A Note on $q$-Analogue of Boole Polynomials

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Abstract: In this paper, we consider the $q$-extensions of Boole polynomials. From those polynomials, we derive some new and interesting properties and identities related to special polynomials.

Keywords: $q$-Boole number, $q$-Boole polynomial, $q$-Euler number, $q$-Euler polynomial

1 Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $Z_p$, $Q_p$ and $C_p$ will denote the ring of $p$-adic integers, the field of $p$-adic numbers and the completion of algebraic closure of $Q_p$. The $p$-adic norm $|·|_p$ is normalized as $|p|_p = 1/p$. The space of continuous functions on $Z_p$ is denoted by $C(Z_p)$. Let $q$ be an indeterminate in $C_p$ with $|1 - q|_p < p^{-1/p - 1}$. The $q$-number of $x$ is defined by $[x]_q = 1 - q^x$. Note that $\lim_{q \to 1} [x]_q = x$. For $f \in C(Z_p)$, the fermionic $p$-adic $q$-integral on $Z_p$ is defined by Kim to be

$$I_q(f) = \int_{Z_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{(pN)_q} \sum_{x=0}^{p^{N-1}} f(x)(-1)^x, \tag{1.1}$$

where $[x]_q = \frac{1 - (-q)^x}{1 + q} \ (\text{see} \ [1 - 9]).$

From (1.1), we note that

$$q^n I_q(f_n) + (-1)^n I_q(f) = [2]_q \sum_{l=0}^{n-1} (-1)^n-1-1 q^l f(l), \tag{1.2}$$

where $f_n(x) = f(x + n), (n \geq 1) \ (\text{see} \ [4]).$

In particular, for $n=1$,

$$q I_q(f_1) + I_q(f) = [2]_q f(0). \tag{1.3}$$

As is well known, the Boole polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty} B_n(x|\lambda) \frac{t^n}{n!} = \frac{1}{1 + (1 + t)^\lambda (1 + t)^x}, \ (\text{see} \ [2, 12]). \tag{1.4}$$

When $\lambda = 1, 2B_n(x|1) = C_n(x)$ are Changhee polynomials which are defined by

$$\frac{2}{l+2} (1+t)^x = \sum_{n=0}^{\infty} C_n(x) \frac{t^n}{n!} \ (\text{see} \ [2, 3, 13, 14]). \tag{1.5}$$

The Euler polynomials of order $\alpha$ are defined by the generating function to be

$$\left( \frac{2}{e^x + 1} \right)^\alpha e^{xu} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \ (\text{see} \ [2, 11]). \tag{1.6}$$

When $x = 0, E_n^{(\alpha)} = E_n^{(\alpha)}(0)$ are called the Euler numbers of order $\alpha$.

In particular, for $\alpha = 1, E_n(x) = E_n^{(1)}(x)$ are called the ordinary Euler polynomials.

The Stirling number of the first kind is given by the generating function to be

$$\log (1 + t)^m = m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!}, (m \geq 0), \tag{1.7}$$

and the Stirling number of the second kind is defined by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!}, \ (\text{see} \ [11, 12]). \tag{1.8}$$

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In this paper, we consider the \( q \)-extensions of Boole polynomials. From those polynomials, we derive new and interesting properties and identities related to special polynomials.

2 \( q \)-analogue of Boole polynomials

In this section, we assume that \( t \in \mathbb{C}_p \) with \( |t|_p < q^{-1} \) and \( \lambda \in \mathbb{Z}_p \) with \( \lambda \neq 0 \). From (1.3), we note that

\[
\int_{\mathbb{Z}_p} (1+t)^{x+\lambda y}d\mu_{-q}(y) = \frac{1+q}{1+q(1+t)}(1+t)^x
\]

\[
= \sum_{n=0}^{\infty} \left[ 2 \right]_q B_{n,q}(x|\lambda) \frac{t^n}{n!}, \tag{2.1}
\]

where \( B_{n,q}(x|\lambda) \) are the \( q \)-Boole polynomials which are defined by

\[
\frac{1}{1+q(1+t)^\lambda} (1+t)^x = \sum_{n=0}^{\infty} B_{n,q}(x|\lambda) \frac{t^n}{n!}. \tag{2.2}
\]

From (2.1), we can derive the following equation:

\[
\int_{\mathbb{Z}_p} \binom{x+\lambda y}{n} d\mu_{-q}(y) = \frac{2}{n!} B_{n,q}(x|\lambda). \tag{2.3}
\]

When \( x = 0, B_{n,q}(\lambda) = B_{n,q}(0|\lambda) \) are called the \( q \)-Boole numbers.

Now, we observe that

\[
(1+t)^{x+\lambda y} = e^{(x+\lambda y)\log(1+t)}
\]

\[
= \sum_{m=0}^{\infty} \frac{(x+\lambda y)^m}{m!} \log(1+t)^m
\]

\[
= \sum_{m=0}^{\infty} \frac{(x+\lambda y)^m}{m!} \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!}
\]

\[
= \sum_{m=0}^{\infty} \left\{ \frac{n}{m} \binom{x+\lambda y}{m} S_1(n,m) \right\} \frac{t^n}{n!}. \tag{2.4}
\]

The \( q \)-Euler polynomials are defined by the generating function to be

\[
\frac{2}{[2]_q qe^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.5}
\]

Note that \( \lim_{q \to 1} E_{n,q}(x) = E_n(x) \).

When \( x = 0, E_{n,q} = E_{n,q}(0) \) are called the \( q \)-Euler numbers.

By (1.3), we easily get

\[
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{2}{[2]_q qe^t + 1}e^{xt}
\]

\[
= \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.6}
\]

Thus, by (2.6), we get

\[
\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = E_{n,q}(x), \quad (n \geq 0). \tag{2.7}
\]

From (2.1), (2.4) and (2.7), we have

\[
\int_{\mathbb{Z}_p} (1+t)^{x+\lambda y} d\mu_{-q}(y)
\]

\[
= \sum_{n=0}^{\infty} \left\{ \frac{n}{m} \binom{x+\lambda y}{m} S_1(n,m) \right\} \frac{t^n}{n!}. \tag{2.8}
\]

Therefore, by (2.1), (2.3) and (2.8), we obtain the following theorem.

**Theorem 1** For \( n \geq 0 \), we have

\[
B_{n,q}(x|\lambda) = \frac{1}{2} \sum_{m=0}^{n} \lambda^m E_{m,q}(\binom{x}{m} S_1(n,m)),
\]

and

\[
\int_{\mathbb{Z}_p} \binom{x+\lambda y}{n} d\mu_{-q}(y) = \frac{2}{n!} B_{n,q}(x|\lambda). \tag{2.9}
\]

From (2.3), we note that

\[
B_{n,q}(x|\lambda) = \frac{1}{2} \int_{\mathbb{Z}_p} \binom{x+\lambda y}{n} d\mu_{-q}(y).
\]

When \( \lambda = 1 \), we have

\[
B_{n,q}(x|1) = \frac{1}{2} \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y). \tag{2.9}
\]

As is known, \( q \)-Chandhep polynomials are defined by the generating function to be

\[
\frac{2}{[2]_q q^t + 1} = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}. \tag{2.10}
\]

Thus, by (2.10), we get

\[
\int_{\mathbb{Z}_p} (1+t)^{x+\gamma y} d\mu_{-q}(y) = \frac{2}{[2]_q q^t + 1} \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}. \tag{2.11}
\]

From (2.11), we have

\[
\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = Ch_{n,q}(x), \tag{2.12}
\]

where \( (x)_n = x(x-1) \cdots (x-n+1) \).

By (2.9) and (2.12), we get

\[
B_{n,q}(x|1) = \frac{1}{2} Ch_{n,q}(x). \tag{2.13}
\]
By replacing \( t \) by \( e^t - 1 \) in (2.2), we see that
\[
\frac{[2]_q}{qe^{t\lambda} + 1} e^{t\lambda} = \frac{[2]_q}{qe^{t\lambda} + 1} \left( \frac{x\lambda}{\lambda} \right)^m (e^t - 1)^n = \frac{[2]_q}{qe^{t\lambda} + 1} \left( \frac{x\lambda}{\lambda} \right)^m S_2(m,n)\frac{t^n}{n!} = \sum_{m=0}^{\infty} \frac{m!}{m!} \left( \frac{x\lambda}{\lambda} \right)^m S_2(m,n)\frac{t^n}{n!}.
\]
(2.14)
and
\[
\frac{[2]_q}{qe^{t\lambda} + 1} e^{t\lambda} = \frac{[2]_q}{qe^{t\lambda} + 1} \left( \frac{x\lambda}{\lambda} \right)^m \sum_{m=0}^{\infty} \frac{m!}{m!} \left( \frac{x\lambda}{\lambda} \right)^m S_2(m,n)\frac{t^n}{n!}.
\]
(2.15)
Therefore, by (2.14) and (2.15), we obtain the following theorem.

**Theorem 2** For \( m \geq 0 \), we have
\[
\sum_{n=0}^{\infty} B_{n,q}(x|\lambda) S_2(m,n) = \frac{1}{[2]_q} \sum_{n=0}^{\infty} E_{m,q} \left( \frac{x}{\lambda} \right)^m.
\]

Let us define the \( q \)-Boole numbers of the first kind with order \( k \) \((\in N)\) as follows:
\[
[2]_q B_{n,q}^{(k)}(\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda(x_1, \cdots, x_k))^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k), \quad (n \geq 0).
\]
(2.16)
Thus, by (2.16), we see that
\[
[2]_q \sum_{n=0}^{\infty} B_{n,q}^{(k)}(\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda(x_1, \cdots, x_k))^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k), \quad (n \geq 0).
\]
(2.17)
Therefore, by (2.17), we obtain the following corollary.

**Corollary 3** For \( n \geq 0 \), we have
\[
B_{n,q}^{(k)}(\lambda) = \sum_{l_1+\cdots+l_k=n} \binom{n}{l_1, \cdots, l_k} B_{l_1,q} \cdots B_{l_k,q}.
\]

The \( q \)-Euler polynomials of order \( k \) are defined by the generating function to be
\[
\frac{\left( \frac{2}{q} \right)}{qe^{t\lambda} + 1} e^{t \lambda} = \frac{\left( \frac{2}{q} \right)}{qe^{t\lambda} + 1} \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.
\]
(2.18)
Thus, by (2.18), we get
\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = E_{n,q}(x).
\]
When \( x = 0, E_{n,q}(x) = E_{n,q}(0) \) are called the \( q \)-Euler numbers of order \( k \).

From (2.16), we note that
\[
[2]_q B_{n,q}^{(k)}(\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda(x_1, \cdots, x_k))^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = \sum_{l=0}^{\infty} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \lambda^l(x_1, \cdots, x_k) d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = \sum_{l=0}^{\infty} S_1(n,l) \lambda^l E_{l,q}^{(k)}.
\]
(2.19)
Therefore, by (2.19), we obtain the following theorem.

**Theorem 4** For \( n \geq 0 \), we have
\[
B_{n,q}^{(k)}(\lambda) = \frac{1}{[2]_q} \sum_{l=0}^{n} S_1(n,l) \lambda^l E_{l,q}^{(k)}.
\]

By replacing \( t \) by \( e^t - 1 \) in (2.17), we get
\[
\frac{[2]_q}{qe^{t\lambda} + 1} e^{t\lambda} = \frac{[2]_q}{qe^{t\lambda} + 1} \left( \frac{x\lambda}{\lambda} \right)^m \sum_{m=0}^{\infty} \frac{m!}{m!} \left( \frac{x\lambda}{\lambda} \right)^m S_2(m,n)\frac{t^n}{n!} = \sum_{m=0}^{\infty} \frac{m!}{m!} \left( \frac{x\lambda}{\lambda} \right)^m S_2(m,n)\frac{t^n}{n!}.
\]
(2.20)
and
\[
[2]_q \sum_{n=0}^{\infty} B_{n,q}^{(k)}(\lambda) = \frac{1}{[2]_q} \sum_{m=0}^{\infty} S_2(m,n)\frac{t^n}{n!}.
\]
(2.21)
Therefore, by (2.20) and (2.21), we obtain the following theorem.

**Theorem 5** For \( m \geq 0 \), we have
\[
\sum_{n=0}^{m} B_{n,q}^{(k)}(\lambda) S_2(m,n) = \frac{1}{[2]_q} E_{m,q}^{(k)} \lambda^m.
\]

Let us define the higher-order \( q \)-Boole polynomials of the first kind as follows:
\[
[2]_q B_{n,q}^{(k)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda(x_1, \cdots, x_k)+x)^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k), \quad \text{where} \quad n \geq 0 \quad \text{and} \quad k \in N.
\]
(2.22)
From (2.22), we can derive the generating function of the higher-order $q$-Boole polynomials of the first kind as follows:

$$\left[2q\right]^k \sum_{n=0}^{\infty} B_{n,q}(x) \frac{(\lambda t)^n}{n!} = \int_{Z_p} \cdots \int_{Z_p} (1+t)^{\lambda x_1 + \cdots + \lambda x_k + x_n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

By (2.17), we easily get

$$\left(\frac{[2q]}{1+q(1+t)^x}\right)^k (1+t)^x$$

Thus, by (2.28), we get

$$\mathcal{B}_{n,q}(x) = \left[\frac{2q}{2q}\right] \sum_{n=0}^{\infty} S_1(n,l) \lambda^l E_{l,q} \left(\frac{x}{\lambda}\right).$$

Therefore, by (2.27), we obtain the following theorem.

**Theorem 8** For $n \geq 0, k \in \mathbb{N}$, we have

$$B_{n,q}^{(k)}(x|\lambda) = \frac{1}{2q} \sum_{l=0}^{n} S_1(n,l) \lambda^l E_{l,q} \left(\frac{x}{\lambda}\right).$$

Now, we consider the $q$-analogue of Boole polynomials of the second kind as follows:

$$\mathcal{B}_{n,q}(x) = \frac{1}{2q} \int_{Z_p} (-\lambda y + x) d\mu_{-q}(y), (n \geq 0).$$

Thus, by (2.28), we get

$$\mathcal{B}_{n,q}(x) = \frac{1}{2q} \sum_{l=0}^{n} S_1(n,l) \lambda^l E_{l,q} \left(\frac{x}{\lambda}\right).$$

When $x = 0, \mathcal{B}_{n,q}(\lambda) = \mathcal{B}_{n,q}(0|\lambda)$ are called the $q$-Boole numbers of the second kind. From (2.28), we can derive the generating function of $\mathcal{B}_{n,q}(x|\lambda)$ as follows:

$$\sum_{n=0}^{\infty} \frac{\mathcal{B}_{n,q}(x|\lambda)}{n!} = \frac{1}{2q} \int_{Z_p} (1+t)^{-\lambda y + x} d\mu_{-q}(y)$$

By replacing $t$ by $e^{rt}$ in (2.30), we get

$$\sum_{n=0}^{\infty} \frac{\mathcal{B}_{n,q}(x|\lambda)}{n!} = \frac{e^{rt} \lambda^t}{q e^{-\lambda t} + 1}.$$
Therefore, by (2.31) and (2.32), we obtain the following theorem.

**Theorem 9** For \( m \geq 0 \), we have

\[
\frac{(-1)^m \lambda^m}{2^m q} E_{m,q}(-\frac{x}{\lambda}) = \sum_{n=0}^{m} \tilde{B}_{n,q}(x|\lambda) S_2(m,n),
\]

and

\[
\tilde{B}_{n,q}(x|\lambda) = \frac{1}{2^m q} \sum_{l=0}^{m} S_2(l,m) (-1)^l E_{l,q}(-\frac{x}{\lambda}).
\]

For \( k \in \mathbb{N} \), let us define the \( q \)-Boole polynomials of the second kind with order \( k \) as follows:

\[
\tilde{B}^{(k)}_{n,q}(x|\lambda) = \frac{1}{2^m q} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( (-\lambda x_1 + \cdots + \lambda x_k) + x \right)_n \prod_{l=1}^{k} d\mu_{-q}(x_l).
\]

Then we have

\[
[2^k q] \tilde{B}^{(k)}_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_2(l,m) (-1)^l E_{l,q}(-\frac{x}{\lambda}).
\]

From (2.33), we can derive the generating function of \( \tilde{B}^{(k)}_{n,q}(x|\lambda) \) as follows:

\[
\sum_{n=0}^{\infty} \tilde{B}^{(k)}_{n,q}(x|\lambda) \frac{t^n}{n!} = \frac{1}{2^k q} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{-(\lambda x_1 + \cdots + \lambda x_k) + x} \prod_{l=1}^{k} d\mu_{-q}(x_l) = \left( \frac{(1+t)^{\lambda}}{q(1+t)^{\lambda+1}} \right)^k (1+t)^x.
\]

Thus, by (2.34), we get

\[
\tilde{B}^{(k)}_{n,q}(x|\lambda) = B^{(k)}_{n,q}(x - \lambda), \quad (n \geq 0).
\]

Indeed,

\[
(-1)^n \frac{[2^k q] \tilde{B}_{n,q}(x|\lambda)}{n!} = (-1)^n \int_{\mathbb{Z}_p} \sum_{m=0}^{n} \frac{n-1}{m} \frac{m}{n-m} \left( \frac{-\lambda - x + n - 1}{n} \right) d\mu_{-q}(y) = \frac{[2^k q]}{n!} \sum_{m=1}^{n} \frac{(n-1)!}{m!} \int_{\mathbb{Z}_p} \left( \frac{-\lambda - x + m}{m} \right) d\mu_{-q}(y).
\]

References


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