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Three-Dimensional Williamson Fluid Flow over a Linear Stretching Surface

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Abstract: In this article three-dimensional flow of Williamson (pseudoplastic) fluid over linearly stretching surface with magnetic field effects are investigated. Transformation method has been utilized for reduction of partial differential equations in to dimensionless coupled system of non-linear ordinary differential equations and solved by numerical scheme named as shooting technique. The dimensionless velocities and shear stresses are obtained in both lateral directions. Pertinent results are presented graphically and discussed quantitatively to analyze the variation of different parameters of interest on velocity in both directions. The effects of governing parameters on skin friction are also illustrated in tabular way. The results for the parameters involved in problem are in total covenant with literature survey presented by kudenatti et. al [22].

Keywords: Williamson fluid; Three-Dimensional; Stretching Surface;Skin-Friction Coefficient; Shooting Method; MHD.

1 Introduction

Researchers have shown enormous interest in three-dimensional flow of non-Newtonian fluid due to its extensive practical importance in industry and engineering. Such interest in fact is enlarged because of diverse applications in various disciplines, for instance in biological sciences, geophysics, chemical and petroleum industries. Such flows broadly appear in plastic manufacture, food processing, performance of lubricants, movement of biological fluids, polymer processing, ice and magma flows. The fluid flow over a stretching surface is significant in solicitations such as extrusion, cord depiction, copper spiraling, warm progressing, and melts of high molecular weight polymers. Non-Newtonian fluid is not described modestly by Navier strokes equations. Due to versatile nature of these fluids constitutive equations consists of many rheological factors in this way it will become more complex than the equations which are describing viscous fluid flow. Hence various non-Newtonian fluid models have been introduced in the literature. In general these fluids have been presented under three classes namely, the differential, the integral and the rate type fluids. In resulting differential equations of these equations contain extra rheological parameters and terms which make it complex from viscous fluid flow models.

There is extensive literature available on the two-dimensional flows over a stretching surface since the seminal works of Sakadais and crane [1,2]. Most recently three-dimensional flow over stretching surface has gained considerable interest. Wang [3] found exact solution for Navier strokes equation for three-dimensional flow. He discussed that three-dimensional flow are reduced to two-dimensional flow by taking stretching ratio equals to zero. Also asymmetric flow is reduced from three-dimensional flow when stretching ratio equalizes to unity. Magneto hydrodynamic three-dimensional flow and heat transfer over a stretching surface in a viscoelastic fluid was discussed by Ahmed et al. [4]. Nadeem et al. [5] investigated the peristaltic motion of an incompressible Williamson fluid with constant and radially varying magneto hydrodynamics (MHD)in an endoscope. Rajeswari et al. [6] investigated unsteady laminar incompressible boundary layer flow over stretching surface. They assumed that velocity of stretching surface varies arbitrarily with time. In their analysis they considered nodal and saddle point regions of flow. They utilized quasilinearization method with an implicit finite-difference scheme for nodal points $(0 \preceq C \preceq 1)$ and this technique failed during saddle points $(-1 \preceq C \preceq 0)$.

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Williamson et al. [7] presented theory of pseudo plastic fluids. He described practical significance of plastic flows. He recognized that plastic flows are very different from viscous flows. He found that there are certain of dispersions which does not follows ideal plastic and viscous fluids and these type of materials are called pseudo plastic flows. Ariel [8] obtained approximate analytical solution of steady laminar flow of three dimensional flow over surface. He applied homotopy perturbation method to first order expansion to compute results and compared his results from previous results. Ariel et. al [9] investigated on extended homotopy solution for laminar boundary layer flow over radially stretching sheet. The coefficients of the stretching parameter were determined by ensuring that the resulting solution is free of the secular terms. Very freshly in another article, Nadeem et al. [10] examined the magneto hydrodynamic (MHD) boundary layer flow of a Casson fluid over an exponentially penetrable shrinking sheet. They computed the analytical solutions for arising differential system by Adomian decomposition method (ADM). Nadeem et al. [11] investigated peristaltic flow of a Williamson model in an asymmetric channel. The governing equations of Williamson model in two dimensional peristaltic flow phenomena are constructed under long wave length and low Reynolds number approximations. A regular perturbation expansion method was used to obtain the analytical solution of the non-linear problem. Pop et al. [12] probed unsteady boundary layer flow of an incompressible micro polar fluid over a stretching sheet. Ariel [13] found the homotopy perturbation and exact solutions for the three-dimensional flow of a viscous fluid over a stretched surface. Tsou et al. [14] probed the solution for unsteady flow of fluid over continuous moving surfaces along with heat transfer. They analyzed heat transfer in two cases for prescribed heat flux and prescribed surface temperature. They concluded that by increasing stretching parameter fluid velocity increases. Initial work for the boundary layer flow on continuous surfaces was discussed by Crane [15] in which he examined the boundary layer flow on continuous surface. He represented new class of boundary layer problem with solution substantially different from these of boundary layer flow on surface of finite length. Nadeem et al. [16] analyzed the peristaltic transport of MHD Newtonian fluid in a symmetric two dimensional channel with variable viscosity along with the heat transfer analysis. Ellahi et al. [17] explored solution for non-Newtonian nano fluid with coaxial cylinders for constant and space-dependent viscosity. Nadeem et al. [18] examined the flow of Williamson fluid caused due to linear stretching, in this article they described the behavior of pseudo plastic model of flow varies with different parameters and specially the non-newtonian parameter (Weissenberg) parameter. It was found from the investigation that it shows declination in velocity distribution. Nadeem et al. [19] reconnoitered the effect of heat transfer on the Williamson fluid flow over a porous exponentially stretching surface . Heat transfer in two cases was deliberated prescribed exponential surface temperature (PEST) and prescribed exponential heat flux (PEHF). Also the velocity distribution in case of suction and injection was discussed. Nadeem et al. investigated self-similar solutions for MHD Casson fluid flow in two lateral directions and analyzed that magnetic field, Casson fluid parameter and porosity parameter reduce velocity profile in both directions. While opposite trends of velocity along x- and y- directions were found with an increase of stretching ratio parameter. Nadeem et al. [21] examined the effects of partial slip on the peristaltic flow of a MHD Newtonian fluid in an asymmetric channel and solved by Adomian decomposition method. Kartini et al. [22] probed the solution for heat transfer over a stretching surface in a viscoelastic fluid by using finite difference scheme known as the Keller box Method and it is found after describing effects of material parameter and magnetic parameter both show opposite behavior on velocity profile. Rao et al. [23] investigated combined effects of Hall and ion slip current in a magneto hydrodynamic boundary layer flow. They included these terms from generalized Ohm's law. They found that skin friction decreases and displacement thickness increases due to Hall currents. Rahman et al. [24] explored about the effects of mixed convection boundary layer flow past vertical flat plate with convective boundary conditions. They introduced specific forms of outer flow and surface heat transfer parameter. They probed the solution for different values of Prandtl number and concluded that $Pr = 1/5$ is transitional case. They also made findings that this phenomenon reduces to uniform flow for $Pr = 0$. Malik et al. [25] elaborated analytical treatment of steady flow of Eyring Powel fluid due to stretching cylinder with temperature dependent viscosity. Heat transfer analysis was taken in to account along with variable viscosity of Vogels and Reynolds model was considered. During their analysis they probed that thermal boundary layer increases by increasing Reynold number and Prandtl number. Cortell [26] examined flow of viscous fluid over non-linear stretching. They included thermal radiation effects in their heat transfer equation and studied the physical effects of Prandtl number, Power law index, Radiation parameter on both momentum and temperature profile along with pertinent coefficients related to heat and momentum transfer. Ishak et al. [27] numerically probed solution for heat transfer in case of uniform and variable heat flux. They discussed the effects of material parameter, velocity and heat flux exponent parameters resp. They concluded from their investigation that Coefficient of convective heat transfer is more for non-newtonian model (Micro polar fluid) than for viscous fluid flow. Sedeek et al. [28] computed exact and numerical solution for velocity and temperature profile. They considered thermal diffusivity effect on profiles and assumed it to be linear. Zeeshan et al. [29] addressed MHD fluid flow in pipe surrounded by porous space accompanied by partial slip. They probed analytical solution for two different viscosity models named as constant model and variable model. Ellahi accounted the effects of unsteady and incompressible flow of micro polar fluid through composite stenosis. Analytic solutions of velocity and volumetric flow flux were developed in terms of modified Bessel functions Ellahi et al. [31] discussed the peristaltic flow of Jeffery fluid between two eccentric tubes. Low Reynolds approximation is employed to obtained analytical solution. Ellahi et al. [32] deliberated their investigation for unsteady three dimensional Williamson fluid flow in rectangular duct. The flow is caused due to peristaltic pumping of propagating sinusoidal waves. They expressed pressure rise numerically. Nadeem et al. [33] examined peristaltic flow of viscous fluid flow in rectangular duct with complaint walls. They found the solution of problem by eigenfunction expansion method. They also discussed trapping phenomenon. Akbar et al. [34] theoretical studied unsteady blood flow of a Williamson fluid through composite stenosed arteries with permeable walls. Perturbation solutions are computed for velocity, flow impedance, wall shear stress and shearing stress at the stenosis throat.

The aim of this investigation is to venture further in regime of three dimensional Williamson (pseudo-plastic) flow over linearly stretching surface with magnetic field. It is obvious that three-dimensional flows are more suitable in giving physical insight of real world when compared with two-dimensional flows. It is important to mention that numerical solution of such type of three-dimensional problems and physical interpretation are big challenge. To the best of our knowledge the results of this paper are originally new, very interesting and they have not been published before.

2 Mathematical formulation

We consider three-dimensional (3D) unsteady incompressible three dimensional Williamson fluid flow over linear stretching surface. It is considered that surface is stretched along xy-plane, the fluid occupies the space z >0 and the motion of fluid is caused due to stretching surface. Furthermore magnetic field is applied normal to fluid flow and induced magnetic field is presumed to be negligible. The constitutive equations in Williamson fluid flow for continuity and momentum after using boundary layer approximation is given by

$$\operatorname{div} V = 0 \tag{1}$$

$$\rho\,\frac{dV}{dt} = \operatorname{div} S + \rho b \tag{2}$$

where ρ is the density, V is the velocity vector, S is the Cauchy stress tensor, b represents the specific body force vector and d/dt represents the material time derivative. The constitutive equations of the Williamson fluid model are given as:

$$S = pI + \tau \tag{3}$$

$$\tau = [\frac{(\mu_0 - \mu_\infty)}{1 - \Gamma\gamma}]A_1 \tag{4}$$

τ is the extra stress tensor, μ_0 and μ_∞ are the limiting viscosities at zero and at infinite shear rate, $\Gamma \succ 0$ is the time constant, A_1 is the first Rivlin-Erickson tensor and γ is defined as follows

The continuity equation and equation of motion under the assumptions associated with the boundary layer flow yield

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} + \sqrt{2}\nu\Gamma\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} - \frac{\sigma B_o^2}{\rho}u \tag{6}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} + \sqrt{2}\nu\Gamma\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + \frac{\sigma B_o^2}{\rho}v \tag{7}$$

$$u = U_{w=}\ ax,\ v = V_w = by \text{ at } z = 0 \qquad u = 1,\ v = 0,\ z \to \infty \tag{8}$$

In the above expressions, u, v and w denote the respective velocities in the x- , y- and z -directions, respectively Γ is the Williamson fluid parameter, B_o is the magnetic induction, ν is the kinematic viscosity, ρ is the density where as a and b are positive constants, and U_w and V_w are stretching velocities in x- and y -directions, respectively. Introducing the following similarity transformations.

$$u = axf'(\eta),\ v = byg'(\eta),\quad w = -\sqrt{a\nu}(f(\eta)+g(\eta)),\ \eta = \sqrt{\frac{a}{\upsilon}}z \tag{1}$$

where c = b /a is the ratio of the velocities in y- and x-directions, and prime denote differentiation with respect to η. Making use of Eq. (9), equation of continuity is identically satisfied. After using transformation the equations become eq 6 to 7 becomes

$$f_{\eta\eta\eta} - f_\eta^2 + f f_{\eta\eta} + We\ f_{\eta\eta} \quad f_{\eta\eta\eta} + g f_{\eta\eta} - M^2 f_\eta = 0 \tag{10}$$

$$g_{\eta\eta\eta} - g_{\eta}^{2} + g g_{\eta\eta} + We\, g_{\eta\eta} g_{\eta\eta\eta} + f g_{\eta\eta} - M^{2} g_{\eta} = 0 \quad (11)$$

$$f(0) = 0,\ \ f'(0) = 1,\ g(0) = 1, \qquad g'(0) = c \quad (12)$$
$$f'(\infty) = 0,\ g'(\infty) = 0.$$

Where We denotes Williamson parameter

$$We = \Gamma x \sqrt{\frac{2a^{3}}{\upsilon}}, M = \frac{\sigma B_{o}^{2}}{\rho} \quad (13)$$

Where $\Gamma \succ 0$ is time rate constant and M is the magnetic parameter

$$C_{fx} = \frac{\tau_{wx}}{\rho u_{w}^{2}}$$
$$C_{fy} = \frac{\tau_{wy}}{\rho u_{w}^{2}} \quad (14)$$

After using boundary layer approximation shear stress rate and coefficient of skin friction will become

$$\tau_{wx} = \mu_{o} \left(\frac{\partial u}{\partial z} + \frac{\Gamma}{\sqrt{2}} \left(\frac{\partial u}{\partial z}\right)^{2}\right)_{z=0},$$
$$\tau_{wy} = \mu_{o} \left(\frac{\partial v}{\partial z} + \frac{\Gamma}{\sqrt{2}} \left(\frac{\partial v}{\partial z}\right)^{2}\right)_{z=0} \quad (15)$$

While the dimensionless forms of skin friction and local Nusselt number are

$$\frac{1}{\sqrt{2}} C_{fx} \operatorname{Re}_{x}^{1/2} = [f''(0) + We f''^{2}(0)],$$
$$\frac{1}{\sqrt{2}} C_{fy} \operatorname{Re}_{x}^{1/2} = [g''(0) + We g''^{2}(0)\ ] \quad (16)$$

where Rex=u_x(x)/v is local Reynolds number based on the stretching velocity u_w(x).

3 Shooting Method

The non linear ordinary differential equations $(11) - (12)$ along with the boundary conditions (13) can be solved using Runge-Kutta-Fehlberg method. The computation is done by program which uses a symbolic and computer language MATLAB. The required boundary value is converted in to initial value problem by using shooting method. In order to integrate (11) -(12) as initial conditions are required for the value of $f''(0)$ and $g''(0)$ but no such value is given at boundary. The suitable guess value are chosen and then integration is carried out. The numerical solution is obtained by using step size $\Delta\eta = 0.01$ obtained the numerical solution with $\eta_{\max}$, and accuracy to fifth decimal place is chosen as criterion of convergence.

First we reduce the original O. D. E to system of 1^{st} order ODE'^{s} by substituting

$$y_1 = f',\ y_2',\ y_3' = f''', y_4 = g, y_5 = g' y_6 = g'', \quad (17)$$

$$f'''(1 + We\, f'') = n\, f'(f' + g') - \left(\frac{n+1}{2}\right) f''(f' + g') \quad (2)$$

$$g'''(1 + We\, g'')g''' = n\, g'(f' + g') - \left(\frac{n+1}{2}\right) g''(f' + g') \quad (3)$$

After using above substitutions Eqs. $(19-20)$ becomes

$$y_3' = \frac{n y_2 (y_2 + y_5) - \left(\frac{n+1}{2}\right) y_3 (y_2 + y_5)}{1 + We\, y_2} \quad (20)$$

$$y_6' = \frac{n y_5 (y_2 + y_5) - \left(\frac{n+1}{2}\right) y_6 (y_2 + y_5)}{1 + We\, y_6} \quad (21)$$

As both the Momentum equations are of third. To solve it in more content manner through a numerical technique shooting is conjunction with Runge- kutta-Fehlberg method. The non-linear momentum equation is converted in to system of six first order simultaneous equations and boundary condition are transformed. Boundary conditions are transformed after using numerical approximation becomes

$$y_1(0) = 0,\ y_2(0) = 1,\ , y_2(\infty) \to 0 \quad (22)$$

In order to solve non - linear equations it is must to have three initial conditions one in f and one initial condition in f' are known i.e. one initial condition of f'' is missing. However the value of f' is known at $\eta \longrightarrow \infty$.The most important step of this method is to choose the appropriate finite approximation value of η_∞. Thus to estimate value of η_∞starting with some initial guess and solve the boundary layer value problem consisting of eq. (11)-(12) to obtain $f''(0)$. The solution process is repeated with another large value of f" differ only after derived significant digit. The last value of η is taken as finite value of limit η_∞ for particular set of physical parameter for determining velocity $f(\eta)$in boundary layer. After getting all three initial condition, the equation is reckoned through Runge-Kutta-Fehlberg integration scheme.

4 Graphs and Tables

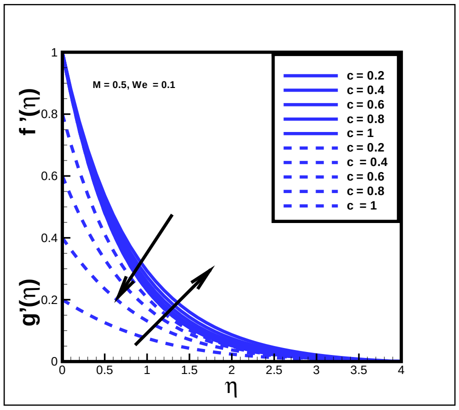

Fig. 1: Effect of stretching ratio parameter on velocity fields

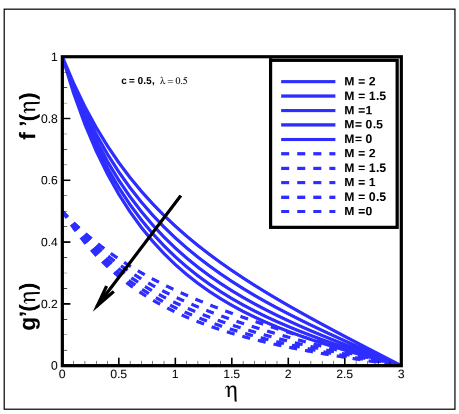

Fig. 2: Effect of magnetic parameter on velocity fields

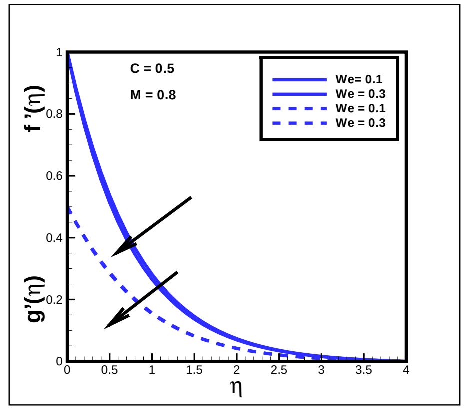

Fig. 3: Effect of Williamson parameter on velocity fields

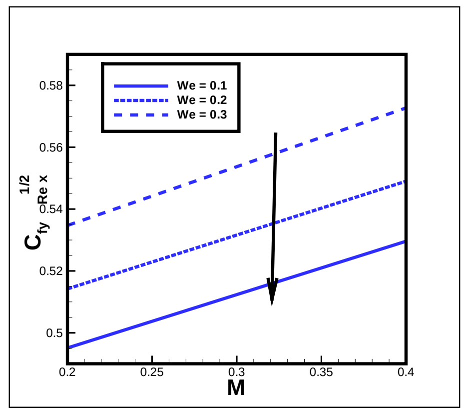

Fig. 4: Effect of M and We on Skin friction coefficient along y-direction

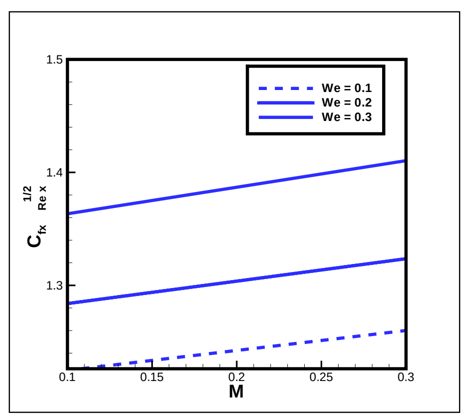

Fig. 5: Effect of M and We on Skin friction coefficient along x-direction

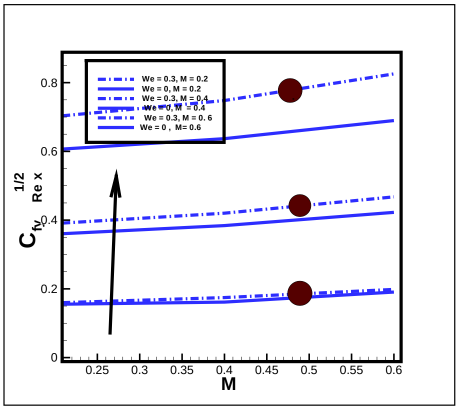

Fig. 6: Comparison of skin friction coefficient along y-direction

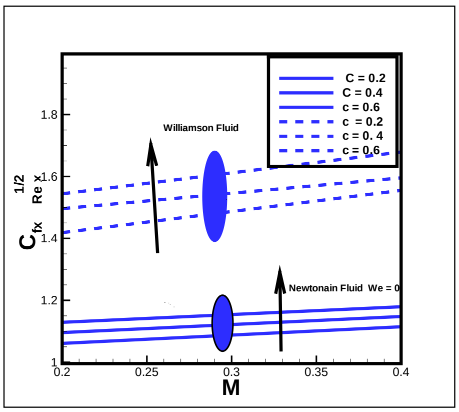

Fig. 7: Comparison of skin friction coefficient along x-direction

Table 1: Validation of present result with previous published literature for skin friction values along x- direction

M	We	C = 0 Ref. [23]	Present Result
		$f''(0)+ We f''^2(0)$	
0	0	1.0042	1.0043
	0.1	1.0080	1.0082
	0.2	1.1395	1.1393
	0.3	1.1987	1.1988

Table 2: Validation of present result with previous published literature for values of skin friction coefficient

M	We	C = 0.5 Ref. [23]	Present Results	C = 0.5 Ref. [23]	Present Reults
		$-[f''(0)+We f''^2(0)]$		$-[g''(0)+We g''^2(0)]$	
0	0.1	1.0932	1.0934	0.4653	0.4661
	0.2	1.2695	1.2695	0.4841	0.4841
	0.3	1.3340	1.3340	0.5024	0.5025
	0.4	1.4915	1.4915	0.5218	0.5220

Table 3: Validation of present result with previous published literature for values of skin friction coefficient

M	We	C = 1 Ref. [23]	Present Results
		$-[g''(0) + We g''^2(0)]$	
0	0	1.1748	1.749
	0.1	1.0880	1.086
	0.2	1. 1395	1.1398
	0.3	1.1987	1.1980

Table 4: Effect of c, We and M on skin friction coefficient in x- and y- direction

M	C	We	$-[f''(0)+We f''^2(0)]$	$-[g''(0)+We g''^2(0)]$
0.2	0.2	0.3	1.5563	0.1591
0.4			1.6238	0.1747
0.6			1.7030	0.1984
0.2	0.2	0.3	1.3755	0.4951
	0.4		1.4328	0.5269
	0.6		1.5592	0.5771
0.2	0.5	0.1	1.2225	0.4843
		0.2	1.3755	0.5147
		0.3	1.7340	0.6278

5 Discussion

The numerical computation of the problem has been carried out for various values of involving parameters. The calculated solution is compared via tables with some previous computed solutions. **Table. 1** represents comparison of skin friction coefficient with kudenatti et al. [23] for $M = 0$ and $We = 0$. It is observed from **Table. 1** that for c = 0 the present phenomenon reduces to two dimensional. This table reflects that both results have good agreement with each other. It is also depicted from the values of table that coefficient of skin friction increases by increasing Williamson parameter. **Table. 2** represents comparison of skin friction coefficient with kudenatti et al. [23] for M =0 ,We = 0 and at C = 0.5. It is concluded from the **Table. 2** that at C = 0.5 skin friction effects fluid flow in both lateral directions and behavior of parameter on skin friction is same as that for **Table. 1**. Then Additionally, comparison of Skin friction flow for C = 1 is discussed in **Table. 3**. In this case axi-symmetric flow is obtained for 3-dimensional flow. It can be seen from the table that the computed results is accurate.

Fig. 1 discloses that enhancement in stretching ratio parameter c causes decrease in velocity along x- direction whereas velocity distribution along y- direction increases. Physically it holds because of the fact that stretching parameter is ratio of fluid velocity along y-direction to the fluid velocity in x-direction. So stretching parameter directly relates to the y- component of velocity and indirectly relates to the x-component of velocity. Due to this reason the vertical component of velocity increases with an enhancement in C on the other velocity distribution along x-direction decreases, the results for two dimensional flow can be obtained from three dimensional flow by reducing c = 0 also result for asymmetric flow can be deduced if c = 1.

Fig. 2 depicts that for higher values of magnetic parameter boundary layer thickness and the magnitude of the velocity in both lateral directions decreases. Physically it occurs due to the reason that magnetic field can induced current in the fluid which causes a resistive-type of force among the fluid particles, which slows down the motion of the fluid.

Fig. 3 is plotted to study the behavior of Williamson parameter on velocity field in x- and y- directions. It is observed that velocity of the fluid declines in x- and y- directions with an increase in Williamson parameter. This is due to the fact that Williamson parameter is ratio of relaxation time to the retardation time. So by increasing Williamson parameter relaxation time increases due to which fluid particles take more time to restore their position so as a result viscosity increases and velocity of fluid particles decreases.

Skin friction coefficients are computed through formula given in $Eq.(16)$. The influence of different parameters on skin friction coefficient is shown via Fig.(4)-(7) **Fig. 4** reflects variations in skin friction coefficient for different values of Williamson parameter We and magnetic field parameter M. As imposed magnetic field thickens momentum boundary layer and reduced fluid velocity which as an outcome causes increase in skin friction coefficient. It is also observed that by increasing Williamson parameter We skin friction coefficient along y- direction enhances. Fig. (5)-(7) characterizes the comportment of magnetic parameter and stretching ratio parameter on coefficient of skin friction in x-direction. It is noticed that for stronger magnetic field the skin friction coefficient increases monotonically. From physical viewpoint, it can be noticed that the Lorentz force increases the values of local skin friction coefficient. It can also be noticed from figure that by increasing Williamson parameter coefficient of skin friction increases along x- direction. It is due to fact that by increasing Williamson parameter it thickens the fluid and it become more viscous. Also the impact of stretching ratio parameter on skin friction is displayed and it is concluded that stretching ratio parameter increases the coefficient of skin friction in x- and y- direction. During the analysis it is found that Skin friction in case of non-newtonian fluid enhances due to increase in viscosity as compared to newtonian fluid. For this model specially the effects of Williamson parameter on skin friction is analyzed and conclusion is made on that as this is rate type of fluid describes both relaxation to retardation time and causes declination in momentum transport so it enhances the skin friction and it is compared graphically that skin friction values are increased in case of Williamson fluid flow as compared to newtonian fluid flow.

6 Concluding remarks

The contemporary study delivers numerical study of three-dimensional Williamson fluid flow over stretching surface and possessions of emerging parameters are conversed for momentum transport in both x-, y- directions.

– Momentum transport decreases with an increase in Williamson parameter, magnetic parameter.
– Stretching ratio parameter show opposite behavior for velocity distribution in lateral directions.
– Shear stresses increases with an increase in Williamson parameter, magnetic parameter, stretching ratio parameter.

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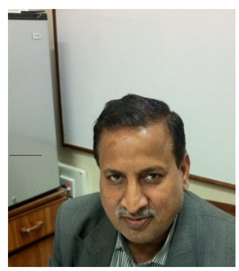

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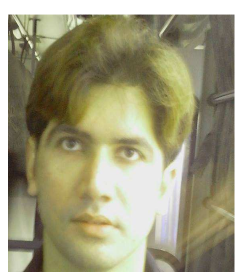

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