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On $\wedge^P_{\mathscr{I}}$-Sets in Ideal Topological Spaces

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Abstract: In this paper, we define and study the notions of pre-$\mathscr{I}$-kernel for any set (briefly, $\wedge^P_{\mathscr{I}}(\ )$), generalized $\wedge^P_{\mathscr{I}}$-sets, $\wedge^P_{\mathscr{I}}$-closed sets and $\mathscr{I}$-generalized pre-closed (briefly, $\mathscr{I}$-gp-closed) sets by using pre-$\mathscr{I}$-open sets in ideal topological spaces. The family of $\wedge^P_{\mathscr{I}}$-sets, which is stronger than the class of pre-$\mathscr{I}$-open sets, is introduced. The collection of $\wedge^P_{\mathscr{I}}$-sets is Alexandroff space is proven. Also, we propose and characterize some relevant low separation axioms, namely pre-$\mathscr{I}$-τ_1 and pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$. The concepts $\wedge^P(\ )$ (resp. $\wedge(\ )$, $\wedge_{\mathscr{I}}(\ )$) from pre-$\mathscr{I}$-kernel of any set with different kinds of ideals are deduced. Variants of continuity, namely $\wedge^P_{\mathscr{I}}$-continuous, quasi-$\wedge^P_{\mathscr{I}}$-continuous, $\wedge^P_{\mathscr{I}}$-irresolute and strongly pre-$\mathscr{I}$-irresolute functions in terms of $\wedge^P_{\mathscr{I}}$-open sets are characterized. Moreover, the relationships between these classes of functions are studied. Some properties and characterizations of them are obtained.

Keywords: pre-$\mathscr{I}$-open sets, $\mathscr{I}$-gp-closed sets, $\wedge^P_{\mathscr{I}}$-continuous functions

1 Introduction

Kuratowski [10] defined the concept of ideals on topological spaces. Jankovic and Hamlett [9] introduced the notion of $\mathscr{I}$-open sets in topological spaces. Several kinds of $\mathscr{I}$-openness are initiated. Abd El-Monsef et al. [1] investigated further properties of $\mathscr{I}$-open sets and $\mathscr{I}$-continuous functions. Dontchev [3] introduced the notion of pre-$\mathscr{I}$-open sets and obtained a decomposition of $\mathscr{I}$-continuity. In 2002, Hatir and Noiri [7] presented the concept of semi-$\mathscr{I}$-open sets in ideal topological spaces. Recently, Noiri and Keskin [14] introduced the notions of $\bigwedge_{\mathscr{I}}$-sets, $\mathscr{I}$-g-closed sets and $\bigwedge_{\mathscr{I}}$-closed sets by using $\mathscr{I}$-open sets. They used these notions to characterize some related separation axioms.

In this paper, we define the notions of $\wedge^P_{\mathscr{I}}$-sets, generalized $\wedge^P_{\mathscr{I}}$-sets, $\wedge^P_{\mathscr{I}}$-closed sets and $\mathscr{I}$-generalized pre-closed (briefly $\mathscr{I}$-gp-closed) sets by using pre-$\mathscr{I}$-open sets in ideal topological spaces. Several characteristics are studied. Also, two low separation axioms, namely pre-$\mathscr{I}$-τ_1 and pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$ are presented. Moreover, we characterize variants of continuity, namely $\wedge^P_{\mathscr{I}}$-continuous, quasi-$\wedge^P_{\mathscr{I}}$-continuous, $\wedge^P_{\mathscr{I}}$-irresolute and strongly pre-$\mathscr{I}$-irresolute functions in terms of $\wedge^P_{\mathscr{I}}$-open sets and investigates related features. The concepts $\wedge^P(\ )$ (resp. $\wedge(\ )$, $\wedge_{\mathscr{I}}(\ )$) from pre-$\mathscr{I}$-kernel of any set with different kinds of ideals are deduced.

2 Preliminaries

Throughout this paper, $\mathscr{P}$(X), cl(A) and int(A) denote the power set of X, the closure of A and the interior of A, respectively.

An ideal $\mathscr{I}$ [10] on a topological space (X, τ) is a nonempty collection of subsets of X, which satisfies the following two properties:
(i) A$\in\mathscr{I}$ and B$\subseteq$A implies B$\in\mathscr{I}$.
(ii) A$\in\mathscr{I}$ and B$\in\mathscr{I}$ implies A$\cup$B$\in\mathscr{I}$.

It is obvious that the simplest ideals are $\{\emptyset\}$ and $\mathscr{P}$(X). Furthermore, $\mathscr{I}_f$ is the ideal of finite sets in (X, τ). A topological space (X, τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by (X, τ, $\mathscr{I}$). Given an ideal topological space (X, τ, $\mathscr{I}$), a set operator$()^\star$: $\mathscr{P}(X)\longrightarrow\mathscr{P}(X)$, is called a local function [10] of A with respect to τ and $\mathscr{I}$, is defined as follows: for A$\subseteq$X, $A^\star(\mathscr{I},\ \tau)=\{x\in X \mid U\cap A\notin\mathscr{I}$ for every $U\in\tau(x)\}$, where $\tau(x)=\{U\in\tau \mid x\in U\}$. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathscr{I},$

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τ). In general, $X^\star$ is a proper subset of X. The hypothesis $X=X^\star$ [8] is equivalent to the hypothesis $\tau\cap\mathscr{I}=\emptyset$ [15]. For any ideal topological space (X, τ, $\mathscr{I}$), there exist a topology $\tau^\star(\mathscr{I}, \tau)$, is called the $\star$-topology, finer than τ, generated by the collection $\beta(\mathscr{I}, \tau)=\{V - E \mid V\in\tau$ and $E\in\mathscr{I}\}$, but in general $\beta(\mathscr{I}, \tau)$ is not always a topology [9]. Additionally, $cl^\star(A)=A\cup A^\star(\mathscr{I}, \tau)$ [9] defines a Kuratowski closure operator for a topology $\tau^\star(\mathscr{I}, \tau)$. For a subset A of an ideal topological space (X, τ, $\mathscr{I}$), A is said to be $\tau^\star$-closed [9] (resp. $\star$-perfect [8]) if $A^\star\subseteq A$ (resp. $A^\star=A$).

Lemma 2.1. [9] Let (X, τ) be a topological space with ideals $\mathscr{I}$ and $\mathscr{J}$ on X. For subsets A and B of X, we have the following assertions:
(i) $A^\star\subseteq B^\star$ if $A\subseteq B$.
(ii) $A^\star=cl(A^\star)\subseteq cl(A)$.
(iii) $A^{\star\star}\subseteq A^\star$.
(iv) $A^\star(\mathscr{J})\subseteq A^\star(\mathscr{I})$ if $\mathscr{I}\subseteq\mathscr{J}$.
(v) $(A\cup B)^\star=A^\star\cup B^\star$ and $(A\cap B)^\star\subseteq A^\star\cup B^\star$.
(vi) $U\cap A^\star\subseteq(U\cap A)^\star$ if $U\in\tau$.

Definition 2.2. A subset A of an ideal topological space (X, τ, $\mathscr{I}$) is said to be
(i) pre-open [12] if $A\subseteq int\ cl(A)$.
(ii) $\mathscr{I}$-open [1] if $A\subseteq int\ (A^\star)$.
(iii) semi-$\mathscr{I}$-open [7] if $A\subseteq cl^\star\ int(A)$.
(iv) pre-$\mathscr{I}$-open [1] if $A\subseteq int\ cl^\star(A)$.

The complement of semi- (resp. pre-) $\mathscr{I}$-open set is said to be semi- (resp. pre-) $\mathscr{I}$-closed. The family of all semi- (resp. pre-) $\mathscr{I}$-open sets of an ideal topological space (X, τ, $\mathscr{I}$) is denoted by SIO(X, τ) (resp. PIO(X, τ)).

Lemma 2.3. [1] For an ideal topological space (X, τ, $\mathscr{I}$), the following statements hold:
(i) Arbitrary union of pre-$\mathscr{I}$-open sets is pre-$\mathscr{I}$-open.
(ii) Intersection of pre-$\mathscr{I}$-open set and open set is pre-$\mathscr{I}$-open.
(iii) Every pre-$\mathscr{I}$-open set is pre-open.
(iv) Every $\mathscr{I}$-open set is pre-$\mathscr{I}$-open.

Definition 2.4. [1] Let A be a subset of an ideal topological space (X, τ, $\mathscr{I}$). Then,
(i) Pre-$\mathscr{I}$-closure of A, denoted by $cl_{PI}(A)$, is the intersection of all pre-$\mathscr{I}$-closed sets that contain A.
(ii) Pre-$\mathscr{I}$-interior of A, denoted by $int_{PI}(A)$, is the union of all pre-$\mathscr{I}$-open sets contained in A.

Definition 2.5. A set A of an ideal topological space (X, τ, $\mathscr{I}$) is $\tau^\star$-dense if $cl^\star(A)=X$.

Definition 2.6. [2] A topological space (X, τ) is an Alexandroff space if arbitrary intersections of sets in τ belongs to τ.

Definition 2.7. [4] An ideal topological space (X, τ, $\mathscr{I}$) is said to be $\star$-extremally disconnected if the $\tau^\star$-closure of every open subset A of X is open. Equivalently, $cl^\star\ int(A)\subseteq int\ cl^\star(A)$ for every $A\subseteq X$.

3 $\wedge^P_{\mathscr{I}}$-Sets and Its Properties

Definition 3.1. The pre-$\mathscr{I}$-kernel of a set A in an ideal topological space (X, τ, $\mathscr{I}$), denoted by $\wedge^P_{\mathscr{I}}(A)$, is the intersection of all pre-$\mathscr{I}$-open superset of A, i.e. $\wedge^P_{\mathscr{I}}(A)=\cap\{U\in PIO(X, \tau) \mid A\subseteq U\}$.

Lemma 3.2. Let A be a subset of an ideal topological space (X, τ, $\mathscr{I}$). Then,
(i) If $\mathscr{I}=\{\emptyset\}$, then $\wedge^P_{\mathscr{I}}(A)=\wedge^P(A)$, (where $\wedge^P(A)=\cap\{U \mid A\subseteq U$, U pre-open$\}$ [5]).
(ii) If $\mathscr{I}=\mathscr{I}_f$ and (X, τ) is τ_1-space, then $\wedge^P_{\mathscr{I}}(A)=\wedge^P(A)$.
(iii) If $\mathscr{I}=\mathscr{P}(X)$, then $\wedge^P_{\mathscr{I}}(A)=\wedge(A)$, (where $\wedge(A)=\cap\{U\in\tau \mid A\subseteq U\}$ [11]).
(iv) If $A\in PIO(X, \tau)$, then $\wedge^P_{\mathscr{I}}(A)=\wedge_{\mathscr{I}}(A)$, (where $\wedge_{\mathscr{I}}(A)=\cap\{U \mid A\subseteq U$, U $\mathscr{I}$-open$\}$[14]).

proof. Straightforward.

Lemma 3.3. Let A be a subset of an ideal topological space (X, τ, $\mathscr{I}$). Then, $\wedge^P_{\mathscr{I}}(A)=\wedge(A)$ is true if one of the following statements holds:
(i) Every pre-$\mathscr{I}$-open set is $\tau^\star$-closed.
(ii) Every pre-$\mathscr{I}$-open set is $\star$-perfect.
(iii) $PIO(X, \tau)\subseteq\mathscr{I}$.

Proof. Straightforward.

Some of fundamental properties of pre-$\mathscr{I}$-kernel of sets will be shown in the next Lemma.

Lemma 3.4. For sets A, B and A_α ($\alpha\in\Gamma$) of an ideal topological space (X, τ, $\mathscr{I}$), the following properties hold:
(i) $\wedge^P_{\mathscr{I}}(A)\subseteq\wedge_{\mathscr{I}}(A)$.
(ii) $\wedge^P_{\mathscr{I}}(\emptyset)=\emptyset$ and $\wedge^P_{\mathscr{I}}(X)=X$.
(iii) $A\subseteq\wedge^P_{\mathscr{I}}(A)$.
(iv) $A\subseteq B$, then $\wedge^P_{\mathscr{I}}(A)\subseteq\wedge^P_{\mathscr{I}}(B)$.
(v) $\wedge^P_{\mathscr{I}}\wedge^P_{\mathscr{I}}(A)=\wedge^P_{\mathscr{I}}(A)$.
(vi) $A\in PIO(X, \tau)$, then $A=\wedge^P_{\mathscr{I}}(A)$.
(vii)$\wedge^P_{\mathscr{I}}(\cup_{\alpha\in\Gamma}A_\alpha)=\cup_{\alpha\in\Gamma}\wedge^P_{\mathscr{I}}(A_\alpha)$.
(viii)$\wedge^P_{\mathscr{I}}(\cap_{\alpha\in\Gamma}A_\alpha)\subseteq\cap_{\alpha\in\Gamma}\wedge^P_{\mathscr{I}}(A_\alpha)$.
(ix) $\wedge^P_{\mathscr{I}}(A\cap B)\subseteq\wedge^P_{\mathscr{I}}(A)\cup\wedge(B)$.

Proof. We prove only (vii) and the rest of the proof follows directly from Definition 3.1.
(vii) Suppose $x\notin\cup_{\alpha\in\Gamma}(\wedge^P_{\mathscr{I}}(A_\alpha))$, then $x\notin\wedge^P_{\mathscr{I}}(A_\alpha)$ for each $\alpha\in\Gamma$. Therefore, for each $\alpha\in\Gamma$ there exists $U_\alpha\in PIO(X, \tau)$ such that $x\notin U_\alpha$ and $A_\alpha\subseteq U_\alpha$. Thus $\cup_{\alpha\in\Gamma}(A_\alpha)\subseteq\cup_{\alpha\in\Gamma}(U_\alpha)$ and $\cup_{\alpha\in\Gamma}(U_\alpha)\in PIO(X, \tau)$ which does not contain x. Which implies that $x\notin\wedge^P_{\mathscr{I}}(\cup_{\alpha\in\Gamma}A_\alpha)$. Consequently, $\wedge^P_{\mathscr{I}}(\cup_{\alpha\in\Gamma}A_\alpha)\subseteq\cup_{\alpha\in\Gamma}\wedge^P_{\mathscr{I}}(A_\alpha)$. Obviously $\cup_{\alpha\in\Gamma}\wedge^P_{\mathscr{I}}(A_\alpha)\subseteq\wedge^P_{\mathscr{I}}(\cup_{\alpha\in\Gamma}A_\alpha)$. Hence, $\wedge^P_{\mathscr{I}}(\cup_{\alpha\in\Gamma}A_\alpha)=\cup_{\alpha\in\Gamma}\wedge^P_{\mathscr{I}}(A_\alpha)$.

Corollary 3.5. (i) $int_{PI}(A)\subseteq\wedge^P_{\mathscr{I}}(A)$.
(ii) $int_{PI}(A)=\wedge^P_{\mathscr{I}}(A)$ if $A\in PIO(X, \tau)$.
(iii) $\wedge^P_{\mathscr{I}}(A)\subseteq cl_{PI}(A)$ if $A\in PIO(X, \tau)$.
(iv) $cl_{PI}(A)\subseteq\wedge^P_{\mathscr{I}}(A)$ if A is pre-$\mathscr{I}$-closed set.

In view of Lemma 3.4, the next theorem hold.

Theorem 3.6. The collection of all pre-$\mathscr{I}$-kernel of sets is supra topological space.

Lemma 3.7. Let (X, τ) be a topological space with two ideals $\mathscr{I}$, $\mathscr{J}$ on X. For a subset A of X, the following statements hold:
(i) $\wedge^P_{\mathscr{I}}(A)\subseteq\wedge^P_{\mathscr{J}}(A)$ if $\mathscr{I}\subseteq\mathscr{J}$.
(ii) $\wedge^P_{\mathscr{I}\cap\mathscr{J}}(A)\subseteq\wedge^P_{\mathscr{I}}(A)\cap\wedge^P_{\mathscr{J}}(A)$.
(iii) $\wedge^P_{\mathscr{I}}(A)\cup\wedge^P_{\mathscr{J}}(A)\subseteq\wedge^P_{\mathscr{I}\cup\mathscr{J}}(A)$.

Proof. (i) $\mathscr{I}\subseteq\mathscr{J}$, then $A^\star(\mathscr{J})\subseteq A^\star(\mathscr{I})$. Hence, every Pre-$\mathscr{J}$-open is pre-$\mathscr{I}$-open. Therefore, $\wedge^P_{\mathscr{I}}(A)\subseteq\wedge^P_{\mathscr{J}}(A)$ by using Definition 3.1. The rest of the proof follows directly from (i).

Definition 3.8. [16] Let A be a subset of an ideal topological space (X, τ, $\mathscr{I}$). Then, $\wedge^S_{\mathscr{I}}(A)=\cap\{U\in SIO(X,\tau) \mid A\subseteq U\}$.

Theorem 3.9. Let (X, τ, $\mathscr{I}$) be $\star$-extremally disconnected space and A$\subseteq$X. Then, $\wedge^P_{\mathscr{I}}(A)\subseteq\wedge^S_{\mathscr{I}}(A)$.

Proof. Immediate consequence of Definition 2.7.

Definition 3.10. The concept of $\wedge^P_{\mathscr{I}}$-sets in an ideal topological space (X, τ, $\mathscr{I}$) is the set that coincide with their pre-$\mathscr{I}$-kernel. In other words, A is called $\wedge^P_{\mathscr{I}}$-set if $A=\wedge^P_{\mathscr{I}}(A)$.

Theorem 3.11. Let (X, τ, $\mathscr{I}$) be an ideal topological space. Then, the following statements hold:
(i) $\emptyset$, X are $\wedge^P_{\mathscr{I}}$-sets.
(ii) $\wedge^P_{\mathscr{I}}(A)$ is $\wedge^P_{\mathscr{I}}$-set, for any set A of X.
(iii) Every pre-$\mathscr{I}$-open is $\wedge^P_{\mathscr{I}}$-set.
(iv) Every $\tau^\star$-dense is $\wedge^P_{\mathscr{I}}$-set.
(v) Union of $\wedge^P_{\mathscr{I}}$-sets is $\wedge^P_{\mathscr{I}}$-set.
(vi) Intersection of $\wedge^P_{\mathscr{I}}$-sets is $\wedge^P_{\mathscr{I}}$-set.

Proof. Follows directly from Lemma 3.4 and Definition 3.10.

Corollary 3.12. The class of all $\wedge^P_{\mathscr{I}}$-sets is finer than PIO(X, τ).

Theorem 3.13. Let (X, τ, $\mathscr{I}$) be an ideal topological space, then $\tau^{\wedge^P_{\mathscr{I}}}=\{A \mid A \text{ is } \wedge^P_{\mathscr{I}}\text{-set}\}$ is an Alexandroff topology on X.

Proof. Immediate consequence of (i), (v), (vi) of Theorem 3.11.

Corollary 3.14. PIO(X, τ)$\subseteq\tau^{\wedge^P_{\mathscr{I}}}$.

Lemma 3.15. Let (X, τ, $\mathscr{I}$) be an ideal topological space. If A is $\wedge^P_{\mathscr{I}}$-set and $A\subseteq B\subseteq\wedge^P_{\mathscr{I}}(A)$, then B is $\wedge^P_{\mathscr{I}}$-set.

Proof. Let A be $\wedge^P_{\mathscr{I}}$-set, then $A=\wedge^P_{\mathscr{I}}(A)$. Since $A\subseteq B\subseteq\wedge^P_{\mathscr{I}}(A)$, then $B=\wedge^P_{\mathscr{I}}(A)$ and so $\wedge^P_{\mathscr{I}}(B)=\wedge^P_{\mathscr{I}}\wedge^P_{\mathscr{I}}(A)=\wedge^P_{\mathscr{I}}(A)=B$.

Definition 3.16. A subset A of an ideal topological space (X, τ, $\mathscr{I}$) is said to be generalized $\wedge^P_{\mathscr{I}}$-set (briefly g $\wedge^P_{\mathscr{I}}$-set) if $\wedge^P_{\mathscr{I}}(A)\subseteq F$ whenever A$\subseteq$F and F is τ-closed set.

Lemma 3.17. Let (X, τ, $\mathscr{I}$) be an ideal topological space and A$\subseteq$X. Then, the following properties hold:
(i) Union of g $\wedge^P_{\mathscr{I}}$-sets is g $\wedge^P_{\mathscr{I}}$-set.
(ii) A is $\wedge^P_{\mathscr{I}}$-set if it is both τ-closed and g $\wedge^P_{\mathscr{I}}$-set.
(iii) If A is g $\wedge^P_{\mathscr{I}}$-set and $A\subseteq B\subseteq\wedge^P_{\mathscr{I}}(A)$, then B is g $\wedge^P_{\mathscr{I}}$-set.

Proof. (i) Let A, B be g $\wedge^P_{\mathscr{I}}$-sets, (A$\cup$B)$\subseteq$F and F is τ-closed set, then A$\subseteq$F and B$\subseteq$F. Hence, $\wedge^P_{\mathscr{I}}(A)\subseteq F$ and $\wedge^P_{\mathscr{I}}(B)\subseteq F$ and so $[\wedge^P_{\mathscr{I}}(A)\cup\wedge^P_{\mathscr{I}}(B)]\subseteq F$. Therefore, $\wedge^P_{\mathscr{I}}(A\cup B)\subseteq F$ and so (A$\cup$B) is g $\wedge^P_{\mathscr{I}}$-set.
(ii) It is clear that $A\subseteq\wedge^P_{\mathscr{I}}(A)$. Since A is τ-closed and g $\wedge^P_{\mathscr{I}}$-set, then $\wedge^P_{\mathscr{I}}(A)\subseteq A$. Hence, $A=\wedge^P_{\mathscr{I}}(A)$ and so A is $\wedge^P_{\mathscr{I}}$-set.
(iii) Assume that B$\subseteq$F and F is τ-closed set. Since A$\subseteq$B and A is g $\wedge^P_{\mathscr{I}}$-set, then $\wedge^P_{\mathscr{I}}(A)\subseteq F$ and so $\wedge^P_{\mathscr{I}}(B)\subseteq\wedge^P_{\mathscr{I}}\wedge^P_{\mathscr{I}}(A)\subseteq F$. Consequently, B is g $\wedge^P_{\mathscr{I}}$-set.

Lemma 3.18. Let (X, τ, $\mathscr{I}$) be an ideal topological space, then a subset A of X is g $\wedge^P_{\mathscr{I}}$-set if and only if $\wedge^P_{\mathscr{I}}(A)\cap U=\emptyset$ whenever $A\cap U=\emptyset$ and $U\in\tau$.

Proof. Obvious

Theorem 3.19. A subset A of an ideal topological space (X, τ, $\mathscr{I}$) is g $\wedge^P_{\mathscr{I}}$-set if and only if $\wedge^P_{\mathscr{I}}(A)\subseteq cl(A)$.

Proof. Let $x\notin cl(A)$, then there is $U\in\tau$ such that $A\cap U=\emptyset$ and $x\in U$. Since A is g $\wedge^P_{\mathscr{I}}$-set, then by Lemma 3.18, $\wedge^P_{\mathscr{I}}(A)\cap U=\emptyset$. Consequently, $x\notin\wedge^P_{\mathscr{I}}(A)$. On the other hand, assume that A$\subseteq$F, F is τ-closed set and $\wedge^P_{\mathscr{I}}(A)\subseteq cl(A)$. Then, by hypothesis $\wedge^P_{\mathscr{I}}(A)\subseteq cl(A)\subseteq cl(F)=F$. Hence, A is g $\wedge^P_{\mathscr{I}}$-set.

Lemma 3.20. Let (X, τ, $\mathscr{I}$) be an ideal topological space and A be g $\wedge^P_{\mathscr{I}}$-set of X. Then, F=X holds for every τ-closed set F such that $(X - \wedge^P_{\mathscr{I}}(A))\cup A\subseteq F$.

Proof. Immediate.

Lemma 3.21. Let (X, τ, $\mathscr{I}$) be an ideal topological space and A be g $\wedge^P_{\mathscr{I}}$-set of X. Then, $(X - \wedge^P_{\mathscr{I}}(A))\cup A$ is τ-closed set if and only if A is $\wedge^P_{\mathscr{I}}$-set.

Proof. By Lemma 3.20, $(X - \wedge^P_{\mathscr{I}}(A))\cup A=X$. Thus, $\wedge^P_{\mathscr{I}}(A)\cap(X - A)=\emptyset$ i.e., $\wedge^P_{\mathscr{I}}(A)\subseteq A$. Hence, $\wedge^P_{\mathscr{I}}(A)\subseteq A$ and so A is $\wedge^P_{\mathscr{I}}$-set. The other side is obvious.

Definition 3.22. A subset A of an ideal topological space (X, τ, $\mathscr{I}$) is said to be $\mathscr{I}$-generalized pre-closed (briefly $\mathscr{I}$-gp-closed) if $A^\star\subseteq U$ whenever A$\subseteq$U and U$\in$PIO(X, τ). The complement of an $\mathscr{I}$-gp-closed set is said to be $\mathscr{I}$-gp-open.

Lemma 3.23. (i) Finite union of $\mathscr{I}$-gp-closed sets is $\mathscr{I}$-gp-closed.
(ii) Every $\tau^\star$-closed set is $\mathscr{I}$-gp-closed.

Proof. (i) Let A, B be $\mathscr{I}$-gp-closed sets, (A$\cup$B)$\subseteq$U and U$\in$PIO(X, τ), then A$\subseteq$U and B$\subseteq$U. Hence, $A^\star\subseteq U$ and $B^\star\subseteq U$ and so $A^\star\cup B^\star\subseteq U$. Therefore, $(A\cup B)^\star\subseteq F$ by using Lemma 2.1. So (A$\cup$B) is $\mathscr{I}$-gp-closed-set.
(ii) Obvious.

Theorem 3.24. A subset A of an ideal topological space (X, τ, $\mathscr{I}$) is $\mathscr{I}$-gp-closed if and only if $A^\star\subseteq\wedge^P_{\mathscr{I}}(A)$.

Proof. Let $x\notin\wedge^P_{\mathscr{I}}(A)$, then there is U$\in$PIO(X, τ) such that A$\subseteq$U and $x\notin U$. Since A is $\mathscr{I}$-gp-closed set, then $A^\star\subseteq U$.

Hence x∉$A^\star$ and so $A^\star \subseteq \wedge^{P}_{\mathscr{I}}(A)$. On the other hand, assume that A⊆U and U∈PIO(X, τ). Then, by hypothesis $A^\star \subseteq \wedge^{P}_{\mathscr{I}}(A) \subseteq \wedge^{P}_{\mathscr{I}}(U)=U$. Hence A is $\mathscr{I}$-gp-closed.

Lemma 3.25. For each x∈X, either {x} is pre-$\mathscr{I}$-closed or {x} is $\mathscr{I}$-gp-open set.

Proof. Suppose that {x} is not pre-$\mathscr{I}$-closed. Then, (X - {x}) is not pre-$\mathscr{I}$-open and the only pre-$\mathscr{I}$-open set containing (X - {x}) is X itself. Thus, $(X - \{x\})^\star \subseteq X$ and hence (X - {x}) is $\mathscr{I}$-gp-closed. Therefore, {x} is $\mathscr{I}$-gp-open set.

Definition 3.26. An ideal topological space (X, τ, $\mathscr{I}$) is said to be pre-$\mathscr{I}$-τ_1 if for any pair of distinct points x and y of X there exist pre-$\mathscr{I}$-open sets U, V of X such that x∈U, y∉U and y∈V, x∉V.

Lemma 3.27. For an ideal topological space (X, τ, $\mathscr{I}$), the following properties are equivalent:
(i) (X, τ, $\mathscr{I}$) is pre-$\mathscr{I}$-τ_1.
(ii) For each x∈X, the singleton {x} is $\wedge^{P}_{\mathscr{I}}$-set.
(iii) For each x∈X, the singleton {x} is pre-$\mathscr{I}$-closed set.

Proof. (i) $\Longrightarrow$ (ii) Let x be any point of X. For each y∈X, y≠x, there exists pre-$\mathscr{I}$-open set U such that x∈U and y∉U. Thus, we have $y \notin \wedge^{P}_{\mathscr{I}}\{x\}$. This shows that $\wedge^{P}_{\mathscr{I}}\{x\} \subseteq \{x\}$. Since $\{x\} \subseteq \wedge^{P}_{\mathscr{I}}\{x\}$, we obtain $\{x\}=\wedge^{P}_{\mathscr{I}}\{x\}$.
(ii) $\Longrightarrow$ (iii) Let x be any point of X and y∈X - {x}. Then, we have $\{y\}=\wedge^{P}_{\mathscr{I}}\{y\}$. So there exists pre-$\mathscr{I}$-open set V_y such that x∉V_y and y∈V_y. Thus, y∈V_y⊆(X - {x}) and so (X - {x})=∪{V_y | y∈(X - {x})}. By Lemma 2.3, (X - {x}) is pre-$\mathscr{I}$-open set and so {x} is pre-$\mathscr{I}$-closed.
(iii) $\Longrightarrow$ (i) Straightforward.

Theorem 3.28. For an ideal topological space (X, τ, $\mathscr{I}$), the following properties are equivalent
(i) (X, τ, $\mathscr{I}$) is pre-$\mathscr{I}$-τ_1.
(ii) Every subset of X is $\wedge^{P}_{\mathscr{I}}$-set.
(iii) Every pre-$\mathscr{I}$-closed set of X is $\wedge^{P}_{\mathscr{I}}$-set.

Proof. (i)⇒(ii) Let A be any subset of X, then by Lemma 3.27, for any point x in A, the singleton {x} is $\wedge^{P}_{\mathscr{I}}$-set. Therefore, A is $\wedge^{P}_{\mathscr{I}}$-set in view of Theorem 3.11.
(ii)⇒(iii) Obvious.
(iii)⇒(i) Obvious.

Corollary 3.29. Let (X, τ, $\mathscr{I}$) be an ideal topological space, then (X, $\tau^{\wedge^{P}_{\mathscr{I}}}$) is always pre-$\mathscr{I}$-$\tau_1$.

4 $\wedge^{P}_{\mathscr{I}}$-Closed Sets

Definition 4.1. A set A of an ideal topological space (X, τ, $\mathscr{I}$) is called $\wedge^{P}_{\mathscr{I}}$-closed set if there exist $\wedge^{P}_{\mathscr{I}}$-set B and $\tau^\star$-closed set C such that A=B∩C. A set is said to be a $\wedge^{P}_{\mathscr{I}}$-open set if its complement is $\wedge^{P}_{\mathscr{I}}$-closed.

In view of X is both $\wedge^{P}_{\mathscr{I}}$-set and $\tau^\star$-closed set, then proof of next lemma is immediate.

Lemma 4.2. (i) Every $\wedge^{P}_{\mathscr{I}}$-set is $\wedge^{P}_{\mathscr{I}}$-closed set.
(ii) Every $\tau^\star$-closed set is $\wedge^{P}_{\mathscr{I}}$-closed set.

Next we show some results related with $\wedge^{P}_{\mathscr{I}}$-closed sets.

Lemma 4.3. For a set A of an ideal topological space (X, τ, $\mathscr{I}$), the following statements are equivalent
(i) A is $\wedge^{P}_{\mathscr{I}}$-closed set.
(ii) A=B∩$cl^\star$(A), where B is $\wedge^{P}_{\mathscr{I}}$-set.
(iii) $A=\wedge^{P}_{\mathscr{I}}(A)\cap cl^\star(A)$.

Proof. (i)$\Longrightarrow$(ii) Let A be $\wedge^{P}_{\mathscr{I}}$-closed set, then there exist $\wedge^{P}_{\mathscr{I}}$-set B and $\tau^\star$-closed set C such that A=B∩C. Since A⊆C, then $cl^\star(A)\subseteq C$ and so $A\subseteq B\cap cl^\star(A)\subseteq B\cap C=A$. Hence $A=B\cap cl^\star(A)$.
(ii)$\Longrightarrow$(iii) Assume that $A=B\cap cl^\star(A)$ and B is $\wedge^{P}_{\mathscr{I}}$-set. Since A⊆B, then $\wedge^{P}_{\mathscr{I}}(A)\subseteq\wedge^{P}_{\mathscr{I}}(B)=B$ follows from Lemma 3.4. Therefore, $A\subseteq\wedge^{P}_{\mathscr{I}}(A)\cap cl^\star(A)\subseteq B\cap cl^\star(A)=A$ and so $A=\wedge^{P}_{\mathscr{I}}(A)\cap cl^\star(A)$.
(iii)$\Longrightarrow$(i) $A=\wedge^{P}_{\mathscr{I}}(A)\cap cl^\star(A)$ is $\wedge^{P}_{\mathscr{I}}$-closed follows immediately from $\wedge^{P}_{\mathscr{I}}(A)$ is $\wedge^{P}_{\mathscr{I}}$-set and $cl^\star(A)$ is $\tau^\star$-closed.

Lemma 4.4. For a set A of an ideal topological space (X, τ, $\mathscr{I}$), the following statements are equivalent
(i) A is $\tau^\star$-closed.
(ii) A is $\wedge^{P}_{\mathscr{I}}$-closed and $\mathscr{I}$-gp-closed.

Proof. (i)$\Longrightarrow$(ii) Follows directly from Lemmas 3.23, 4.2.
(ii)$\Longrightarrow$(i) Since A is $\mathscr{I}$-gp-closed set, by Theorem 3.24, $A^\star\subseteq\wedge^{P}_{\mathscr{I}}(A)$. Hence, $cl^\star(A)\subseteq\wedge^{P}_{\mathscr{I}}(A)$. Since A is $\wedge^{P}_{\mathscr{I}}$-closed, then $A=\wedge^{P}_{\mathscr{I}}(A)\cap cl^\star(A)$ by Lemma 4.3. Thus $A=cl^\star(A)$ and so A is $\tau^\star$-closed set.

Lemma 4.5. The intersection of $\wedge^{P}_{\mathscr{I}}$-closed sets is $\wedge^{P}_{\mathscr{I}}$-closed set.

Proof. Suppose that A_α is $\wedge^{P}_{\mathscr{I}}$-closed set for each $\alpha\in\Gamma$. Then, for each $\alpha\in\Gamma$, there exist $\wedge^{P}_{\mathscr{I}}$-set B_α and $\tau^\star$-closed set C_α such that $A_\alpha=B_\alpha\cap C_\alpha$. Hence we have $\cap_{\alpha\in\Gamma}A_\alpha=\cap_{\alpha\in\Gamma}(B_\alpha\cap C_\alpha)=(\cap_{\alpha\in\Gamma}B_\alpha)\cap(\cap_{\alpha\in\Gamma}C_\alpha)$. Since $\cap_{\alpha\in\Gamma}B_\alpha$ is $\wedge^{P}_{\mathscr{I}}$-set and $\cap_{\alpha\in\Gamma}C_\alpha$ is $\tau^\star$-closed set. This shows that $\cap_{\alpha\in\Gamma}A_\alpha$ is $\wedge^{P}_{\mathscr{I}}$-closed set.

Subsequently, we present certain notions that will allow us to obtain some results related with the $\wedge^{P}_{\mathscr{I}}$-closed sets.

Theorem 4.6. For subsets A and B of an ideal topological space (X, τ, $\mathscr{I}$), the following properties hold:
(i) If A is $\mathscr{I}$-gp-closed and pre-$\mathscr{I}$-open set, then it is $\tau^\star$-closed.
(ii) If A is $\mathscr{I}$-gp-closed set and $A\subseteq B\subseteq A^\star$, then B is $\mathscr{I}$-gp-closed set.

Proof. (i) Immediate consequence of Theorem 3.11, Lemmas 4.2, 4.4.
(ii) Let B⊆U and U∈PIO(X, τ). Since A⊆B and A is $\mathscr{I}$-gp-closed set, then $A^\star\subseteq U$. Thus, $A^\star\subseteq B^\star\subseteq A^{\star\star}\subseteq A^\star$. Hence, $A^\star=B^\star$. Therefore, $B^\star\subseteq U$ and so B is $\mathscr{I}$-gp-closed set.

Theorem 4.7. A subset A of an ideal topological space (X, τ, $\mathscr{I}$) is $\mathscr{I}$-gp-closed if and only if (A* - A) does not contain any nonempty pre-$\mathscr{I}$-closed set.

Proof. Let A be $\mathscr{I}$-gp-closed set. Assume that F$\subseteq$(A* - A) and F is pre-$\mathscr{I}$-closed set. Observe that A$\subseteq$(X - F) and (X - F) is pre-$\mathscr{I}$-open. Then, A$^\star\subseteq$(X - F) and F$\subseteq$(X - A*). Since F$\subseteq$A*, we have F$\subseteq$(X - A*)$\cap$A*=$\emptyset$. So F=$\emptyset$. Thus, (A* - A) does not contain any nonempty pre-$\mathscr{I}$-closed set. On the other hand, let A$\subseteq$U and U$\in$PIO(X, τ). Suppose that A$^\star\cap$(X - U)$\neq\emptyset$. Since A* is τ-closed and every τ-closed is pre-$\mathscr{I}$-closed, then A$^\star\cap$(X - U) is nonempty pre-$\mathscr{I}$-closed. Since A$^\star\cap$(X - U)$\subseteq$A* - A, then A$^\star\cap$(X - U)=$\emptyset$ and so A$^\star\subseteq$U.

Theorem 4.8. Let (X, τ, $\mathscr{I}$) be an ideal topological space and A$\subseteq$X. Then, (A* - A) does not contain any nonempty $\tau^\star$-open set.

Proof. Let A$\subseteq$X. Suppose that U is $\tau^\star$-open set and U$\subseteq$(A* - A). Since U$\subseteq$(A* - A)$\subseteq$(X - A), we have A$\subseteq$(X - U) and (X - U) is $\tau^\star$-closed. Then, A$^\star\subseteq$(X - U)$^\star\subseteq$(X - U). Hence, U$\subseteq$(X - A*). Since U$\subseteq$A*, we have U=$\emptyset$.

Definition 4.9. An ideal topological space (X, τ, $\mathscr{I}$) is said to be pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$ if every $\mathscr{I}$-gp-closed set of X is $\tau^\star$-closed.

The proof of the next lemma is obvious in view of Lemma 3.23.

Lemma 4.10. Let (X, τ, $\mathscr{I}$) be a pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$ space, then the concepts $\mathscr{I}$-gp-closed and $\tau^\star$-closed are the same.

Theorem 4.11. An ideal topological space (X, τ, $\mathscr{I}$) is pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$ if and only if every singleton $\{$x$\}$ of X is pre-$\mathscr{I}$-closed or $\tau^\star$-open.

Proof. Suppose that $\{$x$\}$ is not pre-$\mathscr{I}$-closed set. By Lemma 3.25, $\{$x$\}$ is $\mathscr{I}$-gp-open and so (X - $\{$x$\}$) is $\mathscr{I}$-gp-closed. Since (X, τ, $\mathscr{I}$) is pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$, then (X - $\{$x$\}$) is $\tau^\star$-closed. Therefore, $\{$x$\}$ is $\tau^\star$-open. On the other hand, let A be $\mathscr{I}$-gp-closed set and x$\in$A*. Then, we have the following two cases:
If $\{$x$\}$ is pre-$\mathscr{I}$-closed. By Theorem 4.7, (A* - A) does not contain any nonempty pre-$\mathscr{I}$-closed. Hence x$\notin$(A* - A). Since x$\in$A*, then we obtain x$\in$A.
If $\{$x$\}$ is $\tau^\star$-open, we have (X - $\{$x$\}$) is $\tau^\star$-closed, i.e. (X - $\{$x$\}$)$^\star\subseteq$X - $\{$x$\}$ or equivalently $\{$x$\}\subseteq$[X - (X - $\{$x$\}$)*]. It is obvious x$\notin$(X - $\{$x$\}$)* and so there exists open set U such that x$\in$U and U$\cap$(X - $\{$x$\}$)$\in\mathscr{I}$. Since x$\in$A*, then V$\cap$A$\notin\mathscr{I}$ for each open set V containing x. In particular, U$\cap$A$\notin\mathscr{I}$ and U$\cap$(X - $\{$x$\}$)$\in\mathscr{I}$. We claim that $\{$x$\}\cap$A$\neq\emptyset$. If $\{$x$\}\cap$A=$\emptyset$, then A$\subseteq$(X - $\{$x$\}$) and hence U$\cap$A$\subseteq$U$\cap$(X - $\{$x$\}$)$\in\mathscr{I}$. It follows that U$\cap$A$\in\mathscr{I}$. This is a contradiction. Therefore, $\{$x$\}\cap$A$\neq\emptyset$ and so x$\in$A.
Consequently, in both cases A$^\star\subseteq$A. Then, A is $\tau^\star$-closed and so X is pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$.

Definition 4.12. A set A of an ideal topological space (X, τ, $\mathscr{I}$) is called locally pre-$\mathscr{I}\star$-closed if A=B$\cap$C, where B is pre-$\mathscr{I}$-open set and C is $\tau^\star$-closed set.

The proof of the following is obvious and then omitted.

Lemma 4.13. In an ideal topological space, every locally pre-$\mathscr{I}\star$-closed set is $\wedge^P_{\mathscr{I}}$-closed.

Lemma 4.14. Let (X, τ, $\mathscr{I}$) be an ideal topological space. For a subset A of X, the following properties are equivalent
(i) A is $\tau^\star$-closed.
(ii) A is $\mathscr{I}$-gp-closed and locally pre-$\mathscr{I}\star$-closed.
(iii) A is $\mathscr{I}$-gp-closed and $\wedge^P_{\mathscr{I}}$-closed.

Proof. (i)$\Rightarrow$(ii) Since X is pre-$\mathscr{I}$-open and A is $\tau^\star$-closed set, then A is locally pre-$\mathscr{I}\star$-closed. Also, in view of Lemmas 3.23, A is $\mathscr{I}$-gp-closed.
(ii)$\Rightarrow$(iii) Immediate in view of Lemma 4.13.
(iii)$\Rightarrow$(i) Since A is $\mathscr{I}$-gp-closed set, then A$^\star\subseteq\wedge^P_{\mathscr{I}}$(A) by Theorem 3.24. Furthermore, we have cl*(A)$\subseteq\wedge^P_{\mathscr{I}}$(A). Since A is $\wedge^P_{\mathscr{I}}$-closed, then A=$\wedge^P_{\mathscr{I}}$(A)$\cap$cl*(A)=cl*(A) by Lemma 4.3. This shows A is $\tau^\star$-closed set.

5 Some Forms of Continuous Functions via $\wedge^P_{\mathscr{I}}$-Open Sets

In this section we use the notions of τ-open, pre-$\mathscr{I}$-open, $\wedge^P_{\mathscr{I}}$-open and $\tau^\star$-open sets in order to introduce new forms of continuous functions called $\wedge^P_{\mathscr{I}}$-continuous, quasi-$\wedge^P_{\mathscr{I}}$-continuous and $\wedge^P_{\mathscr{I}}$-irresolute. We study the relationships between these classes of functions and also obtain some properties and characterizations of them.

Definition 5.1. A function f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ, $\mathscr{J}$) is called
(i) Pre-$\mathscr{I}$-irresolute if f^{-1}(V) is pre-$\mathscr{I}$-open set in (X, τ, $\mathscr{I}$) for each pre-$\mathscr{J}$-open set V of (Y, σ, $\mathscr{J}$).
(ii) $\wedge^P_{\mathscr{I}}$-continuous if f^{-1}(V) is $\wedge^P_{\mathscr{I}}$-open set in (X, τ, $\mathscr{I}$) for each open set V of (Y, σ, $\mathscr{J}$).
(iii) Quasi-$\wedge^P_{\mathscr{I}}$-continuous if f^{-1}(V) is $\wedge^P_{\mathscr{I}}$-open set in (X, τ, $\mathscr{I}$) for each $\sigma^\star$-open set V of (Y, σ, $\mathscr{J}$).
(iv) $\wedge^P_{\mathscr{I}}$-irresolute if f^{-1}(V) is $\wedge^P_{\mathscr{I}}$-open set in (X, τ, $\mathscr{I}$) for each $\wedge^P_{\mathscr{J}}$-open set V of (Y, σ, $\mathscr{J}$).
(v) Strongly pre-$\mathscr{I}$-irresolute if f^{-1}(V) is τ-open set in (X, τ, $\mathscr{I}$) for each pre-$\mathscr{J}$-open set V of (Y, σ, $\mathscr{J}$).

Theorem 5.2. If a function f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ, $\mathscr{J}$) is pre-$\mathscr{I}$-irresolute, then f : (X, $\tau^{\wedge^P_{\mathscr{I}}}$) $\longrightarrow$ (Y, $\sigma^{\wedge^P_{\mathscr{J}}}$) is continuous.

Proof. Let V$\in\sigma^{\wedge^P_{\mathscr{J}}}$, i.e V is $\wedge^P_{\mathscr{J}}$-set of (Y, σ, $\mathscr{J}$), then V=$\wedge^P_{\mathscr{J}}$(V)= $\cap\{$W | V$\subseteq$W and W is pre-$\mathscr{J}$-open in (Y, σ, $\mathscr{J}$)$\}$. Since f is pre-$\mathscr{I}$-irresolute, then f^{-1}(W) is pre-$\mathscr{I}$-open set in (X, τ, $\mathscr{I}$). Hence $\wedge^P_{\mathscr{I}}$(f^{-1}(V))= $\cap\{$U | f^{-1}(V)$\subseteq$U and U$\in$PIO(X, τ)$\}\subseteq\cap\{f^{-1}$(W) | f^{-1}(V)$\subseteq f^{-1}$(W) and f^{-1}(W)$\in$PIO(X, τ)$\}$=f^{-1}(V). On the other hand, always we have f^{-1}(V)$\subseteq\wedge^P_{\mathscr{I}}$($f^{-1}$(V)) and so f^{-1}(V)=$\wedge^P_{\mathscr{I}}$(f^{-1}(V)). Therefore, f^{-1}(V)$\in\tau^{\wedge^P_{\mathscr{I}}}$ and so f : (X, $\tau^{\wedge^P_{\mathscr{I}}}$) $\longrightarrow$ (Y, $\sigma^{\wedge^P_{\mathscr{J}}}$) is continuous.

Theorem 5.3. If f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ, $\mathscr{J}$) is $\wedge^P_{\mathscr{J}}$-irresolute function, then f is quasi-$\wedge^P_{\mathscr{J}}$-continuous.

Proof. Let V be a $\sigma^\star$-open set of (Y, σ, $\mathscr{J}$), then from Lemma 4.2., V is $\wedge^P_{\mathscr{J}}$-open. Since f is $\wedge^P_{\mathscr{J}}$-irresolute, then f^{-1}(V) is $\wedge^P_{\mathscr{I}}$-open set of (X, τ, $\mathscr{I}$). Therefore, f is quasi-$\wedge^P_{\mathscr{J}}$-continuous.

Theorem 5.4. If f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ, $\mathscr{J}$) is quasi-$\wedge^P_{\mathscr{J}}$-continuous function, then f is $\wedge^P_{\mathscr{J}}$-continuous.

Proof. Let V be an σ-open set of (Y, σ, $\mathscr{J}$), then V is $\sigma^\star$-open. Since f is quasi-$\wedge^P_{\mathscr{J}}$-continuous, then f^{-1}(V) is $\wedge^P_{\mathscr{J}}$-open set of (X, τ, $\mathscr{I}$). This shows that f is $\wedge^P_{\mathscr{J}}$-continuous.

By Theorems 5.3, 5.4, we have the following diagram and none of these implications is reversible.

(i) $\wedge^P_{\mathscr{J}}$-irresolute $\Longrightarrow$ quasi-$\wedge^P_{\mathscr{J}}$-continuous $\Longrightarrow$ $\wedge^P_{\mathscr{J}}$-continuous.

(ii) Strongly pre-$\mathscr{I}$-irresolute $\Longrightarrow$ pre-$\mathscr{I}$-irresolute.

Lemma 5.5. Let f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ, $\mathscr{J}$) and g : (Y, σ, $\mathscr{J}$) $\longrightarrow$ (Z, θ, $\mathscr{K}$) be two functions, where $\mathscr{I}$, $\mathscr{J}$, $\mathscr{K}$ are ideals on X, Y, Z respectively. Then,

(i) $g \circ f$ is $\wedge^P_{\mathscr{J}}$-irresolute if f is $\wedge^P_{\mathscr{J}}$-irresolute and g is $\wedge^P_{\mathscr{J}}$-irresolute.

(ii) $g \circ f$ is $\wedge^P_{\mathscr{J}}$-continuous if f is $\wedge^P_{\mathscr{J}}$-irresolute and g is $\wedge^P_{\mathscr{J}}$-continuous.

(iii) $g \circ f$ is $\wedge^P_{\mathscr{J}}$-continuous if f is $\wedge^P_{\mathscr{J}}$-continuous and g is continuous.

(iv) $g \circ f$ is quasi-$\wedge^P_{\mathscr{J}}$-continuous if f is $\wedge^P_{\mathscr{J}}$-irresolute and g is quasi-$\wedge^P_{\mathscr{J}}$-continuous.

(v) $g \circ f$ is strongly pre-$\mathscr{I}$-irresolute if f is strongly pre-$\mathscr{I}$-irresolute and g is pre-$\mathscr{J}$-irresolute.

(vi) $g \circ f$ is strongly pre-$\mathscr{I}$-irresolute if f is continuous and g is strongly pre-$\mathscr{J}$-irresolute.

Proof. We prove only (i) and the rest of the proof is similar to (i).

(i) Let V be a $\wedge^P_{\mathscr{K}}$-open set in (Z, θ, $\mathscr{K}$) and g is $\wedge^P_{\mathscr{J}}$-irresolute, then g^{-1}(V) is $\wedge^P_{\mathscr{J}}$-open set in (Y, σ, $\mathscr{J}$). Since f is $\wedge^P_{\mathscr{J}}$-irresolute, we obtain that $f^{-1}(g^{-1}$(V)) is $\wedge^P_{\mathscr{J}}$-open set in (X, τ, $\mathscr{I}$). This shows that $g \circ f$ is $\wedge^P_{\mathscr{J}}$-irresolute.

In the next three theorems, we characterize $\wedge^P_{\mathscr{J}}$-continuous, quasi-$\wedge^P_{\mathscr{J}}$-continuous and $\wedge^P_{\mathscr{J}}$-irresolute functions, respectively.

Theorem 5.6. For a function f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ), the following statements are equivalent:

(i) f is $\wedge^P_{\mathscr{J}}$-continuous.

(ii) f^{-1}(B) is $\wedge^P_{\mathscr{J}}$-closed set in (X, τ, $\mathscr{I}$) for each closed set B in (Y, σ).

(iii) For each x∈X and each open set V in (Y, σ) containing f(x) there exists $\wedge^P_{\mathscr{J}}$-open set U in (X, τ, $\mathscr{I}$) containing x such that f(U)⊆V.

Proof. We prove only the equivalence between (i) and (iii). The rest of the proof is obvious.

(i)⇒(iii) Let x∈X and V be an open set in (Y, σ) such that f(x)∈V, then x∈f^{-1}(V). Since f is a $\wedge^P_{\mathscr{J}}$-continuous function, f^{-1}(V) is $\wedge^P_{\mathscr{J}}$-open set in (X, τ, $\mathscr{I}$). If U=f^{-1}(V), then U is $\wedge^P_{\mathscr{J}}$-open set in (X, τ, $\mathscr{I}$) containing x such that f(U)=f(f^{-1}(V))⊆V.

(iii)⇒(i) Let V be any open set in (Y, σ) and x∈f^{-1}(V), then f(x)∈V. By using (iii), there exists $\wedge^P_{\mathscr{J}}$-open set U_x in (X, τ, $\mathscr{I}$) such that x∈U_x and $f(U_x)$⊆V. Thus, x∈U_x⊆$f^{-1}(f(U_x))$⊆f^{-1}(V) and so f^{-1}(V)=∪{U_x | x∈f^{-1}(V)}. Then, f^{-1}(V) is $\wedge^P_{\mathscr{J}}$-open set in (X, τ, $\mathscr{I}$) and so f is $\wedge^P_{\mathscr{J}}$-continuous.

The proof of the following theorems are similar to Theorem 5.6.

Theorem 5.7. For a function f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ, $\mathscr{J}$), the following statements are equivalent

(i) f is quasi-$\wedge^P_{\mathscr{J}}$-continuous.

(ii) f^{-1}(B) is $\wedge^P_{\mathscr{J}}$-closed set in (X, τ, $\mathscr{I}$) for each $\sigma^\star$-closed set B in (Y, σ, $\mathscr{J}$).

(iii) For each x∈X and each $\sigma^\star$-open set V in (Y, σ, $\mathscr{J}$) containing f(x) there exists $\wedge^P_{\mathscr{J}}$-open set U in (X, τ, $\mathscr{I}$) containing x such that f(U)⊆V.

Theorem 5.8. For a function f : (X, τ, $\mathscr{I}$) $\longrightarrow$ (Y, σ, $\mathscr{J}$), the following statements are equivalent:

(i) f is $\wedge^P_{\mathscr{J}}$-irresolute.

(ii) f^{-1}(B) is $\wedge^P_{\mathscr{J}}$-closed set in (X, τ, $\mathscr{I}$) for each $\wedge^P_{\mathscr{J}}$-closet set B in (Y, σ, $\mathscr{J}$).

(iii) For each x∈X and each $\wedge^P_{\mathscr{J}}$-open set V in (Y, σ, $\mathscr{J}$) containing f(x) there exists $\wedge^P_{\mathscr{J}}$-open set U in (X, τ, $\mathscr{I}$) containing x such that f(U)⊆V.

Theorem 5.9. For spaces (X, τ, $\mathscr{I}$) and (X, $\tau^{\wedge^P_{\mathscr{I}}}$), the following properties hold:

(i) (X, τ, $\mathscr{I}$) is pre-$\mathscr{I}$-τ_1 if and only if (X, $\tau^{\wedge^P_{\mathscr{I}}}$) is the discrete space.

(ii) The identity function I : (X, $\tau^{\wedge^P_{\mathscr{I}}}$) $\longrightarrow$ (X, τ, $\mathscr{I}$) is strongly pre-$\mathscr{I}$-irresolute.

Proof. (i) Suppose that (X, τ, $\mathscr{I}$) is pre-$\mathscr{I}$-τ_1. According to Lemma 3.27, {x} is $\wedge^P_{\mathscr{J}}$-set for each x∈X. So {x}∈$\tau^{\wedge^P_{\mathscr{I}}}$. For any subset A of X and by using Theorem 3.11., we have A is $\tau^{\wedge^P_{\mathscr{I}}}$-open. This shows that (X, $\tau^{\wedge^P_{\mathscr{I}}}$) is the discrete. Conversely, for each x∈X, {x}∈$\tau^{\wedge^P_{\mathscr{I}}}$ and so {x} is $\wedge^P_{\mathscr{J}}$-set. Therefore, from Lemma 3.27, we have (X, τ, $\mathscr{I}$) is pre-$\mathscr{I}$-τ_1.

(ii) Let V be any pre-$\mathscr{I}$-open set of (X, τ, $\mathscr{I}$), then by using Theorem 3.11., I^{-1}(V)=V∈$\tau^{\wedge^P_{\mathscr{I}}}$. Hence, I is strongly pre-$\mathscr{I}$-irresolute.

6 Conclusion

In this paper, we defined the notions of pre-$\mathscr{I}$-kernel for any set, generalized $\wedge^P_{\mathscr{J}}$-sets, $\wedge^P_{\mathscr{J}}$-closed sets and

$\mathscr{I}$-generalized pre-closed sets by using pre-$\mathscr{I}$-open sets in ideal topological spaces. We proved that the collection of $\wedge^P_{\mathscr{I}}$-sets is Alexandroff space. Moreover, we proposed some relevant low separation axioms, namely pre-$\mathscr{I}$-τ_1 and pre-$\mathscr{I}$-$\tau_{\frac{1}{2}}$. We characterized variants of continuity, namely $\wedge^P_{\mathscr{I}}$-continuous, quasi-$\wedge^P_{\mathscr{I}}$-continuous, $\wedge^P_{\mathscr{I}}$-irresolute and strongly pre-$\mathscr{I}$-irresolute functions in terms of $\wedge^P_{\mathscr{I}}$-open sets.

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