Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090524

On the Periodic Structure of the Planar Photogravitational Hill Problem

Elbaz I. Abouelmagd[1,2,3,*], *M. S. Alhothuali*[3], *Juan L. G. Guirao*[4,*] *and H. M. Malaikah*[3]

[1] Mathematics Department, Faculty of Science and Arts , University of Jeddah, Saudi Arabia.
[2] National Research Institute of Astronomy and Geophysics, Cairo, Egypt.
[3] Nonlinear Analysis and Applied Mathematics Research Group (NAAM), Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia.
[4] Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203–Cartagena, Región de Murcia, Spain

Received: 25 Jan. 2015, Revised: 26 Apr. 2015, Accepted: 27 Apr. 2015
Published online: 1 Sep. 2015

Abstract: Using the averaging theory of dynamical systems we describe in an analytical way the periodic structure and the stability of periodic orbits for the planar photogravitational hill problem. Moreover, we present a proof of the $\mathscr{C}^1$–non integrability for this problem.

Keywords: Hamiltonian system, Photogravitational Hill problem, Integrability, Averaging theory.

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction and statement of the main results

The three–body problem is one of the most important problems in space dynamics. There are various dynamical systems composed of three bodies. Consequently this problem has many applications in scientific researches especially in astrophysics and astrodynamics. In general the three–body problem is classified into two classes, first is the so–called *general problem* and second is the *restricted problem* in which the mass of the third body is so small in comparison to the masses of the other two that it does not affect their motion. The general problem describes the motion of three bodies of arbitrary masses under their mutual attraction due to the gravitational field. However our knowledge about the general problem is considerably less than on the restricted problem and limited, It has some applications in celestial mechanics (one is the dynamics of triple star systems) and only a very few in space dynamics and solar system dynamics.

The restricted problem plays an important role in the study of the motion of artificial satellites. It can be used also to evaluate the motion of the planets, minor planets and comets. The restricted problem gives an accurate description not only for the motion of the Moon but also for the motion of other natural satellites. It is worth noting that the restricted problem has many applications not only in celestial mechanics researches but also in physics, mathematics and engineering. In quantum mechanics a general form of the restricted problem is formed to solve the Schrodinger equation of helium-like ions, see Barcza [3]. Furthermore, in modern solid state physics the restricted problem can be used to discuss the motion of an infinitesimal mass affected not only by the gravitational field but also by light pressure from one (or both) of the primaries which is called the photogravitational problem see Kunitsyn and Polyakhova [11]. The restricted problem is modified into five versions:

1. The *planar circular restricted three–body problem*, when the primaries revolve in a circular orbit around their common center of mass as well as the third body moves in the same plane.
2. If the motion of the primaries is not circular and have a conic section. The important case when the primaries

* Corresponding author e-mail: eabouelmagd@gmail.com, juan.garcia@upct.es

move in elliptical orbits around center of their masses. In this case It is called *elliptical* or *pseudo restricted problem*.

3. When the third body moves out of the plane of the primaries the problem is called *three–dimensional problem*. This problem is applicable in the study of the orbits for some minor planets with large inclination to the ecliptic.
4. If the masses of the primaries or third body vary with time. It is called *restricted problem with variable mass* and it has important applications in stellar dynamics and cosmology.
5. If the mass ratio of the smaller primary to the sum masses of the primaries is very small and may be tends to zero, the problem, is called *Hill problem*. In this case the problems with a very small of mass ratio appear as a perturbed problems for the two-body problem.

Although the restricted problem is consider simpler than the general one, there are great difficulties to find its solution in the general case and it is also a non–integrable problem. Therefore numerical methods and perturbation techniques provide us versatile tools to try to solve the problem.

As a seminal problem related with the effect of the pressure of the radiation generated by the primaries with the gravitational force, in the frame of the restricted three body problem, we underline the work of Simmons [17]. In this paper a study of the librance points and their stability is presented. Regarding the Hill problem, there are different studies mainly numerical, see for instance Chauvineau [5], Markellos et alt. [13,**?**], Kanavos et alt. [10], Papadakis [15] and Voyatzis [20].

The aim of our work in to study in an analytical way the periodic orbits and the $\mathscr{C}^1$ integrability of the planar Hill problem with radiation pressure completing some numerical studies on this problem like [12].

The Hamiltonian of the planar Hill problem with radiation pressure of the main primary is given by

$$\mathscr{H}=\frac{1}{2}(P_1^2+P_2^2)-\frac{1}{\sqrt{Q_1^2+Q_2^2}}+(Q_2P_1-Q_1P_2)+ \quad (1)$$
$$KQ_1-Q_1^2+\frac{Q_2^2}{2}$$

with K is the parameter of the radiation pressure of the primary. After Levi–Civita regularization the Hamiltonian (1) is

$$\begin{aligned}\mathscr{H}_{\text{Reg}}=&\frac{1}{2}(X^2+Y^2+x^2+y^2)+2(x^2+y^2)(yX-xY)\\&+4K(x^4-y^4)-4(x^2+y^2)(x^4-2x^2y^2+y^4)\end{aligned} \quad (2)$$

We rescale (2) by mean of

$$(x,y,X,Y)\to(\sqrt{\varepsilon}x,\sqrt{\varepsilon}y,\sqrt{\varepsilon}X,\sqrt{\varepsilon}Y)$$

in order to obtain a perturbed problem.

$$\begin{aligned}\mathscr{H}_{\text{Reg}}=&\frac{1}{2}(X^2+Y^2+x^2+y^2)+2\varepsilon(x^2+y^2)(yX-xY)\\&+4K\varepsilon(x^4-y^4)-4\varepsilon^2(x^2+y^2)(x^4-2x^2y^2+y^4).\end{aligned} \quad (3)$$

We do that, in the standard way, via the introduction of a small parameter ε which allows us to study the dynamics of the perturbed system in a small environment of the point $(0,0)$. Note that if we find a periodic orbit for the perturbed system it can be continuously extended to a periodic orbit the initial system, see Table 1. For more details on the equations of motion for this problem, see Appendix II.

The idea is to use the averaging method of first order to compute isolated periodic orbits, see Appendix I. One of the main difficulties in practice for applying the averaging method is to express the system in the normal form stated in the results of Appendix. The use of proper variables in each concrete situation can simplify a lot this process. In particular, in the present paper we shall use canonical variables of Lissajous introduced by Deprit [6] to obtain information on the existence of non–degenerated isolated periodic orbits for the system, see for instance [4,**?**,**?**,9].

The Lissajous variables are defined by mean of the following canonical transformation $\lambda:(l,g,L,G)\to(x,y,X,Y)$ in the domain

$$\Omega=T^2\times\{L>0\}\times\{|G|\leq L\}$$

by

$$\begin{aligned}x&=\sqrt{L+G}\sin(l+g) & X&=\sqrt{L+G}\cos(l+g)\\ y&=\sqrt{L-G}\sin(l-g) & Y&=\sqrt{L-G}\cos(l-g)\end{aligned} \quad (4)$$

In these variables $\mathscr{H}_{\text{Reg}}$ is given by

$$\mathscr{H}_1=L+\varepsilon\mathscr{P}_1(L,G,l,g)$$

with $\mathscr{P}_1(L,G,l,g)$ is the pullback of the transformation (4) with the perturbed polynomial

$$\begin{aligned}&2\varepsilon(x^2+y^2)(yX-xY)+4\varepsilon K(x^4-y^4)-\\&4\varepsilon^2(x^2+y^2)(x^4-2x^2y^2+y^4).\end{aligned}$$

We denote by $\langle f\rangle$ the averaged map of a smooth function f with respect to the variable l. The first result of the work is the following.

Theorem 1. *At the level of energy $\mathscr{H}_1=h$ with the differential system given by the Hamiltonian $\mathscr{H}_1$ is*

$$\begin{aligned}\frac{dG}{dl}&=\varepsilon f_1(g,G)\\ \frac{dg}{dl}&=\varepsilon f_2(g,G)\end{aligned}. \quad (5)$$

with

$$f_1(g,G) = -\varepsilon \frac{\partial \langle \mathscr{P}_1 \rangle}{\partial g}$$
$$f_2(g,G) = \varepsilon \frac{\partial \langle \mathscr{P}_1 \rangle}{\partial g}.$$

Now we are ready to state a corollary of previous result which provides sufficient conditions for the existence and the kind of stability of the periodic orbits in this problem.

Corollary 1.*System (5) is the Hamiltonian system taking l as independent variable of the Hamiltonian $\mathscr{H}_1$ on the fixed energy level $\mathscr{H} = h$. If $\varepsilon \neq 0$ is sufficiently small then for every solution $\mathbf{p} = (g_0, G_0)$ of the system $f_i(g,G) = 0$ for $i = 1,2$ satisfying that*

$$\det \left(\frac{\partial(f_1, f_2)}{\partial(g,G)} \bigg|_{(g,G)=(g_0,G_0)} \right) \neq 0, \tag{6}$$

there exists a 2π–periodic solution $\gamma_\varepsilon(l) = (g(l,\varepsilon), L(l,\varepsilon), G(l,\varepsilon))$ such that $\gamma_\varepsilon(0) \to (g_0, h, G_0)$ when $\varepsilon \to 0$. The stability or instability of the periodic solution $\gamma_\varepsilon(l)$ is given by the stability or instability of the equilibrium point $\mathbf{p}$ of system (5). In fact, the equilibrium point $\mathbf{p}$ has the stability behavior of the Poincaré map associated to the periodic solution $\gamma_\varepsilon(l)$.

The main result of the paper is the following

Theorem 2.*On every energy level $\mathscr{H}_1 = h$ the Hamiltonian $\mathscr{H}_1$ for $\varepsilon \neq 0$ sufficiently small has four 2π–periodic solutions $\gamma_\varepsilon^k(l) = (g(l,\varepsilon), L(l,\varepsilon), G(l,\varepsilon))$ such that*

$$\gamma_\varepsilon^1(0) \to \left(\frac{\pi}{4}, h, \frac{3hK}{\sqrt{1+9K^2}}\right)$$
$$\gamma_\varepsilon^2(0) \to \left(-\frac{\pi}{4}, h, \frac{-3hK}{\sqrt{1+9K^2}}\right)$$
$$\gamma_\varepsilon^3(0) \to \left(\frac{\pi}{4}, h, \frac{-3hK}{\sqrt{1+9K^2}}\right)$$
$$\gamma_\varepsilon^4(0) \to \left(-\frac{\pi}{4}, h, \frac{3hK}{\sqrt{1+9K^2}}\right)$$

when $\varepsilon \to 0$. The four periodic orbits are linearly stable.

The previous result states that at any positive energy level there exists at least four isolated periodic orbit in such concrete energy level. We note that we shall use this information as a key point to prove other important aspect: the $\mathscr{C}^1$ non–integrability of the model in the sense of Liouville–Arnold.

Theorem 3.*The Hamiltonian of the planar Hill problem with radiation pressure is not Liouville–Arnold integrable with any second first integral of class $\mathscr{C}^1$.*

*Proof.*The proof is a direct consequence of Theorem 3 using Theorems 5 and 8 of Appendix I.

(i)	$x(t) = \sqrt{ah}\sin\left(t+\frac{\pi}{4}\right)$,	$y(t) = \sqrt{bh}\sin\left(t-\frac{\pi}{4}\right)$
(ii)	$x(t) = \sqrt{bh}\sin\left(t-\frac{\pi}{4}\right)$,	$y(t) = \sqrt{ah}\sin\left(t+\frac{\pi}{4}\right)$
(iii)	$x(t) = \sqrt{bh}\sin\left(t+\frac{\pi}{4}\right)$,	$y(t) = \sqrt{ah}\sin\left(t-\frac{\pi}{4}\right)$
(iv)	$x(t) = \sqrt{ah}\sin\left(t-\frac{\pi}{4}\right)$,	$y(t) = \sqrt{bh}\sin\left(t+\frac{\pi}{4}\right)$

Table 1: The four periodic orbits $(x(t), y(t))$ of the unperturbed Hamiltonian system $\mathscr{H}_{\text{Reg}}$ on each energy level $\mathscr{H}_{\text{Reg}} = h > 0$. In the formulae $a = 1+3K(1+9K^2)^{-1}$ and $b = 1-3K(1+9K^2)^{-1}$.

2 Proof of Theorem 1

The Hamiltonian system associated to the Hamiltonian $\mathscr{H}_1$ can be written as

$$\begin{aligned} \frac{dl}{dt} &= 1+\varepsilon\frac{\partial \mathscr{P}_1}{\partial L} & \frac{dL}{dt} &= -\varepsilon\frac{\partial \mathscr{P}_1}{\partial l} \\ \frac{dg}{dt} &= \varepsilon\frac{\partial \mathscr{P}_1}{\partial G} & \frac{dG}{dt} &= -\varepsilon\frac{\partial \mathscr{P}_1}{\partial g} \end{aligned} \tag{7}$$

Taking as new independent variable l, the equations (7) become

$$\begin{aligned} \frac{dG}{dl} &= \frac{-\varepsilon\dfrac{\partial \mathscr{P}_1}{\partial g}}{1+\varepsilon\dfrac{\partial \mathscr{P}_1}{\partial L}} = -\varepsilon\frac{\partial \mathscr{P}_1}{\partial g} + O(\varepsilon^2) \\ \frac{dg}{dl} &= \frac{\varepsilon\dfrac{\partial \mathscr{P}_1}{\partial G}}{1+\varepsilon\dfrac{\partial \mathscr{P}_1}{\partial L}} = \varepsilon\frac{\partial \mathscr{P}_1}{\partial G} + O(\varepsilon^2) \end{aligned} \tag{8}$$

Fixing the energy level of $\mathscr{H}_1 = h$ we obtain

$$h = L + \varepsilon\mathscr{P}_1(L,G,l,g).$$

Using the Implicit Function Theorem for ε sufficiently small, we get $L = h + O(\varepsilon)$, and the equations are reduced to (8).

Proof(Proof of Theorem 1). The averaged system in the angle l obtained from (8) is

$$\begin{aligned} \frac{dG}{dl} &= -\frac{\varepsilon}{2\pi}\int_0^{2\pi}\frac{\partial \mathscr{P}_1}{\partial g}dl + O(\varepsilon^2) \\ \frac{dg}{dl} &= \frac{\varepsilon}{2\pi}\int_0^{2\pi}\frac{\partial \mathscr{P}_1}{\partial G}dl + O(\varepsilon^2). \end{aligned} \tag{9}$$

See the Appendix for a short introduction to the averaging theory used in this paper.

Since,

$$\frac{\partial \langle \mathscr{P}_1 \rangle}{\partial g} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \mathscr{P}_1}{\partial g} dl$$

$$\frac{\partial \langle \mathscr{P}_1 \rangle}{\partial G} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \mathscr{P}_1}{\partial G} dl$$

the differential system (9) coincides with the system (5).

Once we have obtained the averaged system (5) it is immediate to check that it satisfies the assumptions of Theorem 4 of the Appendix, then applying the conclusions of Theorem 4 to our averaged system (5) the rest of the statement of Theorem 1 follows immediately.

3 Proof of Theorem 2

The function $\mathscr{P}_1$ in Deprit canonical variables is given by

$$\begin{aligned}\mathscr{P}_1(L,G,l,g) = {} & 2\sqrt{L^2-G^2}(L - L\cos 2g\cos 2l \\ & + G\sin 2g\sin 2l) + 4K((G+L)^2\sin^4(g+l) \\ & -(G-L)^2\sin^4(g-l)) + O(\varepsilon)\end{aligned}$$

Fixed the energy level h for the Hamiltonian $\mathscr{H}_1$ we obtain the following system

$$\begin{aligned}\frac{dG}{dl} &= \varepsilon f_1(g,G) = \varepsilon 4h\sqrt{h^2-G^2}\cos 2g \\ \frac{dg}{dl} &= \varepsilon f_2(g,G) = \varepsilon h\left(6K + \frac{G\sin 2g}{\sqrt{h^2-G^2}}\right).\end{aligned}$$

The following expressions

$$\begin{aligned}(G_1,g_1) &= \left(\frac{3hK}{\sqrt{1+9K^2}}, \frac{\pi}{4}\right), \\ (G_2,g_2) &= \left(-\frac{3hK}{\sqrt{1+9K^2}}, -\frac{\pi}{4}\right), \\ (G_3,g_3) &= \left(\frac{3hK}{\sqrt{1+9K^2}}, -\frac{\pi}{4}\right), \\ (G_4,g_4) &= \left(-\frac{3hK}{\sqrt{1+9K^2}}, \frac{\pi}{4}\right)\end{aligned}$$

are the only solutions of the system $f_1(g,G) = 0$, $f_2(g,G) = 0$.

On the other hand

$$\left.\frac{\partial(f_1,f_2)}{\partial(G,g)}\right|_{(G_k,g_k)} = \begin{pmatrix} 0 & \dfrac{-8h^2}{\sqrt{1+9K^2}} \\ 2(1+9K^2)^{3/2} & 0 \end{pmatrix}$$

for $k = 1,4$ and

$$\left.\frac{\partial(f_1,f_2)}{\partial(G,g)}\right|_{(G_k,g_k)} = \begin{pmatrix} 0 & \dfrac{8h^2}{\sqrt{1+9K^2}} \\ -2(1+9K^2)^{3/2} & 0 \end{pmatrix}$$

if $k = 2,3$. In all the cases the determinant of $\left.\dfrac{\partial(f_1^1,f_1^2)}{\partial(G,g)}\right|_{(G_k,g_k)}$ are $16h^2(1+9K^2)$.

4 Appendix I

Now we shall present the basic results from averaging theory that we need for proving the results of this paper. In the first part we described the main tools from the averaging theory for ending with some classical results on integrability of two–degrees of freedom systems.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [19].

Consider the differential equation

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon),\ x(0) = x_0 \tag{10}$$

with $x \in D \subset \mathbb{R}^n$, $t \geq 0$. Moreover we assume that $F_1(t,\mathrm{x})$ is T periodic in t. Separately we consider in D the averaged differential equation

$$\dot{y} = \varepsilon f_1(y),\ y(0) = x_0, \tag{11}$$

where

$$f_1(y) = \frac{1}{T}\int_0^T F_1(t,y)dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with T–periodic solutions of equation (11).

Theorem 4.*Consider the two initial value problems (10) and (11). Suppose:*

(i)F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$ are defined, continuous and bounded by an independent constant ε in $[0,\ \infty)\times D$ and $\varepsilon \in (0,\ \varepsilon_0]$.
(ii)F_1 is T–periodic in t (T independent of ε).
(iii)$y(t)$ belongs to Ω on the interval of time $[0,1/\varepsilon]$.

Then the following statements hold.

(a)For $t \in [1,1/\varepsilon]$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.
(b)If p is a singular point of the averaged equation (11) and

$$\det\left(\frac{\partial f_1}{\partial y}\right)\Bigg|_{y=p} \neq 0,$$

then there exists a T–periodic solution $\varphi(t,\varepsilon)$ of equation (10) which is close to p such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.
(c)The stability or instability of the limit cycle $\varphi(t,\varepsilon)$ is given by the stability or instability of the singular point p of the averaged system (11). In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t,\varepsilon)$.

In the sequel we use the idea of the proof of Theorem 4(c). For more details see sections 6.3 and 11.8 of [19]. Suppose that $\varphi(t,\varepsilon)$ is a periodic solution of (10) corresponding to $y = p$ a singular point of the averaged equation (11). We linearize the equation (10) in a neighborhood of the periodic solution $\varphi(t,\varepsilon)$ obtaining a linear equation with T periodic coefficients

$$\dot{x} = \varepsilon A(T,\varepsilon)x, \tag{12}$$

with $A(t,\varepsilon) = \frac{\partial}{\partial x}[F_1(t,x)]_{x=\varphi(t,\varepsilon)}$.

We define the T–periodic matrix

$$B(t) = \frac{\partial F_1}{\partial x}(t,p).$$

From Theorem 4 we have $\lim_{\varepsilon\to 0} A(t,\varepsilon) = B(t)$. We shall use the matrices

$$B_1 = \frac{1}{T}\int_0^T B(t)dt$$

and

$$C(t) = \int_0^t [B(s) - B_1]ds.$$

Note that B_1 is the matrix of the linearized averaged equation. The matrix $C(t)$ is T periodic and it has average zero. The near–identity transformation

$$x \to y = (I - \varepsilon C(t))x \tag{13}$$

allows us to writte the equation (12) as

$$\dot{y} = \varepsilon B_1 y + \varepsilon(A(t,\varepsilon) - B(t))y + O(\varepsilon^2). \tag{14}$$

*Remark.*We note that $A(t,\varepsilon) - B(t) \to 0$ as $\varepsilon \to 0$, and also that the characteristic exponents of equation (14) depend continuously on the small parameter ε. It follows that, for ε sufficiently small, if the determinant of B_1 is not zero, then 0 is not an eigenvalue of the matrix B_1 and then it is not a characteristic exponent of (14). By the near-identity transformation we obtain that system (12) has not multipliers equal to 1.

We shall summarize some facts on the Liouville–Arnold integrability theory for Hamiltonian systems and on the theory of periodic orbits of differential equations, for more details see [1] and subsection 7.1.2 of [2], respectively. We present these results for Hamiltonian systems of two degrees of freedom, because they adjust to the study presented in this paper associated to a perturbed planar Kepler problem under a Newtonian force field, but we remark that these results also work for an arbitrary number of degrees of freedom.

A Hamiltonian system with Hamiltonian $\mathscr{H}$ of two degrees of freedom is called *integrable in the sense of Liouville–Arnold* if it has a first integral $\mathscr{G}$ independent of $\mathscr{H}$ (i.e. the gradient vectors of $\mathscr{H}$ and $\mathscr{G}$ are independent in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in *involution* with $\mathscr{H}$ (i.e. the parenthesis of Poisson of $\mathscr{H}$ and $\mathscr{G}$ is zero).

For Hamiltonian systems with two degrees of freedom the involution condition is redundant, since $\mathscr{G}$ is a first integral of the Hamiltonian system the mentioned Poisson parenthesis is always zero.

Finally, a flow defined on a subspace of the phase space is *complete* if its solutions are defined for all time.

Now are ready for stating the Liouville–Arnold's Theorem restricted to Hamiltonian systems of two degrees of freedom.

Theorem 5(Lioville–Arnold). *Suppose that a Hamiltonian system with two degrees of freedom defined on the phase space M has its Hamiltonian $\mathscr{H}$ and the function $\mathscr{G}$ as two independent first integrals in involution. If $I_{hc} = \{\ p \in M \ /\ H(p) = h$ and $C(p) = c\}$ $\neq \emptyset$ and $(h,\ c)$ is a regular value of the map $(\mathscr{H},\ \mathscr{G})$, then the following statements hold.*

(a)I_{hc} is a two dimensional submanifold of M invariant under the flow of the Hamiltonian system.
(b)If the flow on a connected component I^*_{hc} of I_{hc} is complete, then I^*_{hc} is diffeomorphic either to the torus $\mathbb{S}^1 \times \mathbb{S}^1$, or to the cylinder $\mathbb{S}^1 \times \mathbb{R}$, or to the plane $\mathbb{R}^2$. If I^*_{hc} is compact, then the flow on it is always complete and $I^*_{hc} \approx \mathbb{S}^1 \times \mathbb{S}^1$.
(c)Under the hypothesis (b) the flow on I^*_{hc} is conjugated to a linear flow either on $\mathbb{S}^1 \times \mathbb{S}^1$, on $\mathbb{S}^1 \times \mathbb{R}$, or on $\mathbb{R}^2$.

The main goal of this result is to connect the components of the invariant sets associated, which are generically submanifolds of the phase space, with the two independent first integrals in involution and if the flow on them is complete then they are diffeomorphic to a torus, a cylinder or a plane, where the flow is conjugated to a linear one.

Using the notation of the previous Theorem when a connected component I^*_{hc} is diffeomorphic to a torus, either all orbits on this torus are periodic if the rotation number associated to this torus is rational, or they are quasi-periodic (i.e. every orbit is dense in the torus) if the rotation number associated to this torus is not rational.

For an autonomous differential system, one of the multipliers is always 1, and its corresponding eigenvector is tangent to the periodic orbit.

A periodic orbit of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and the other has again value 1 due to the existence of the first integral given by the Hamiltonian.

Theorem 6(Poincaré). *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnold integrable, and G is a second first* *integral such that the gradients of H and G are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to 1.*

Theorem 6 is due to Poincaré [16]. It gives us a tool to study the non Liouville–Arnold integrability, independently of the class of differentiability of the second first integral. The main problem for applying this result in a negative way is to find periodic orbits having multipliers different from 1.

5 Appendix II

The equations of motion of the restricted three-body problem (Szebehely, [18]) with radiation pressure of the primaries (Simmons et al, [17]), with origin at the center of mass, in a rotating system of coordinates, using dimensionless variables $q_1,\ q_2$, may be written as,

$$\frac{d^2q_1}{dt^2}-2\frac{dq_2}{dt}=\frac{\partial\Omega}{\partial q_1},\quad \frac{d^2q_2}{dt^2}+2\frac{dq_1}{dt}=\frac{\partial\Omega}{\partial q_2},\tag{15}$$

and Ω is the gravitational potential in synodic coordinates defined as,

$$\Omega(q_1,q_2)=\frac{1}{2}(q_1^2+q_2^2)+\frac{k_1(1-\mu)}{r_1}+\frac{k_2\mu}{r_2}$$

with

$$r_1^2=(q_1-\mu)^2+q_2^2,\quad r_2^2=(q_1-\mu+1)^2+q_2^2.$$

Te constant μ, is the ratio of the mass of the smaller primary to the total mass of the primaries and $0<\mu<\dfrac{1}{2}$. The radiation factors of the primaries are denoted by the constants k_1 and k_2.
The Jacobi integral of this problem, has the following expression,

$$C=2\Omega(q_1,q_2)-\left(\left(\frac{dq_1}{dt}\right)^2+\left(\frac{dq_2}{dt}\right)^2\right)$$

where C is the Jacobian constant.
If we put the origin of the synodical system in the smaller primary and change the scale of lengths through the relations

$$q_1=\mu-1+X\mu^{1/3},\ q_2=Y\mu^{1/3},\ k_1=1-K\mu^{1/3}\tag{16}$$

using the relations (16) into (15) and taking the limit for $\mu\to 0$, we obtain

$$\begin{aligned}\frac{d^2X}{dt^2}-2\frac{dY}{dt}&=3X-K-\frac{k_2X}{r^3}\\ \frac{d^2Y}{dt^2}+2\frac{dX}{dt}&=-\frac{k_2Y}{r^3}\end{aligned}$$

with $r=\sqrt{X^2+Y^2}$, the *equations of motion of the Hill problem with radiation pressure of the primaries*.

The Jacobi integral of the Hill problem with radiation pressure of the primaries are

$$C_{HR}=-\left(\left(\frac{dX}{dt}\right)^2+\left(\frac{dY}{dt}\right)^2\right)+3X^2-2KX+\frac{2k_2}{r}$$

and the potential function

$$\Omega_{HR}(X,Y)=\frac{3X^2}{2}-KX+\frac{k_2}{r}$$

and consequently the equations of motion take the form

$$\frac{d^2X}{dt^2}-2\frac{dY}{dt}=\frac{\partial\Omega_{HR}}{\partial X},\quad \frac{d^2Y}{dt^2}+2\frac{dX}{dt}=\frac{\partial\Omega_{HR}}{\partial Y}$$

In this work we consider the case when only the larger primary radiates and consequently $k_2=1$. We now make a canonical transformation, through the set of variables,

$$(Q_1,\ Q_2,\ P_1,\ P_2)\to\left(X,\ Y,\ \frac{dX}{dt}-Y,\ X+\frac{dY}{dt}\right)$$

and we obtain the Hamiltonian function of *the Hill problem with radiation pressure of the larger primary*.

$$\begin{aligned}\mathscr{H}_{HR}=-\frac{C_{HR}}{2}=\frac{1}{2}(P_1^2+P_2^2)-\frac{1}{\sqrt{Q_1^2+Q_2^2}}+\\ (Q_2P_1-Q_1P_2)+KQ_1-Q_1^2+\frac{Q_2^2}{2}\end{aligned}$$

The Hamiltonian function of the Hill problem with radiation pressure of the larger primary. can be also regularized by applying the Levi-Civita transformation. This transformation takes the previous Hamiltonian into the form of two uncoupled harmonic oscillators perturbed by the Coriolis force, the sun action and the radiation action of the main primary. Following the method exposed in (poner los nuestros) we obtain

$$\begin{aligned}\mathscr{H}_{\text{Reg}}=&\frac{1}{2}(X^2+Y^2+x^2+y^2)+2(x^2+y^2)(yX-xY)+\\ &4K(x^4-y^4)-4(x^2+y^2)(x^4-2x^2y^2+y^4).\end{aligned}$$

Fixed the energy $\mathscr{H}_{\text{Reg}}=h$ the following relation hold

$$h=\frac{1}{2}|C_{HR}|^{-3/2}.$$

Acknowledgements

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (59 - 130 - 35 RG). The authors, therefore, acknowledge with thanks DSR technical and financial support.

This work has been partially supported by MICINN/FEDER grant number MTM2011–22587.

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Benjamin, Reading, Masachusets, 1978.

[2] V.I. Arnold, V. Kozlov and A. Neishtadt, *Dynamical Systems III. Mathematical Aspects of Classical and Celestial Mechanics*, Third Edition, Encyclopaedia of Mathematical Science, Springer, Berlin, 2006.

[3] S. Barcza, *Restricted quantum-mechanical three-body problems*, Astrophysics and Space Science, 72(2), 497-507, 1980

[4] M.T. de Bustos, J.L.G. Guirao, J.A. Vera, J. Vigo-Aguiar, *Periodic orbits and $\mathscr{C}^1$ integrability in the planar Stark–Zeeman problem*, Journal of Mathematical Physics, **53(082701)**, 1–9, 2012.

[5] B. Chauvineau, *The generalized hill problem. Case of an external field of force deriving from a central potential*, Celestial Mechanics and Dynamical Astronomy, **51**, 119–129, 1991.

[6] A. Deprit, *The Lissajous transformation I. Basics*, Celestial Mechanics and Dynamical Astronomy, **51**, 201–225, 1991.

[7] J.L.G. Guirao, M.A. López and J.A. Vera, *$\mathscr{C}^1$ non–integrability of a hydrogen atom in a circularly polarized microwave field*, Central European Journal of Physics, **10(4)**, 742–748, 2012.

[8] J.L.G. Guirao, J. Llibre and J.A. Vera, *Periodic solutions induced by an upright position of small oscillations of a sleeping symmetrical gyrostat*, Nonlinear Dynamics, **73**, 417–425, 2013.

[9] J.L.G. Guirao, J. Llibre and J.A. Vera, *On the dynamics of the rigid body with a fixed point: periodic orbits and integrability*, Nonlinear Dynamics, **74**, 327–333, 2013.

[10] S.S. Kanavos, V.V. Markellos, E.A. Perdios, C.N. Douskos, *The Photogravitational Hill Problem Numerical Exploration*, Earth, Moon and Planets, **91**, 223–241, 2002.

[11] A.L. Kunitsyn and E.N. Polyakhova, *The restricted photogravitational three–body problem: A modern state*, Astronomical and Astrophysical Transactions, **6(4)**, 283–293, 1995.

[12] K.E. Papadakis, *The planar Hill problem with oblate primary*, Astrophysics and Space Science, 293(3), 271-287, 2004

[13] V.V Markellos et alt., *A Photogravitational Hill Problem and radiation Effects on Hill Stability of Orbits*, Astrophysics and Space Science, 271, 293-301, 2000

[14] V.V Markellos et alt., *The photogravitational Hill problem with oblateness: equilibrium points and Lyapunov families*, Astrophysics and Space Science, 315, 297-306, 2008

[15] K.E. Pepadakis, *The planar photogravitational Hill problem*, International Journal of Bifurcation and Chaos, **16**, 1809, 2006.

[16] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. I, Gauthier-Villars, Paris 1899.

[17] J.F.L. Simmons et alt., *The restricted 3–body problem with radiation pressure*, Celestial Mechanics and Dynamical Astronomy, **35**, 145-187, 1985.

[18] V.G. Szebehely, *Theory of Orbits*, Academic Press, London and New York, 1967.

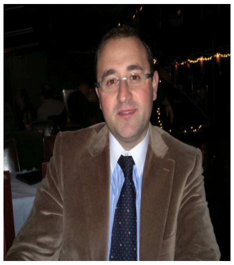
[19] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer, 1991.
Voyatzis G., Gkolias I., Varvoglis H.: The dynamics of the elliptic Hill problem: periodic orbits and stability regions. Celest Mech Dyn Astr (2012) 113:125139

[20] G. Voyatzis, I. Gkolias and H. Varvoglis, *The dynamics of the elliptic Hill problem: periodic orbits and stability regions.*, Celestial Mechanics and Dynamical Astronomy, **113**, 125–139, 2012.

[21] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge Mathematical Library, Cambridge University Press, 1989.

Elbaz I. Abouelmagd is assistant professor of celestial mechanics at King Abdulaziz University. His research interests are in the field of celestial mechanics, especially studying problems coming from Astrophysics and Astrodynamics. His researches also include motion of artificial satellites and the effects of various perturbing forces on the stability of motion. He has published several papers in this field. He has been funded for many research projects by the Deanship of Scientific Research (DSR) at King Abdulaziz University. He is referee of several astrophysics and space science journals.

M. S. Alhothuali works in the field of Mechanics and theory of ordinary and partial differential equations. He is actively engaged in research problems of recent interest. In particular, he has published several papers in the journals like Journal of Mechanics, Commun. Nonlinear Sci. Numer. Simulat., Int. J. Heat Mass Transfer, Applied Mathematics and Mechanics, Mathematical Methods in the Applied Sciences.



Juan Luis García Guirao is Full Professor of Applied Mathematics at Universidad Politécnica de Cartagena in Spain. In 2010, he became in the youngest Mathematics Full Professor in Spain with 33 years old. Author of more than 60 research papers published in the best journal he has supervised 4 PhD's and more than 10 Master Thesis. He belongs to the Editorial Board of several journals, between them MATCH Commun. Math. Comput. Chem. ranked in first position of the JCR 2010 list (Interdisciplinary Mathematics).

Honaida Mohammed Malaikah works in the field of Mechanics and Numerical Simulation. She received her PhD recently from the University of Manchester, UK. Presently, she has been working on some problems of Mechanics. In view of her strong background on numerical computations, she has successfully accomplished some new results. In particular, she has studied the flow of viscoelastic fluids in the following paper: ?Newtonian heating effects in three-dimensional flow of viscoelastic fluid, Chinese Physics B. Since she is new in the field of research and it is the policy of Deanship of Scientific Research (DSR) to encourage the upcoming Saudi researchers, her involvement in the present project will help her a lot to learn more and establish herself as a competent KAU researcher on the topic of research of the project under grant no. (59 -130 - 35RG).