Quantum Information Review
An International Journal

http://dx.doi.org/10.12785/qir/paper

Hopf Bifurcations in a Delayed-Energy-Based Model of Capital Accumulation

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Received: 7 Jun. 2012, Revised: 21 Sep. 2012, Accepted: 23 Sep. 2012
Published online: 1 Jan. 2013

Abstract: Building on a contribution by Dalgaard and Strulik [C.L. Dalgaard and H. Strulik, Resource and Energy Economics **33**, 782 (2011)], this paper deals with the mathematical modelling for an economy viewed as a transport network for energy in which the law of motion of capital occurs with a time delay. By choosing time delay as a bifurcation parameter, it is proved that the system loses stability and a Hopf bifurcation occurs when time delay passes through critical values. An important scenario arising from the analysis is the existence of limit cycles generated by supercritical Hopf bifurcations. The results are of great interest for the analysis of the asymptotic economic growth.

Keywords: Dalgaard-Strulik model, energy, economic growth, time delay, limit cycle

1 Introduction

The research of basic principles for the modelling of the economic growth are nowadays a new and hot challenge which has been object of several investigations. Since 1982 Spencer has advocated that the economic growth of societies depends on their capability to exploit the increasing amounts of energy [1]. Accordingly, the quantity of energy that a society consumes becomes an economic tool to measure its progress and thus the capital accumulation represents an important strategy for the growth process, see [2] and [3]. In particular the Solow model [3], which involves the aggregate production function, has given an important contribution to the economic growth theory especially because it has been proven to be able to explain the cross-country differences in GDP per worker. However, as discussed in [4], the derivation of a law of motion for capital without recurring to the existence of an aggregate production function could be more appropriated.

Recently, some principles of the physics and biology have been proposed for the modelling of the law of motion for capital per worker, see, among others, [5,7,8,9]. Similarly to paper [10] where a growth model for living tissue has been derived by assuming that energy is required to cells for their survival and reproduction (*thermodynamics conservation principle*), it is assumed that the capital stock increases if total energy expenditure exceeds the energy costs. Another principle is referred to Kleiber's law [11], which states the correlation between the energy consumption of biological organisms (*basal metabolism*) and their energy requirements (*body mass*). Specifically the biological systems are viewed as energy transporting networks and the Kleiber's law models the diffusion and absorbtion of energy. The previous principles refer to biological networks that have been developed through natural selection, which has produced more efficient networks. Similarly, these principles can be also applied to man-made networks, see [12], where the authors have applied these principles to artificial networks with the aim to discover universal laws with applications to human societies. Moreover mathematical models have been developed in [5] and [6] for an economy viewed as a transport network for energy. In these models the energy consumption per worker is seen as the counterpart to metabolism, and capital per worker as the counterpart to body size.

Recently, Dalgaard and Strulik [6] have developed a mathematical model of an economy viewed as a transportation network for electricity that is mathematically isomorphic to the Solow-Swan model proposed in papers [3,13]. The model is based on the

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main assumption that there exists a supply relation, which is a concave and log-linear Kleiber's law relation, between the electricity consumption per capita (viewed as the economic counterpart to metabolism) and capital per capita (viewed as the counterpart to body size). This paper is concerned with a generalization of the Dalgaard and Strulik model analyzed in [6]. Specifically, the energy conservation equation contains a time delay which take cares of the previous occurring dynamics. Generally, delay in dynamical systems is exhibited whenever the system's behavior is dependent at least in part on its history. The introduction of time delay is a common approach used in biology for instance in the modelling of gene expression, cell division, as well as cell differentiation and cell maturation, with the aim to be more consistent with the cell growth kinetics, see the review paper [14], papers [15,16,17] and the references therein. This work is motivated by economical applications to plan the asymptotic economic growth [22].

The present paper is organized as follows. After this introduction Section 2 reviews the original model by Dalgaard and Strulik and deals with the generalization which includes the delay. In Section 3 by choosing time delay as a bifurcation parameter, and applying the local Hopf bifurcation theory (see e.g. [18]), we investigate the existence of stable periodic oscillations for equation. More specifically, we prove that, as the delay T increases, the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur. Furthermore in Section 4 by using the Lindstedt's perturbation method [19], we prove that the Hopf bifurcation is supercritical and the bifurcating solutions are stable. Finally Section 5 is devoted to research perspective.

2 The delayed Dalgaard and Strulik model

As already mentioned in the introduction the mathematical model of Dalgaard and Strulik [6] is concerned with the modelling of an economy viewed as a transportation network for electricity. Electricity is used to run, maintain, and create capital.

Assuming that time is continuous, and let μ be the energy requirement to operate and maintain the generic capital good while ν is the energy costs to create a new capital good, energy conservation implies

$$e(t) = \mu k(t) + \nu \frac{dk(t)}{dt}, \tag{1}$$

where $k(t)$ denotes capital stock. Equation (1), which provides a metric for aggregation of capital, captures the electricity at any given instant in time; the right-hand side of (1) summarizes the instantaneous electricity requirements (the size of population has been normalized to one).

It is worth stressing that if we were to shut off energy supply entirely, namely $e(t) = 0$, the capital stock would shrink over time, due to lack of maintenance and replacement. The rate at which the stock shrink is $-\mu/\nu$, which therefore can be viewed as the mirror image of the depreciation rate, commonly introduced in models of growth and capital accumulation.

Bearing all above in mind the Dalgaard and Strulik mathematical model [6] is derived by modelling the energy as $e(t) = \varepsilon[k(t)]^a$ where $0 < a < 1$ is a real constant proportional to the dimension and efficiency of the network, and $\varepsilon > 0$ is a real constant in the sense that it is independent of capital per worker. The model thus reads:

$$\frac{dk(t)}{dt} = \frac{\varepsilon}{\nu}[k(t)]^a - \frac{\mu}{\nu}k(t). \tag{2}$$

The Dalgaard and Strulik model shares the technical properties with the Solow model. In particular, there exists a unique globally stable steady-state to which the economy adjusts.

In what follows we consider a generalization of the Dalgaard and Strulik model [6]. Specifically, it is assumed that the energy conservation equation contains a time delay T which is introduced in the equation (1) as follows:

$$e(t-T) = \mu k(t-T) + \nu \frac{dk(t)}{dt}, \tag{3}$$

Consequently, the law of motion for capital is described by the following non-linear delay differential equation:

$$\frac{dk(t)}{dt} = \frac{\varepsilon}{\nu}[k(t-T)]^a - \frac{\mu}{\nu}k(t-T), \tag{4}$$

for some initial function $k(t) = \phi(t), t \in [-T, 0]$.

According to the mathematical model (4), at any given instant in time t, the capital stock $k(t)$ is determined by the electricity at the instant in time $t-T$.

3 Existence and analysis of Hopf bifurcations

Equilibria (or steady states in the language of the economical sciences) of equation (4), of course, coincide with the corresponding points for zero delay, $T = 0$. Hence, there exists a unique positive steady state k_* satisfying the relation $\varepsilon k_*^{a-1} = \mu$. After setting the following translation $x(t) = k(t) - k_*$, equation (4) is rewritten as follows:

$$\frac{dx(t)}{dt} = \frac{\varepsilon}{\nu}[x(t-T)+k_*]^a - \frac{\mu}{\nu}[x(t-T)+k_*]. \tag{5}$$

The following theorem characterizes the nature of the equilibrium point k_*.

Theorem 3.1. Let k_* be the unique positive equilibrium for the mathematical model (4). Then there exists a positive number T_0 such that the equilibrium k_* is asymptotically stable for $T \in [0, T_0)$ and unstable for $T > T_0$. Moreover equation (4) undergoes a Hopf bifurcation at k_* when $T = T_j$ where

$$T_j = \frac{1}{\omega_0}\left(\frac{\pi}{2}+2j\pi\right), \quad j\in\{0,1,2,\dots\},$$

and

$$\omega_0 = \frac{(1-a)\,\mu}{v}.$$

Proof. As is well known, the stability of the positive steady state and local Hopf bifurcations can be determined by the distribution of the roots associated with the characteristic equation of its linearization [20]. The linearization of (5) at zero is

$$\frac{dx(t)}{dt} = \frac{(a-1)\mu}{v}x(t-T). \tag{6}$$

By substituting candidate solutions of the form $e^{-\lambda T}$ into equation (6), we get that the corresponding characteristic equation of (6) is given by

$$\lambda = \frac{(a-1)\mu}{v}e^{-\lambda T}. \tag{7}$$

Equation (7) is a quasi-polynomial, which exhibits an infinite number of (complex) roots. Notice that, when $T = 0$, $x_* = 0$ is asymptotically stable because $\lambda = (a-1)\mu/v < 0$.
Let $i\omega$ $(\omega > 0)$ be a root of equation (7). Then we have

$$i\omega = \frac{(a-1)\mu}{v}e^{-i\omega T}.$$

Separation in the real and imaginary parts implies that

$$\frac{(a-1)\mu}{v}\cos\omega T = 0, \quad \omega = -\frac{(a-1)\,\mu}{v}\sin\omega T. \tag{8}$$

Squaring and adding the both equations in (8), we get $\omega^2 = (a-1)^2\mu^2/v^2$. Consequently, we can conclude that equation (7) has a unique pair of purely imaginary roots $\pm i\omega_0$, where

$$\omega_0 = \frac{(1-a)\,\mu}{v}. \tag{9}$$

From the equations in (8), we can define

$$T_j = \frac{1}{\omega_0}\left(\frac{\pi}{2}+2j\pi\right), \quad j\in\{0,1,2,\dots\}.$$

Let $\lambda_j(T) = \alpha_j(T)+i\omega_j(T)$ denote a root of equation (7) near $T = T_j$ satisfying $\alpha_j(T_j) = 0$ and $\omega_j(T_j) = \omega_0$. Differentiating the characteristic equation (7) with respect to T, we obtain

$$\frac{d\lambda}{dT} = \frac{(1-a)\mu}{v}e^{-\lambda T}\left(T\frac{d\lambda}{dT}+\lambda\right).$$

Hence, we have

$$\left(\frac{d\lambda}{dT}\right)^{-1} = -\frac{1}{\lambda^2}-\frac{T}{\lambda}.$$

This implies that

$$\begin{aligned} sign\left[\left.\frac{dv_k(T)}{dT}\right|_{T=T_j}\right] &= sign\left[Re\left.\left(\frac{d\lambda}{dT}\right)^{-1}\right|_{T_j}\right] \\ &= sign\left[Re\left.\left(-\frac{1}{\lambda^2}-\frac{T}{\lambda}\right)\right|_{T_j}\right] \\ &= sign\left(\frac{1}{\omega_0^2}\right). \end{aligned} \tag{10}$$

Thus, from (10) we have that $\alpha_j'(T_j) > 0$, implying that all the roots crossing the imaginary axis at $i\omega_0$ cross from left to right as T increases and thus this results in the loss of stability. We have found that if $T \in [0,T_0)$, then all roots of equation (7) have negative real parts. If $T = T_0$, then all roots of equation (7), except $\pm i\omega_0$, have negative real parts. Finally, if $T \in (T_j,T_{j+1})$ for $j \in \{0,1,2,\dots\}$, then equation (7) has $2(j+1)$ roots with positive real parts. Recalling that spectral properties of equation (7) lead immediately to the properties of the positive equilibrium k_* for equation (4), the conclusion holds. □.

4 On the direction and stability of Hopf bifurcation

In this section, we investigate the direction and stability of bifurcating periodic solutions of equation (4) at T_0 given by Theorem 3.1, using the method based on the perturbation theory introduced by Lindstedt [19].

Theorem 4.1. The mathematical model (4) admits a stable limit cycle. Moreover the Hopf bifurcation is supercritical.

Proof. We start by considering the Taylor expansion of equation (5) up to the third order at the zero equilibrium:

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{(a-1)\mu}{v}x(t-T) \\ &+ \frac{a(a-1)\mu k_*^{-1}}{2v}[x(t-T)]^2 \\ &+ \frac{a(a-1)(a-2)\mu k_*^{-2}}{6v}[x(t-T)]^3+\cdots \end{aligned} \tag{11}$$

Next, we stretch time by replacing the independent variable t by $s = \omega(\eta)t$, where ω is a parameter close to ω_0 and η is a small positive number. In this way, solutions which are $2\pi/\omega$ periodic in t become periodic with period 2π. With this change of variable equation (11) becomes

$$\begin{aligned} \omega\frac{dx(s)}{ds} &= \frac{(a-1)\mu}{v}x(s-\omega T) \\ &+ \frac{a(a-1)\mu k_*^{-1}}{2v}[x(s-\omega T)]^2 \\ &+ \frac{a(a-1)(a-2)\mu k_*^{-2}}{6v}[x(s-\omega T)]^3+\cdots. \end{aligned} \tag{12}$$

As a final step in the perturbation method, we expand $x(s)$, ω and T in power series of η as follows:

$$\begin{cases} x(s) = \eta x_0(s) + \eta^2 x_1(s) + \eta^3 x_2(s) + \cdots, \\ \omega = \omega_0 + \eta\omega_1 + \eta^2\omega_2 + \cdots, \\ T = T_0 + \eta T_1 + \eta^2 T_2 + \cdots, \end{cases} \tag{13}$$

with the obvious definition of $x_0, x_1, \ldots$.

According to (13), $x(s-\omega\tau)$ can be expanded as follows:

$$x(s-\omega T) = \eta x_0(s-\omega T) + \eta^2 x_1(s-\omega T) + \eta^3 x_2(s-\omega T) + \cdots,$$

where

$$\begin{aligned} & x_j(s-\omega T) = x_j(s-\omega_0 T_0) \\ & -x_j'(s-\omega_0 T_0)[\eta(\omega_1 T_0 + \omega_0 T_1) \\ & +\eta^2(\omega_2 T_0 + \omega_1 T_1 + \omega_0 T_2) + \cdots] \\ & +\frac{1}{2}x_j''(s-\omega_0 T_0)[\eta(\omega_1 T_0 + \omega_0 T_1) + \cdots]^2 - \cdots. \end{aligned}$$

Recalling (9) and the fact that $\omega_0 T_0 = \pi/2$, by substituting the above series expansions in (12), and regrouping into contributions at each order in η, we obtain a system of differential equations, omitted here for brevity. After tedious and long calculations, we can derive (see Rand and Verdugo [21] for details) that $\omega_1 = 0$, $T_1 = 0$ as well as the amplitude A of the limit cycle that was born in the Hopf bifurcation. Therefore we have

$$A^2 = \frac{P}{Q}\eta^2 T_2, \tag{14}$$

where $P = 20\omega_0^7 > 0$ and

$$Q = \left[\frac{5\pi a(a-2)}{4} - \frac{11\pi a^2}{4} - a^2\right] k_*^{-2}\omega_0^6 < 0.$$

In (14), A is real so that $A^2 > 0$, which means from (14) that T_2 must have the same sign as P/Q. Therefore the proof of the theorem is concluded.□

5 Perspective

This section lays out some research perspective of the Dalgaard and Strulik model with time-delay introduced in the present paper. The model is based on the thermodynamic assumption according which the capital is generated and maintained by human and non-human energy.

The first issue to be developed is the comparison of the delayed model introduced in the present paper with the experimentally measurable quantities. Indeed the mathematical models should reproduce both qualitatively and quantitatively empirical data. The economic growth is a complex phenomenon from which emerges a collective behaviour that cannot be explained by the analysis of the single elements. Therefore the model should reproduce, at least at a qualitative level, the relative emerging collective behaviours. Accordingly our model should be able to match the data on electricity consumption per capita, which is an observable variable.

The mathematical model proposed in this paper could be also adapted for the analysis of the asymptotic economic growth. This is an interesting research perspective since, if it is reached, allows the possibility to perform predictions of future economical disasters.

The energy-based method used to derive the mathematical model of this paper can be further specialized by taking into account the possibility to include the conservation of global resources. The conservation of global quantities in the system can be performed by using the framework of the thermostatted kinetic theory for active particles [23,24]. This new framework has been developed for the modelling of complex systems where the kinetic energy (in general a moment of the distribution function) must be preserved. The framework has been adopted to model large systems of physical and living systems, e.g. to semiconductor devices, nanosciences, biological phenomena, vehicular traffic, social and economics systems, crowds and swarms dynamics, see the review paper [25]. Therefore perspective include also the possibility of generalizing the Dalgaard and Strulik model within this new framework.

Acknowledgement

The first author acknowledges the financial support by the FIRB project-RBID08PP3J-Metodi matematici e relativi strumenti per la modellizzazione e la simulazione della formazione di tumori, competizione con il sistema immunitario, e conseguenti suggerimenti terapeutici.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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