left.\times\left(\int_0^{\frac12}t^n\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{1/q}\right].\end{aligned}\tag{26}$$

From (14), Lemma 4, Lemma 5 and the same reasoning as in proving Theorem 4, we have the required inequality (25). This completes the proof of the theorem.

Corollary 5.*Suppose the assumptions of Theorem 6 are fulfilled and if $q=1$, we have*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)^n\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}}{n!}\\&\times\left\{F_3(\mu,n)+F_4(\mu,n)\right\},\end{aligned}\tag{27}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{s}$ *and* $F_3(\mu,n)$, $F_4(\mu,n)$ *are defined as in Theorem 6, and* (δ,θ) *are defined as in Theorem 4.*

Corollary 6.*Suppose the assumptions of Theorem 6 are fulfilled and if $s=1$, we have*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)^n\left|f^{(n)}(b)\right|}{n!2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\\&\times\left\{[F_3(\mu,n)]^{1-1/q}+[F_4(\mu,n)]^{1-1/q}\right\},\end{aligned}\tag{28}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{q}$ *and* $F_3(\mu,n)$, $F_4(\mu,n)$ *are defined as in Theorem 6.*

Corollary 7.*Suppose the assumptions of Theorem 6 are fulfilled and if $n=1$, we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)}{2^{3(1-1/q)}}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\\&\times\left\{[F_3(\mu,1)]^{1/q}+[F_4(\mu,1)]^{1/q}\right\},\end{aligned}\tag{29}$$

where $\mu=\left|\frac{f'(a)}{f'(b)}\right|^{sq}$,

$$F_3(\mu,1)=\begin{cases}\frac{2+\mu^{1/2}(\ln\mu-2)}{2(\ln\mu)^2}, & \mu\neq 1,\\ \frac18, & \mu=1,\end{cases}$$

$$F_4(\mu,1)=\begin{cases}\frac{2\mu-\mu^{1/2}(\ln\mu-2)}{2(\ln\mu)^2}, & \mu\neq 1,\\ \frac18, & \mu=1,\end{cases}$$

and

$$(\delta,\theta)=\begin{cases}(0,s), & \text{if } 0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1,\\ (1-s,1), & \text{if } 1\leq\left|f'(a)\right|,\left|f'(b)\right|,\\ (0,1), & \text{if } 0<\left|f'(a)\right|\leq 1\leq\left|f'(b)\right|,\\ (1-s,s), & \text{if } 0<\left|f'(b)\right|\leq 1\leq\left|f'(a)\right|.\end{cases}$$

Theorem 7.*Let $I\subseteq[0,\infty)$ be an open real interval and let $f:I\to(0,\infty)$ be a function such that $f^{(n)}$ exists on I, a, $b\in I$ with $a<b$ and $f^{(n)}$ is integrable on $[a,b]$ for $n\in\mathbb{N}$,*

$n \geq 1$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$ for $q \in (1,\infty)$, $s \in (0,1]$, we have the inequality

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right.\\&\left.-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)^n\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}}{2^{n+1/p}(np+1)^{1/p}n!}\\&\times\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\},\end{aligned}\tag{30}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$,

$$F_5(\mu)=\begin{cases}\frac{\mu^{1/2}-1}{\ln\mu}, & \mu\neq1,\\ \frac{1}{2}, & \mu=1,\end{cases}\quad F_6(\mu)=\begin{cases}\frac{\mu-\mu^{1/2}}{\ln\mu}, & \mu\neq1,\\ \frac{1}{2}, & \mu=1,\end{cases}$$

(δ,θ) are defined as in Theorem 4 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, the Hölder integral inequality and s-logarithmically convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right.\\&\left.-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)^n}{n!}\left[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\right.\\&\times\left(\int_0^{\frac{1}{2}}\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(1-t)^{np}dt\right)^{\frac{1}{p}}\\&\left.\times\left(\int_{\frac{1}{2}}^1\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{\frac{1}{q}}\right].\end{aligned}\tag{31}$$

Using (14) and similar arguments as in proving Theorem 4, we get (30). This completes the proof of the theorem.

Corollary 8. *Under the assumptions of Theorem 7, if $n=1$, we have the inequality*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}}{2^{1+1/p}(p+1)^{1/p}}\\&\times\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\},\end{aligned}\tag{32}$$

where $\mu=\left|\frac{f'(a)}{f'(b)}\right|^{sq}$,

$$F_5(\mu)=\begin{cases}\frac{\mu^{1/2}-1}{\ln\mu}, & \mu\neq1,\\ \frac{1}{2}, & \mu=1,\end{cases}\quad F_6(\mu)=\begin{cases}\frac{\mu-\mu^{1/2}}{\ln\mu}, & \mu\neq1,\\ \frac{1}{2}, & \mu=1,\end{cases}$$

(δ,θ) are defined as in Corollary 7 and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 9. *Under the assumptions of Theorem 7, if $s=1$, we have the inequality*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right.\\&\left.-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)^n\left|f^{(n)}(b)\right|}{2^{n+1/p}(np+1)^{1/p}n!}\\&\times\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\},\end{aligned}\tag{33}$$

where

$$F_5(\mu)=\begin{cases}\frac{\mu^{1/2}-1}{\ln\mu}, & \mu\neq1,\\ \frac{1}{2}, & \mu=1,\end{cases}\quad F_6(\mu)=\begin{cases}\frac{\mu-\mu^{1/2}}{\ln\mu}, & \mu\neq1,\\ \frac{1}{2}, & \mu=1,\end{cases}$$

$\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^q$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

3 Applications to Special Means

For positive numbers $a>0$, $b>0$, define

$$A(a,b)=\frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ H(a,b)=\frac{2ab}{a+b},$$

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\\ a & a=b,\end{cases}$$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\neq0,-1\text{ and }a\neq b,\\ \frac{b-a}{\ln b-\ln a}, & p=-1\text{ and }a\neq b,\\ I(a,b), & p=0\text{ and }a\neq b,\\ a, & a=b.\end{cases}$$

It is well known that A, G, H, L=L_{-1}, $I=L_0$ and L_p are called the arithmetic, geometric, harmonic, logarithmic, exponential and generalized logarithmic means of positive numbers a and b.

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

Theorem 8. *Let $0<a<b\leq 1$, $r<0$, $r\neq -1,-2$, $s\in(0,1]$ and $q\geq 1$.*

1. If $r\neq -3$, then

$$\begin{aligned}&\left|A\left(a^{r+2},b^{r+2}\right)-\left[L_{r+2}(a,b)\right]^{r+2}\right|\\&\leq \frac{(b-a)^2}{4}\left(\frac{1}{3}\right)^{1-1/q}|(r+2)(r+1)|\\&\times\left[\frac{2G\left(a^{rq(1-s)},b^{rq(1-s)}\right)}{rqs(\ln b-\ln a)}\right]^{2/q}\\&\times\left[A\left(a^{rqs},b^{rqs}\right)-L\left(a^{rqs},b^{rqs}\right)\right]^{1/q}.\end{aligned}$$

2. If $r=-3$, then

$$\begin{aligned}&\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right|\\&\leq \frac{(b-a)^2}{2}\left(\frac{1}{3}\right)^{1-1/q}\left[\frac{2G\left(a^{-3q(1-s)},b^{-3q(1-s)}\right)}{3qs(\ln a-\ln b)}\right]^{2/q}\\&\times\left[A\left(a^{-3qs},b^{-3qs}\right)-L\left(a^{-3qs},b^{-3qs}\right)\right]^{1/q}.\end{aligned}$$

Proof. Let $f(x)=\frac{x^{r+2}}{(r+2)(r+1)}$ for $0<x\leq 1$. Then $\left|f''(x)\right|=x^r$ and

$$\begin{aligned}&\ln\left|f''(\lambda x+(1-\lambda)y)\right|^q\\&\leq \lambda^s\ln\left|f''(x)\right|^q+(1-\lambda)^s\ln\left|f''(y)\right|^q\end{aligned}$$

for $x,y\in(0,1]$, $\lambda\in[0,1]$, $s\in(0,1]$ and $q\geq 1$. This shows that $\left|f''(x)\right|^q=x^{rq}$ is s-logarithmically convex function on $(0,1]$. Since $\left|f''(a)\right|>\left|f''(b)\right|=b^r\geq 1$, hence

$$\mu=\left|\frac{f''(a)}{f''(b)}\right|^{qs}=\left(\frac{a}{b}\right)^{rqs}$$

and

$$\begin{aligned}&\left|f''(b)\right|^q\left|f''(a)\right|^{q(1-s)}F_1(\mu,2)=2a^{rq(1-s)}b^{rq(1-s)}\\&\times\left[\frac{rqs\left(a^{rqs}+b^{rqs}\right)(\ln a-\ln b)+2\left(b^{rqs}-a^{rqs}\right)}{r^3q^3s^3(\ln a-\ln b)^3}\right]\\&=\left[\frac{4a^{rq(1-s)}b^{rq(1-s)}}{r^2q^2s^2(\ln a-\ln b)^2}\right]\left[\frac{a^{rqs}+b^{rqs}}{2}-\frac{b^{rqs}-a^{rqs}}{rqs(\ln b-\ln a)}\right]\\&=\left[\frac{2G\left(a^{rq(1-s)},b^{rq(1-s)}\right)}{rqs(\ln b-\ln a)}\right]^2\left[A\left(a^{rqs},b^{rqs}\right)-L\left(a^{rqs},b^{rqs}\right)\right].\end{aligned}$$

Substituting the above quantities in Corollary 2, we get the required inequality.

Remark. The other results given above may also give very interesting inequalities containing means and the details are left to the interested reader.

4 Conclusion

In the manuscript, we have provided more general Hermite-Hadamard type inequalities by using the notion of s-logarthimic convexity of the nth derivative of $|f^((n))|^q$, where $q\geq 1$. In order to prove our results, we also have evaluated the integrals of the form $\int_0^1 t^n\mu^t dt$, $\int_0^{\frac{1}{2}} t^n\mu^t dt$ and $\int_{\frac{1}{2}}^1 (1-t)^n\mu^t dt$ for $\mu>0$, $\neq 1$ and $n\geq 1$. Such integrals have not been evaluated in previous works. The results presented in the manuscript not only contain results proved in Xi *et al.* [24] for $n=1$ but also provide refinements of those results concerning Hermite-Hadamard type inequality for the class of s-logarthimically convex functions. We have also given some applications of our results to special means of positive real numbers.

Acknowledgement

We acknowledge and are tankful to the reviewers for the useful comments made. This has enhanced the presentation of the paper with further insight.

References

[1] M. W. Alomari, M. Darus and U. S. Kirmaci, Some inequalities of Hadamard type inequalities for s-convex , Acts Math. Sci. Ser. B Engl. Ed. 31 (2011), no. 4, 1643-1652.
[2] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, Tamkang J. Math. 33 (2002) 45–55.
[3] S. S. Dragomir and S. Fitzpatric, The Hadamard inequalities for s-convex functions in the second sense, Demonstratio Math. 32 (1999), no. 4 687-696.
[4] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91-95.
[5] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.
[6] J. Deng and J. R. Wang, Fractional Hermite-Hadamard inequalities for (α,m)-logarithmically convex functions, J Inequal Appl 2013, 2013:364.
[7] W. D. Jiang, D. W. Niu, Y. Hua and F. Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense, Analysis (Munich) 32 (2012), 1001–1012.

[8] R. F. Bai, F. Qi and B. Y. Xi, Hermite-Hadamard type inequalities for the *m*- and (α,m)-logarithmically convex functions, Filomat 27:1 (2013), 1-7.

[9] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58 (1893), 171–215.

[10] C. Hermite, Sur deux limites d'une intégrale définie, Mathesis 3 (1883), 82.

[11] D. Y. Hwang, Some inequalities for *n*-time differentiable mappings and applications, Kyugpook Math. J. 43(2003), 335-343.

[12] U. S. Kırmacı Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.

[13] U. S. Kırmacı and M. E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153 (2004), 361-368.

[14] U. S. Kırmacı Improvement and further generalization of inequalities for differentiable mappings and applications, Comp and Math. with Appl., 55 (2008), 485-493.

[15] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications, Acta Univ. M. Belii, ser. Math. (2013), 24–39.

[16] M. A. Latif and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for *n*-times differentiable (α,m)-logarithmically convex functions, Miskolc Mathematical Notes, in press.

[17] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, Aequationes Math. 28 (1985), 229–232.

[18] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000), 51–55.

[19] M. Z. Sarikaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for *h*-convex functions, J Math Inequal, Vol. 2, No. 3 (2008), 335-341.

[20] M. Z. Sarikaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, Volume 54, Issues 9-10, November 2011, Pages 2175-2182.

[21] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving *h*-convex functions, Acta Mathematica Universitatis Comenianae, Vol. LXXIX, 2(2010), pp. 265-272.

[22] A. Saglam A, M. Z. Sarikaya and H. Yıldırım, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50 (2010), 399-410.

[23] J. Wang, J. Deng and M. Fečkan, Exploring s-e-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals, Math. Slovaca 64 (2014), No. 6, 1381-1396.

[24] S. H. Wang, B. Y. Xi and F. Qi, Some new inequalities of Hermite-Hadamard type for *n*-time differentiable functions which are *m*-convex, Analysis (Munich) 32 (2012) 247–262.

[25] B. Y. Xi, R. F. Bai and F. Qi, Hermite-Hadamard type inequalities for the *m*- and (α,m)-geometrically convex functions, Aequationes Math. 84(3) (2012) 261-269.

[26] B. Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for *s*-logarithmically convex functions, Acta Math. Sci. Ser. A Chin. Ed. 34 (2014), in press.

[27] T. Y. Zhang, A. P. Ji AP and F. Qi, On integral inequalities of Hermite-Hadamard type for *s*-geometrically convex functions, Abstr. Appl. Anal. 2012 (2012).

Muhammad Amer Latif has received the M. Phil degree in Mathematics from University of Engineering and Technology Pakistan. His research interests are in the areas of Almost periodicity, almost automorphy, mathematical inequalities. He has published research articles in reputed international journals of mathematics. He is referee of some very reputable mathematical journals. He is currently a PhD student under the supervision of Professor S. S. Dragomir at the School of Computational and Applied Mathematics, University of Witwatersrand, Johannesburg South of Africa.

Sever Silvestru Dragomir He is the author of more than 850 papers and 10 books. He is the chair of RGMIA, http//rgmia.org comprising more than 1200 associated members. He is the Editor in Chief of The Australian Journal of Mathematical Analysis and Applications. His research interest are in the areas of Classical Mathematical Analysis, Convex Functions, Best Approximation, Numerical Integration, Geometry of Banach Spaces, Operator Theory, Variational Methods, Volterra Integral Equations, Qualitative Theory of Differential Equations, Theory and Coding, Guessing Theory, Adaptive Quadrature Rules, Adaptive Cubature Rules, Numerical Methods for Differential Equations, Numerical Methods for PDE's, Game Theory, Kolmogorov Complexity. He has been nominated for the Vice-Chancellor Award - Excellence in Staff Research, Team Category, Victoria University and Vice-Chancellors Citation for Excellence in Research, in the category Research Staff, Victoria University in 2001.