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New Prospective of Hypergeometric Summation

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Abstract: In this paper, we have developed the explicit expression of

$$ {}_2F_1\left[\begin{matrix} a, & n-a & ; & \frac{1}{2} \\ c & & ; & 2 \end{matrix}\right] $$

for $n = 13, 14, 15, 16$ & 17. For $n = 0, 1, 2, 3, 4, 5, 6$, the results were established by Prudnikov et al[6,p.414] and for $n = 7, 8, 9, 10, 11, 12$,the results were established by Salahuddin et al[7,p.193-194] .The results are derived with the help of Contiguous relation[1,p.558]and the result from Salahuddin et al[7,p.193-194].The results established in this research paper are very useful.

Keywords: Contiguous relation, Summation formulae

1 Introduction

Special functions and their applications are now awe-inspiring in their scope, variety and depth. Not only in their rapid growth in pure Mathematics and its applications to the traditional fields of Physics, Engineering and Statistics but in new fields of applications like Behavioral Science, Optimization, Biology, Environmental Science and Economics, etc. they are emerging.

The discovery of a hypergeometric function has provided an intrinsic stimulation in the world of mathematics. It has also motivated the development of several domains such as complex functions, Riemann surfaces, differential equations, difference equations, arithmetic theory and so forth. The global structure of the Gauss hypergeometric function as a complex function, i.e., the properties of its monodromy and the analytic continuation, has been extensively studied by Riemann. His method is based on complex integrals. Moreover, when the parameters are rational numbers, its relation to the period integral of algebraic curves became clear, and a fascinating problem on the uniformization of a Riemann surface was proposed by Riemann and Schwarz. On the other hand, Kummer has contributed a lot to the research of arithmetic properties of hypergeometric functions. But there, the main object was the Gauss hypergeometric function of one variable.

A generalized hypergeometric function ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$ \frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\, z. \tag{1.1} $$

Where $k+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \cdots, a_p \ ; \\ \\ b_1, b_2, \cdots, b_q \ ; \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{1.2} $$

or

$$ {}_pF_q\left[\begin{matrix} (a_p) \ ; \\ \\ (b_q) \ ; \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{((a_p))_k z^k}{((b_q))_k k!} \tag{1.3} $$

where the parameters $b_1, b_2, \cdots, b_q$ are positive integers.
The ${}_pF_q$ series converges for all finite z if $p \le q$, converges for $\mid z \mid < 1$ if $p = q+1$, diverges for all z ,

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$z \neq 0$ if $p > q+1$[Luke p.156(3)].

The function ${}_2F_1(a,b;c;z)$ corresponding to $p=2, q=1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

The hypergeometric functions are solutions of Gaussian hypergeometric linear differential equation of second order

$$z(1-z)y'' + [c-(a+b+1)z]y' - aby = 0 \tag{1.4}$$

The solution of this equation is

$$y = A_0\left[1+\frac{ab}{1!\,c}z+\frac{a(a+1)b(b+1)}{2!\,c(c+1)}z^2+\cdots\cdots\right] \tag{1.5}$$

This is the so-called regular solution, denoted

$$ {}_2F_1(a,b;c;z) = \left[1+\frac{ab}{1!\,c}z+\frac{a(a+1)b(b+1)}{2!\,c(c+1)}z^2+\right.$$

$$\left.+\cdots\cdots\right] = \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k z^k}{(c)_k k!} \tag{1.6}$$

which converges if c is not a negative integer for all $|z|<1$ and on the unit circle $|z|=1$ if $R(c-a-b)>0$.

It is known as Gauss hypergeometric function in terms of Pochhammer symbol $(a)_k$ or generalized factorial function.

Many of the common mathematical functions can be expressed in terms of the hypergeometric function. Some typical examples are

$$(1-z)^{-a} = z\,{}_2F_1(1,1;2;-z) \tag{1.7}$$

$$\sin^{-1}z = z\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2) \tag{1.8}$$

Summation formulae for hypergeometric series are applied to a relation obtained by the elementary manipulation of series. Prudnikov et al[6,p.414] derived the following seven summation formulae

$$ {}_2F_1\left[\begin{matrix} a, & -a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{2^c}\left[\frac{1}{\Gamma(\frac{c+a+1}{2})\,\Gamma(\frac{c-a}{2})}+\right.$$

$$\left.+\frac{1}{\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c-a+1}{2})}\right] \tag{1.9}$$

$$ {}_2F_1\left[\begin{matrix} a, & 1-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-1}}\left[\frac{1}{\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c-a+1}{2})}\right] \tag{1.10}$$

$$ {}_2F_1\left[\begin{matrix} a, & 2-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] =$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{(a-1)\,2^{c-2}}\left[\frac{1}{\Gamma(\frac{c+a-2}{2})\,\Gamma(\frac{c-a+1}{2})}-\frac{1}{\Gamma(\frac{c+a-1}{2})\,\Gamma(\frac{c-a}{2})}\right] \tag{1.11}$$

$$ {}_2F_1\left[\begin{matrix} a, & 3-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] =$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{(a-1)(a-2)\,2^{c-3}}\left[\frac{(c-2)}{\Gamma(\frac{c+a-2}{2})\,\Gamma(\frac{c-a+1}{2})}-\right.$$

$$\left.-\frac{2}{\Gamma(\frac{c+a-3}{2})\,\Gamma(\frac{c-a}{2})}\right] \tag{1.12}$$

$$ {}_2F_1\left[\begin{matrix} a, & 4-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] =$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{(1-a)(2-a)(3-a)\,2^{c-4}}\left[\frac{(a-2c+3)}{\Gamma(\frac{c+a-4}{2})\,\Gamma(\frac{c-a+1}{2})}+\right.$$

$$\left.+\frac{(a+2c-7)}{\Gamma(\frac{c+a-3}{2})\,\Gamma(\frac{c-a}{2})}\right] \tag{1.13}$$

$$ {}_2F_1\left[\begin{matrix} a, & 5-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] =$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-5}\{\prod_{\gamma=1}^{4}(\gamma-a)\}}\left[\frac{\{2(c-2)(c-4)-(a-1)(a-4)\}}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-4}{2})}+\right.$$

$$\left.+\frac{(12-4c)}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-5}{2})}\right] \tag{1.14}$$

$$ {}_2F_1\left[\begin{matrix} a, & 6-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] =$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-6}\{\prod_{\delta=1}^{5}(\delta-a)\}}\left[\frac{(4c^2+2ac-a^2-a-34c+62)}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-5}{2})}-\right.$$

$$\left.-\frac{(4c^2-2ac-a^2+13a-22c+20)}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-6}{2})}\right] \tag{1.15}$$

After Prudnikov et al[6,p.414], Salahuddin et al[7,p.193-194] derived the following six summation formulae

$$ {}_2F_1\left[\begin{matrix} a, & 7-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-7}\{\prod_{\varsigma=1}^{6}(\varsigma-a)\}}\times$$

$$\times\left[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-6}{2})}(-3a^2c+12a^2+21ac-84a+\right.$$

$$+4c^3-48c^2+158c-120)+$$

$$+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-7}{2})}(2a^2-14a-8c^2+64c-108)\Big] \tag{1.16}$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix} a, & 8-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] &= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-8}\{\prod\limits_{\xi=1}^{7}(\xi-a)\}}\times \\
&\times\Big[\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-7}{2})}(-a^3-4a^2c+30a^2+4ac^2-4ac- \\
&-107a+8c^3-124c^2+576c-762)+ \\
&+\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-8}{2})}(-a^3+4a^2c-6a^2+4ac^2-68ac+ \\
&+181a-8c^3+92c^2-288c+210)\Big]
\end{aligned} \tag{1.17}
$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix} a, & 9-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] &= \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-9}\{\prod\limits_{\varpi=1}^{8}(\varpi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-8}{2})}\times \\
&\times(a^4-18a^3-8a^2c^2+80a^2c-85a^2+72ac^2-720ac+ \\
&+1494a+8c^4-160c^3+1056c^2-2560c+1680)+ \\
&+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-9}{2})}(8a^2c-40a^2-72ac+360a-16c^3+ \\
&+240c^2-1072c+1360)\Big]
\end{aligned} \tag{1.18}
$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix} a, & 10-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] &= \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-10}\{\prod\limits_{\upsilon=1}^{9}(\upsilon-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-10}{2})}\times \\
&\times(-a^4-4a^3c+42a^3+12a^2c^2-72a^2c-107a^2+ \\
&+8ac^3-252ac^2+1772ac-3054a-16c^4+312c^3- \\
&-2000c^2+4704c-3024)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-9}{2})}\times \\
&\times(a^4-4a^3c+2a^3-12a^2c^2+192a^2c- \\
&-553a^2+8ac^3-12ac^2-868ac+3406a+16c^4- \\
&-392c^3+3320c^2-11224c+12264)\Big]
\end{aligned} \tag{1.19}
$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix} a, & 11-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] &= \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-11}\{\prod\limits_{\varphi=1}^{10}(\varphi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-10}{2})}\times \\
&\times(5a^4c-30a^4-110a^3c+660a^3-20a^2c^3+360a^2c^2- \\
&-1305a^2c-810a^2+220ac^3-3960ac^2+21010ac- \\
&-31020a+16c^5-480c^4+5240c^3-25200c^2+50544c- \\
&-30240)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-11}{2})}(-2a^4+44a^3+24a^2c^2- \\
&-288a^2c+530a^2-264ac^2+3168ac-8492a-32c^4+ \\
&+768c^3-6352c^2+20928c-22320)\Big]
\end{aligned} \tag{1.20}
$$

$$
\begin{aligned}
{}_2F_1\left[\begin{matrix} a, & 12-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] &= \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-12}\{\prod\limits_{\chi=1}^{11}(\chi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-12}{2})}(a^5-6a^4c+ \\
&+9a^4-12a^3c^2+300a^3c-1103a^3+32a^2c^3- \\
&-408a^2c^2+46a^2c+6351a^2+16ac^4-800ac^3+ \\
&+10364ac^2--46852ac+62182a-32c^5+944c^4- \\
&-10112c^3+47656c^2-93776c+55440)+ \\
&+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-11}{2})}(a^5+6a^4c--69a^4- \\
&-12a^3c^2+12a^3c+769a^3-32a^2c^3++840a^2c^2- \\
&-5662a^2c+8301a^2+16ac^4-32ac^3-4612ac^2+ \\
&+42380ac-96002a+32c^5-1136c^4+15104c^3- \\
&-92536c^2+255392c-245640)\Big]
\end{aligned} \tag{1.21}
$$

Gauss' Relations for Contiguous Functions:

The six functions $F(a\pm 1,b;c;z), F(a,b\pm 1;c;z), F(a,b;c\pm 1;z)$ are called contiguous to $F(a,b;c;z)$. Relation between $F(a,b;c;z)$ and any two contiguous functions have been given by Gauss[Abramowitz p.558(15.2.19)]. The contiguous relation which is used in this paper is defined as

$$
\begin{aligned}
b\,{}_2F_1\left[\begin{matrix} a, & b+1 & ; & z \\ c & & ; & \end{matrix}\right] &= (b-c+1)\,{}_2F_1\left[\begin{matrix} a, & b & ; & z \\ c & & ; & \end{matrix}\right]+ \\
&+(c-1)\,{}_2F_1\left[\begin{matrix} a, & b & ; & z \\ c-1 & & ; & \end{matrix}\right]
\end{aligned} \tag{1.22}
$$

2 Main Summation Formulae

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, & 13-a & ; \\ c & & ; \end{matrix}\ \frac{1}{2}\right] = \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-13}\{\prod_{\beta=1}^{12}(\beta-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-12}{2})}\times \\
&\times(-a^6+39a^5+18a^4c^2-252a^4c+275a^4-468a^3c^2+ \\
&+6552a^3c-18135a^3-48a^2c^4+1344a^2c^3-9834a^2c^2+ \\
&+5964a^2c+74246a^2+624ac^4-17472ac^3+ \\
&+167388ac^2-631176ac+752856a+32c^6-1344c^5+ \\
&+21824c^4-172032c^3+674384c^2-1187424c+ \\
&+665280)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-13}{2})}(-12a^4c+84a^4+ \\
&+312a^3c-2184a^3+64a^2c^3-1344a^2c^2+6620a^2c- \\
&-2436a^2-832ac^3+17472ac^2-112424ac+ \\
&+216216a-64c^5+2240c^4-29312c^3+176512c^2- \\
&-478752c+453600)\Big] \qquad (2.1)
\end{aligned}
$$

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, & 14-a & ; \\ c & & ; \end{matrix}\ \frac{1}{2}\right] = \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-14}\{\prod_{\gamma=1}^{13}(\gamma-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-14}{2})}\times \\
&\times(a^6+6a^5c-87a^5-24a^4c^2+150a^4c+925a^4-32a^3c^3+ \\
&+1392a^3c^2-12706a^3c+24615a^3+80a^2c^4-1728a^2c^3+ \\
&+5368a^2c^2+58986a^2c-242486a^2+32ac^5-2320ac^4+ \\
&+47328ac^3-391568ac^2+1344076ac-1496568a- \\
&-64c^6+2656c^5-42560c^4+330752c^3-1278144c^2+ \\
&+2222160c-1235520)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-13}{2})}(-a^6+ \\
&+6a^5c-3a^5+24a^4c^2-570a^4c+2225a^4-32a^3c^3+ \\
&+48a^3c^2+7454a^3c-39225a^3-80a^2c^4+3072a^2c^3- \\
&-35608a^2c^2+133626a^2c-68104a^2+32ac^5-80ac^4- \\
&-19872ac^3+313808ac^2-1676564ac+2856228a+ \\
&+64c^6-3104c^5+59360c^4-566848c^3+2810304c^2- \\
&-6724560c+5897520)\Big] \qquad (2.2)
\end{aligned}
$$

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, & 15-a & ; \\ c & & ; \end{matrix}\ \frac{1}{2}\right] = \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-15}\{\prod_{\varepsilon=1}^{14}(\varepsilon-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-14}{2})}\times \\
&\times(-7a^6c+56a^6+315a^5c-2520a^5+56a^4c^3- \\
&-1344a^4c^2+5103a^4c++16520a^4-1680a^3c^3+ \\
&+40320a^3c^2-271215a^3c+449400a^3-112a^2c^5+ \\
&+4480a^2c^4-54040a^2c^3+150080a^2c^2+845824a^2c- \\
&-3383296a^2+1680ac^5-67200ac^4+999600ac^3- \\
&-6787200ac^2+20482140ac-21070560a+64c^7- \\
&-3584c^6+80864c^5-940800c^4+5987520c^3- \\
&-20296192c^2+32464368c-17297280)+ \\
&+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-15}{2})}(2a^6-90a^5-48a^4c^2+768a^4c- \\
&-1474a^4+1440a^3c^2-23040a^3c+77970a^3+160a^2c^4- \\
&-5120a^2c^3+46640a^2c^2-90880a^2c-226192a^2- \\
&-2400ac^4++76800ac^3-861600ac^2+3955200ac- \\
&-6138120a-128c^6+6144c^5-116160c^4+1095680c^3- \\
&-5363584c^2+12679168c-11009376)\Big] \qquad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix} a, & 16-a & ; \\ c & & ; \end{matrix}\ \frac{1}{2}\right] = \\
&= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-16}\{\prod_{\zeta=1}^{15}(\zeta-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-16}{2})}\times \\
&\times(-a^7+8a^6c-12a^6+24a^5c^2-792a^5c+3710a^5- \\
&-80a^4c^3+1080a^4c^2++6280a^4c-66600a^4- \\
&-80a^3c^4+5280a^3c^3-85480a^3c^2+435480a^3c- \\
&-458929a^3+192a^2c^5-6240a^2c^4+45200a^2c^3+ \\
&+271560a^2c^2-3746640a^2c+8942052a^2+64ac^6- \\
&-6336ac^5+186000ac^4-2408160ac^3+15005072ac^2- \\
&-42553152ac+41722740a-128c^7+7104c^6- \\
&-158720c^5+1827360c^4-11505152c^3+38596416c^2- \\
&-61194240c+32432400)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-15}{2})}\times \\
&\times(-a^7-8a^6c+124a^6+24a^5c^2-24a^5c-2818a^5+ \\
&+80a^4c^3-3000a^4c^2+26360a^4c-40760a^4-80a^3c^4+ \\
&+160a^3c^3+45080a^3c^2-534760a^3c+1499471a^3- \\
&-192a^2c^5+10080a^2c^4-175760a^2c^3+1189560a^2c^2- \\
&-2226480a^2c--2760884a^2+64ac^6-192ac^5- \\
&-75120ac^4++1782560ac^3-16394608ac^2+
\end{aligned}
$$

$$+65703616ac-93008652a+128c^7-8128c^6+$$

$$+210944c^5-2878240c^4+22080512c^3-94015552c^2+$$

$$+202146816c-165145680)\Big] \qquad (2.4)$$

$$ {}_2F_1\left[\begin{matrix} a, & 17-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]=$$

$$=\frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-17}\{\prod\limits_{\vartheta=1}^{16}(\vartheta-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-16}{2})}\times$$

$$\times(a^8-68a^7-32a^6c^2+576a^6c-638a^6+1632a^5c^2-$$

$$-29376a^5c+101320a^5++160a^4c^4-5760a^4c^3+$$

$$+44640a^4c^2+129600a^4c-1341071a^4-5440a^3c^4+$$

$$+195840a^3c^3-2303840a^3c^2++9743040a^3c-$$

$$-9832052a^3-256a^2c^6+13824a^2c^5-246560a^2c^4+$$

$$+1411200a^2c^3+4297408a^2c^2-64103040a^2c+$$

$$+143207628a^2+4352ac^6-235008ac^5+4977600ac^4-$$

$$-52289280ac^3+282566656ac^2-727036416ac+$$

$$+670152240a+128c^8-9216c^7+275456c^6-$$

$$-4423680c^5+41249792c^4-224907264c^3+$$

$$+683065344c^2-1014128640c+518918400+$$

$$+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-17}{2})}(16a^6c-144a^6-816a^5c+$$

$$+7344a^5-160a^4c^3+4320a^4c^2-22480a^4c-30960a^4+$$

$$+5440a^3c^3--146880a^3c^2+1157360a^3c-2484720a^3+$$

$$+384a^2c^5-17280a^2c^4+247840a^2c^3-1092960a^2c^2-$$

$$-1901760a^2c+15669504a^2-6528ac^5+293760ac^4-$$

$$-4999360ac^3+39804480ac^2-146267456ac+$$

$$+194890176a-256c^7+16128c^6-414976c^5-$$

$$+5610240c^4-42628864c^3+179788032c^2-$$

$$-383195904c+310867200)\Big] \qquad (2.5)$$

3 Derivation of the Main Formulae

Derivation of formula(2.1)

Putting $b=(12-a)$ and $z=\frac{1}{2}$ in (1.22), we have

$$(12-a)\,{}_2F_1\left[\begin{matrix} a, & 13-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]=$$

$$=(13-c-a)\,{}_2F_1\left[\begin{matrix} a, & 12-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]+$$

$$+(c-1)\,{}_2F_1\left[\begin{matrix} a, & 12-a & ; & \frac{1}{2} \\ c-1 & ; & & \end{matrix}\right]$$

Now involving (1.21),we have

$$(12-a)\,{}_2F_1\left[\begin{matrix} a, & 13-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right]=$$

$$=(13-c-a)\frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-12}\{\prod\limits_{\chi=1}^{11}(\chi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-12}{2})}\times$$

$$\times(55440+62182a+6351a^2-1103a^3+9a^4+a^5-$$

$$-93776c-46852ac+46a^2c+300a^3c-6a^4c+$$

$$+47656c^2+10364ac^2-408a^2c^2-12a^3c^2-10112c^3-$$

$$-800ac^3+32a^2c^3+944c^4+16ac^4-32c^5)+$$

$$+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-11}{2})}(-245640-96002a+8301a^2+769a^3-$$

$$-69a^4+a^5+255392c+42380ac-5662a^2c+12a^3c+6a^4c-$$

$$-92536c^2-4612ac^2+840a^2c^2-12a^3c^2+15104c^3-$$

$$-32ac^3-32a^2c^3-1136c^4+16ac^4+32c^5)\Big]$$

$$+\frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-13}\{\prod\limits_{\chi=1}^{11}(\chi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-13}{2})}(207960+$$

$$+120214a+5865a^2-1415a^3+15a^4+a^5-223360c-$$

$$-70044ac+958a^2c+324a^3c-6a^4c++83976c^2+$$

$$+12860ac^2-504a^2c^2-12a^3c^2-14208c^3-864ac^3+$$

$$+32a^2c^3+1104c^4+16ac^4-32c^5)+$$

$$+\frac{1}{\Gamma(\frac{c-a-1}{2})\,\Gamma(\frac{c+a-12}{2})}(-609840-142946a+14835a^2+$$

$$+745a^3-75a^4+a^5++490480c++51444ac-$$

$$-7438a^2c+36a^3c+6a^4c--144984c^2-4420ac^2+$$

$$+936a^2c^2-12a^3c^2+19968c^3-96ac^3-32a^2c^3-1296c^4+$$

$$+16ac^4+32c^5)\Big]$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-13}\{\prod_{\chi=1}^{11}(\chi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-12}{2})}\times$$
$$\times\Big\{\frac{1}{2}(720720+752926a+20381a^2-20690a^3+1220a^4+$$
$$+4a^5-a^6-1274528c-577482ac+41099a^2c+$$
$$+4957a^3c-387a^4c+5a^5c+713304c^2+133928ac^2-$$
$$-15714a^2c^2-48a^3c^2+18a^4c^2-179112c^3-$$
$$-10652ac^3+1624a^2c^3-20a^3c^3+22384c^4+64ac^4-$$
$$-48a^2c^4-1360c^5+16ac^5+32c^6)+$$
$$+(609840+752786a+128111a^2-15580a^3-670a^4+$$
$$+74a^5-a^6-1100320c-684870ac-29171a^2c+$$
$$+8147a^3c-117a^4c-5a^5c+635464c^2+200848ac^2-$$
$$-3954a^2c^2-888a^3c^2+18a^4c^2-164952c^3-24292ac^3+$$
$$+1064a^2c^3+20a^3c^3+21264c^4+1184ac^4-$$
$$-48a^2c^4-1328c^5-16ac^5+32c^6)\Big\}+$$
$$+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-13}{2})}\Big\{(207960+120214a+5865a^2-$$
$$-1415a^3+15a^4+a^5-223360c-70044ac+958a^2c+$$
$$+324a^3c-6a^4c+83976c^2+12860ac^2-504a^2c^2-$$
$$-12a^3c^2-14208c^3-864ac^3+32a^2c^3+1104c^4+16ac^4-$$
$$-32c^5)-(-245640-96002a+8301a^2+769a^3-69a^4+$$
$$+a^5+255392c+42380ac-5662a^2c+12a^3c+6a^4c-$$
$$-92536c^2-4612ac^2+840a^2c^2-12a^3c^2+15104c^3-$$
$$-32ac^3-32a^2c^3-1136c^4+16ac^4+32c^5)\Big\}\Big]$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-13}\{\prod_{\chi=1}^{11}(\chi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-12}{2})}\times$$
$$\times(665280+752856a+74246a^2-18135a^3+275a^4+$$
$$+39a^5-a^6-1187424c-631176ac+5964a^2c+$$
$$+6552a^3c-252a^4c+674384c^2+167388ac^2-$$
$$-9834a^2c^2-468a^3c^2+18a^4c^2--172032c^3-$$
$$-17472ac^3+1344a^2c^3+21824c^4+624ac^4-48a^2c^4-$$
$$-1344c^5+32c^6)+$$
$$+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-13}{2})}(453600+216216a-2436a^2-$$
$$-2184a^3+84a^4-478752c-112424ac+6620a^2c+$$
$$+312a^3c-12a^4c+176512c^2+17472ac^2-1344a^2c^2-$$
$$-29312c^3-832ac^3+64a^2c^3+2240c^4-64c^5)\Big]$$

Now dividing both sides by $(12-a)$, we get

$${}_2F_1\left[\begin{matrix} a, & 13-a & ; & \frac{1}{2} \\ c & & ; & 2\end{matrix}\right] =$$

$$= \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-13}\{\prod_{\beta=1}^{12}(\beta-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-12}{2})}\times$$
$$\times(-a^6+39a^5+18a^4c^2-252a^4c+275a^4-468a^3c^2+$$
$$+6552a^3c-18135a^3-48a^2c^4+1344a^2c^3-9834a^2c^2+$$
$$+5964a^2c+74246a^2+624ac^4-17472ac^3+$$
$$+167388ac^2-631176ac+752856a+32c^6-1344c^5+$$
$$+21824c^4-172032c^3+674384c^2-1187424c+$$
$$+665280)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-13}{2})}(-12a^4c+84a^4+$$
$$+312a^3c-2184a^3+64a^2c^3-1344a^2c^2+6620a^2c-$$
$$-2436a^2-832ac^3+17472ac^2-112424ac+$$
$$+216216a-64c^5+2240c^4-29312c^3+176512c^2-$$
$$-478752c+453600)\Big]$$

On the similar way , other formulae can be derived.

4 Conclusion

In this paper we have derived some summation formulae involving contiguous relation and the result from Salahuddin et al[7,p.193-194] . However, the formulae ascertained herein may be further developed to extend these results .So we can only expect that the development presented in this work will create further interest and research in this important area of special functions.

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