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Topological Defects and Bifurcation Analysis of the DS Equation with Power Law Nonlinearity

Ming Song[1,2,*] *and Anjan Biswas*[3,4]

[1] Department of Mathematics, Shaoxing University, Shaoxing 312000,China
[2] Department of Mathematics, Yuxi Normal University, Yuxi 653100, China
[3] Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA
[4] Department of Mathematics, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia

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Abstract: This paper studies the Davey-Stewartson equation with power law nonlinearity. The bifurcation analysis will be carried out. This analysis will obtain several different solutions to this equation. The phase portraits will also be given. Additionally, the ansatz method will reveal the fact that topological defects, also known as topological solitons, will exist provided the power law nonlinearity collapses to cubic nonlinear medium.

Keywords: Davey-Stewartson equation; Bifurcation analysis; Traveling wave solutions

1 Introduction

There are several nonlinear evolution equations (NLEEs) that are studied globally [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]. These NLEEs appear in various areas of theoretical physics, mathematical physics and engineering sciences. They govern various kinds of physical phenomena and form the fabric of nonlinear sciences. This paper is going to focus on one such NLEE that appears in the study of long-wave short-wave resonances and other patterns of propagating waves and is known as the Davey-Stewartson (DS) equation. In order to maintain it on a generalized setting, DS equation will be considered with power law nonlinearity that condenses to the cubic law for a special value of the power law nonlinearity parameter.

The focus of this paper will be its integrability aspect. The bifurcation analysis will be carried out for the DS equation with power law nonlinearity. The Hamiltonian will be computed and several phase portraits will be displayed. Various solutions to the DS equation will be thus obtained and listed. Additionally, the ansatz method will be applied to extract the topological defects for this equation. Topological defects are alternatively known as topological solitons or shock wave solutions. This method will prove that these topological solitons for the DS equation will exist provided the power law nonlinearity collapses to cubic nonlinear medium. This is a very important observation that will be made for the first time in this paper.

2 Governing equation

The dimensionless form of the DS equation with power law nonlinearity, that will be studied in this paper, is given by

$$\begin{cases} iq_t + a\left(q_{xx} + q_{yy}\right) + b\left|q\right|^{2n} q = \alpha q r, \\ r_{xx} + r_{yy} + \beta\left(\left|q\right|^{2n}\right)_{xx} = 0. \end{cases} \tag{1}$$

In equations (1), $q(x,y,t)$ is the complex valued wave function while $r(x,y,t)$ is the real valued wave function. Also, in (1), x and y are spatial independent variables while t is the temporal independent variable. Thus, this is a nonlinear wave equation in (2+1)-D setting. The constants a, b, α and β are all non-zero real-valued constants. The power law nonlinearity parameter is n. The special case, when $n = 1$, is known as the DS equation with cubic law nonlinearity.

In order to give a brief history of this equation, DS equation was studied before by several authors[1,2,3,4,5,

* Corresponding author e-mail: songming12_15@163.com

6, 7, 14, 15]. The traveling wave hypothesis, ansatz method, G'/G-expansion method were all applied to extract several kinds of wave solutions to this equation[1, 2]. This paper will extract the topological soliton solution to (1) for the first time by the ansatz method. Additionally, the bifurcation analysis for this equation was also studied earlier for the special case when $n=1$ [10]. This paper will now focus on the bifurcation analysis of the DS equation for any arbitrary value of the parameter n.

2.1 Topological soliton solution

In order to solve (1) for topological 1-soliton solution, the starting hypothesis is taken to be

$$\begin{cases} q(x,y,t) = A_1 \tanh^{p_1}\tau e^{i\phi}, \\ r(x,y,t) = A_2 \tanh^{p_2}\tau, \end{cases} \tag{2}$$

where A_1 and A_2 are free parameters of the topological soliton and p_1, p_2 are the unknown exponents whose values will fall out during the course of derivation of the soliton solution to (1). Additionally,

$$\tau = B_1 x + B_2 y - vt \tag{3}$$

and from the phase component,

$$\phi(x,y,t) = -\kappa_1 x - \kappa_2 y + \omega t + \theta. \tag{4}$$

The parameters B_1 and B_2 are also free parameters, while the velocity of the soliton is given by v. Now, from the phase component, κ_1 and κ_2 are frequencies in the x- and y-directions respectively while ω is the wave number and θ is the phase constant.
Substituting the ansatz (2) into (1) and then decomposing into real and imaginary parts reveals the following three relations respectively

$$\begin{aligned} &-\left(\omega + a\kappa_1^2 + a\kappa_2^2\right)\tanh^{p_1}\tau + bA_1^{2n}\tanh^{(2n+1)p_1}\tau \\ &+ ap_1\left(B_1^2 + B_2^2\right)\left\{(p_1-1)\tanh^{p_1-2}\tau - 2p_1\tanh^{p_1}\tau\right. \\ &+ \left.(p_1+1)\tanh^{p_1+2}\tau\right\} = \alpha A_2 \tanh^{p_1+p_2}\tau, \end{aligned} \tag{5}$$

$$(v + 2a\kappa_1 B_1 + 2a\kappa_2 B_2)\left(\tanh^{p_1-1}\tau - \tanh^{p_1+1}\tau\right) = 0, \tag{6}$$

$$\begin{aligned} &p_2 A_2\left(B_1^2 + B_2^2\right)\left\{(p_2-1)\tanh^{p_2-2}\tau - 2p_2\tanh^{p_2}\tau\right. \\ &+ \left.(p_2+1)\tanh^{p_2+2}\tau\right\} - 2np_1\beta A_1^{2n}B_1^2\left\{4np_1\tanh^{2np_1}\tau\right. \\ &- \left.(2np_1-1)\tanh^{2np_1-2}\tau - (2np_1+1)\tanh^{2np_1+2}\tau\right\} \\ &= 0. \end{aligned} \tag{7}$$

From the imaginary part equation (6), the soliton velocity is given by

$$v = -2a(\kappa_1 B_1 + \kappa_2 B_2). \tag{8}$$

From the balancing principle applied to (5), equating the exponents $(2n+1)p_1$ and p_1+2 leads to

$$(2n+1)p_1 = p_1 + 2, \tag{9}$$

that implies

$$p_1 = \frac{1}{n}. \tag{10}$$

Similarly, equating the exponents $(2n+1)p_1$ and p_1+p_2 in (5), leads to

$$p_2 = 2np_1, \tag{11}$$

so that by virtue of (10),

$$p_2 = 2. \tag{12}$$

Now from (5), the stand-alone linearly independent term is $\tanh^{p_1-2}\tau$. whose coefficient must therefore be zero. This yields

$$p_1 = 1, \tag{13}$$

so that from (10),

$$n = 1. \tag{14}$$

Therefore, the DS equation with power law nonlinearity must reduce to DS equation with cubic nonlinearity in order to support the topological soliton solution. Now, setting the coefficients of the other linearly independent functions $\tanh^{p_1+j}\tau$ for $j=0, 2$, from (5) to zero leads to

$$B_1^2 + B_2^2 = -\frac{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)}{2a}, \tag{15}$$

and

$$bA_1^2 - \alpha A_2 = \omega + a\left(\kappa_1^2 + \kappa_2^2\right) \tag{16}$$

. Thus (15), introduces the constraint condition

$$a\left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\} < 0. \tag{17}$$

Now, from (7), the balancing principle yields, upon equating the exponents $2np_1+2$ and p_2+2, the same relation as (11) and thereafter the rest of the analysis follows. Again, setting the coefficients of the linearly independent functions $\tanh^{p_2+j}\tau$ for $j=-2$, 0 and 2, from (7) to zero, all lead to the same relation given by

$$2a\beta A_1^2 B_1^2 - A_2\left\{\omega + a\left(\kappa_1^2 + \kappa_2^2\right)\right\} = 0. \tag{18}$$

Hence, finally the DS equation with power law nonlinearity reduces to DS equation with cubic nonlinearity that is given by

$$\begin{cases} iq_t + a(q_{xx} + q_{yy}) + b|q|^2 q = \alpha qr, \\ r_{xx} + r_{yy} + \beta\left(|q|^2\right)_{xx} = 0, \end{cases} \tag{19}$$

whose topological 1-soliton solution is given by

$$\begin{cases} q(x,y,t) = A_1\tanh(B_1 x + B_2 y - vt)e^{i(-\kappa_1 x - \kappa_2 y + \omega t + \theta)} \\ r(x,y,t) = A_2\tanh^2(B_1 x + B_2 y - vt), \end{cases} \tag{20}$$

where the velocity of the solitons is given by (8) and the relation between the free parameters is given by the coupled equations (15), (16) and (18). These introduce the constraint condition that is given by (17).

3 Bifurcation analysis

This section will carry out the bifurcation analysis of the DS equation with power law nonlinearity. Initially, the phase portraits will be obtained and the corresponding qualitative analysis will be discussed. Several interesting properties of the solution structure will be obtained based on the parameter regimes. Subsequently, the traveling wave solutions will be discussed from the bifurcation analysis.

3.1 Phase portraits and qualitative analysis

We assume that the traveling wave solutions of (1) is of the form

$$\begin{cases} q(x,y,t) = e^{i\eta}\varphi(\xi), \quad r(x,y,t) = \phi(\xi), \\ \eta = kx + ly + wt, \quad \xi = px + my - ct, \end{cases} \tag{21}$$

where $\varphi(\xi)$ and $\phi(\xi)$ are real functions, k, l, w, p, m and c are real constants.
Substituting (21) into (1)), we find that $c = 2a(kp + lm)$, φ and ϕ satisfy the following system:

$$\begin{cases} (ap^2 + am^2)\varphi'' - \alpha\varphi\phi - (w + ak^2 + al^2)\varphi \\ +b\varphi^{2n+1} = 0, \\ (p^2 + m^2)\phi'' + \beta p^2(\varphi^{2n})'' = 0. \end{cases} \tag{22}$$

Integrating the second equation (22) twice and letting the first integral constant be zero, we have

$$\phi = \frac{-\beta p^2 \varphi^{2n}}{p^2 + m^2} + g, \tag{23}$$

where g is the second integral constant.
Substituting (23) into the first equation of (22), we have

$$a(p^2 + m^2)\varphi'' - (w + ak^2 + al^2 + \alpha g)\varphi$$
$$+ (b + \frac{\alpha\beta}{p^2 + m^2})\varphi^{2n+1} = 0. \tag{24}$$

To facilitate discussions, we let

$$\delta = \frac{\alpha\beta p^2 + b(p^2 + m^2)}{a(p^2 + m^2)^2}, \tag{25}$$

$$\theta = \frac{w + ak^2 + al^2 + \alpha g}{a(p^2 + m^2)}. \tag{26}$$

Letting $\varphi' = z$, then we get the following planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = z, \\ \frac{dz}{d\xi} = -\delta\varphi^{2n+1} + \theta\varphi. \end{cases} \tag{27}$$

Obviously, the above system (27) is a Hamiltonian system with Hamiltonian function

$$H(\varphi, z) = z^2 + \frac{\delta}{n+1}\varphi^{2n+2} - \theta\varphi^2. \tag{28}$$

In order to investigate the phase portrait of (31), set

$$f(\varphi) = -\delta\varphi^{2n+1} + \theta\varphi. \tag{29}$$

Obviously, when $\delta\theta > 0$, $f(\varphi)$ has three zero points, φ_-, φ_0 and φ_+, which are given as follows

$$\varphi_- = -\left(\frac{\theta}{\delta}\right)^{\frac{1}{2n}}, \qquad \varphi_0 = 0, \qquad \varphi_+ = \left(\frac{\theta}{\delta}\right)^{\frac{1}{2n}}. \tag{30}$$

When $\delta\theta \leqslant 0$, $f(\varphi)$ has only one zero point

$$\varphi_0 = 0. \tag{31}$$

Letting $(\varphi_i, 0)$ be one of the singular points of system (27), then the characteristic values of the linearized system of system (27) at the singular points $(\varphi_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(\varphi_i)}. \tag{32}$$

From the qualitative theory of dynamical systems, we know that

(I) If $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point.
(II) If $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point.
(III) If $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the bifurcation phase portraits of system (31) in Figure 1.
Let

$$H(\varphi, z) = h, \tag{33}$$

where h is Hamiltonian.
Next, we consider the relations between the orbits of (27) and the Hamiltonian h.
Set

$$h^* = |H(\varphi_+, 0)| = |H(\varphi_-, 0)|. \tag{34}$$

According to Figure 1, we get the following propositions.

Proposition 1.*Suppose that $\delta > 0$ and $\theta > 0$, we have*
(I) When $h \leqslant -h^*$, system (27) does not any closed orbit.
(II) When $-h^* < h < 0$, system (27) has two periodic orbits Γ_1 and Γ_2.
(III) When $h = 0$, system (27) has two homoclinic orbits Γ_3 and Γ_4.
(IV) When $h > 0$, system (27) has a periodic orbit Γ_5.

Proposition 2.*Suppose that $\delta < 0$ and $\theta < 0$, we have*
(I) When $h < 0$ or $h > h^*$, system (27) does not any closed orbit.
(II) When $0 < h < h^*$, system (27)) has three periodic orbits Γ_6, Γ_7 and Γ_8.
(III) When $h = 0$, system (27) has two periodic orbits Γ_9 and Γ_{10}.
(IV) When $h = h^*$, system (27) has two heteroclonic orbits Γ_{11} and Γ_{12}.

Proposition 3.*(I)When $\delta > 0$, $\theta \geqslant 0$ and $h > 0$, system (27) have a periodic orbits.*
(II)When $\delta < 0$, $\theta \leqslant 0$, system (27) does not any closed orbit.

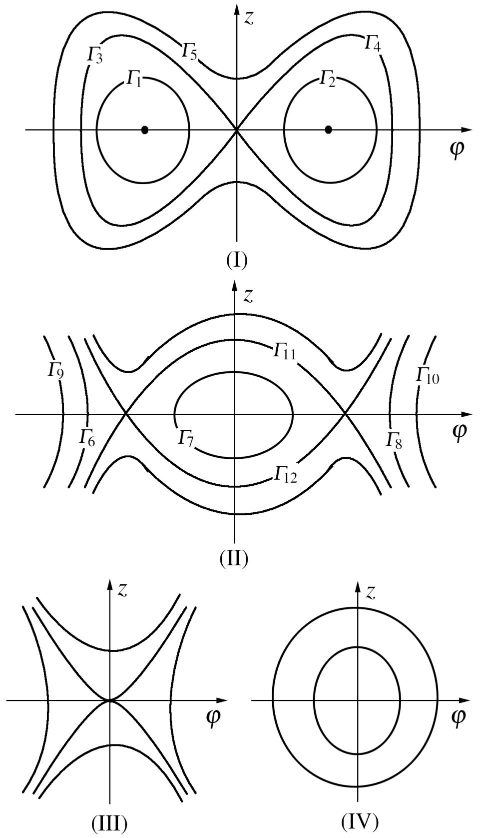

Figure 1 The bifurcation phase portraits of system (31). (I) $\delta > 0$, $\theta > 0$, (II) $\delta < 0$, $\theta < 0$, (III)$\delta < 0$, $\theta \geqslant 0$, (IV) $\delta > 0$, $\theta \leqslant 0$.

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink wave solution or a unbounded wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to above analysis, we have the following propositions.

Proposition 4. *If $\delta > 0$ and $\theta > 0$, we have*

(I) When $-h^* < h < 0$, the equations (1)) has two periodic wave solutions(corresponding to the periodic orbits Γ_1 and Γ_2 in Figure 1).

(II) When $h = 0$, the equations (1)) has two solitary wave solutions(corresponding to the homoclinic orbits Γ_3 and Γ_4 in Figure 1).

(III) When $h > 0$, the equations (1)) has two periodic wave solutions(corresponding to the periodic orbit Γ_5 in Figure 1) .

Proposition 5. *If $\delta < 0$ and $\theta < 0$, we have*

(I)When $0 < h < h^*$, the equations (1)) has two periodic wave solutions (corresponding to the periodic orbit Γ_7 in Fig. 1) and two periodic blow-up wave solutions(corresponding to the periodic orbits Γ_6 and Γ_8 in Figure 1).

(II) When $h = 0$, the equations (1)) has periodic blow-up wave solutions(corresponding to the periodic orbits Γ_9 and Γ_{10} in Figure 1).

(III) When $h = h^*$, the equations (1)) has two kink profile solitary wave solutions (corresponding to the heteroclinic orbits Γ_{11} and Γ_{12} in Figure 1).

3.2 Exact traveling wave solutions

Firstly, we will obtain the explicit expressions of traveling wave solutions for the equations (1)) when $\delta > 0$ and $\theta > 0$. From the phase portrait, we see that there are two symmetric homoclinic orbits Γ_3 and Γ_4 connected at the saddle point $(0,0)$. In (φ, z)-plane the expressions of the homoclinic orbits are given as

$$z = \pm\sqrt{\frac{\delta}{n+1}}\varphi\sqrt{-\varphi^{2n} + \frac{(n+1)\theta}{\delta}}. \tag{35}$$

Substituting (35) into $\frac{d\varphi}{d\xi} = z$ and integrating them along the orbits Γ_3 and Γ_4, we have

$$\pm\int_{\varphi_1}^{\varphi} \frac{1}{s\sqrt{-s^{2n} + \frac{(n+1)\theta}{\delta}}} ds = \sqrt{\frac{\delta}{n+1}}\int_0^{\xi} ds, \tag{36}$$

$$\pm\int_{\varphi_2}^{\varphi} \frac{1}{s\sqrt{-s^{2n} + \frac{(n+1)\theta}{\delta}}} ds = \sqrt{\frac{\delta}{n+1}}\int_0^{\xi} ds, \tag{37}$$

where $\varphi_1 = -\left(\frac{(n+1)\theta}{\delta}\right)^{\frac{1}{2n}}$ and $\varphi_2 = \left(\frac{(n+1)\theta}{\delta}\right)^{\frac{1}{2n}}$.
Completing above integrals we obtain

$$\varphi = \left(\sqrt{\frac{(n+1)\theta}{\delta}} \operatorname{sech} n\sqrt{\theta}\xi\right)^{\frac{1}{n}}, \tag{38}$$

and

$$\varphi = -\left(\sqrt{\frac{(n+1)\theta}{\delta}} \operatorname{sech} n\sqrt{\theta}\xi\right)^{\frac{1}{n}}. \tag{39}$$

Noting that (21) and (23), we get the following solitary wave solutions

$$\begin{cases} q_1(x,y,t) = e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}} \operatorname{sech} n\sqrt{\theta}\xi\right)^{\frac{1}{n}}, \\ r_1(x,y,t) = \frac{-(n+1)\beta\theta(\operatorname{sech} n\sqrt{\theta}\xi)^2}{\delta(p^2+m^2)} + g, \end{cases} \tag{40}$$

and

$$\begin{cases} q_2(x,y,t) = -e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}} \operatorname{sech} n\sqrt{\theta}\xi\right)^{\frac{1}{n}}, \\ r_2(x,y,t) = \frac{-(n+1)\beta\theta(\operatorname{sech} n\sqrt{\theta}\xi)^2}{\delta(p^2+m^2)} + g, \end{cases} \tag{41}$$

where δ is given by (25), θ is given by (26), $\eta = kx + ly + wt$ and $\xi = px + my - ct$.
Secondly, we will obtain the explicit expressions of traveling wave solutions for the equations (1)) when

$\delta < 0$ and $\theta < 0$. From the phase portrait, we note that there are two special orbits Γ_9 and Γ_{10}, which have the same hamiltonian with that of the center point $(0,0)$. In (φ, z)-plane the expressions of the orbits are given as

$$z = \pm\sqrt{-\frac{\delta}{n+1}}\varphi\sqrt{\varphi^{2n} - \frac{(n+1)\theta}{\delta}}. \tag{42}$$

Substituting (42) into $\frac{d\varphi}{d\xi} = z$ and integrating them along the two orbits Γ_9 and Γ_{10}, it follows that

$$\pm\int_{\varphi}^{+\infty} \frac{1}{s\sqrt{s^{2n} - \frac{(n+1)\theta}{\delta}}} ds = \sqrt{-\frac{\delta}{n+1}}\int_0^{\xi} ds, \tag{43}$$

$$\pm\int_{\varphi_4}^{\varphi} \frac{1}{s\sqrt{s^{2n} - \frac{(n+1)\theta}{\delta}}} ds = \sqrt{-\frac{\delta}{n+1}}\int_0^{\xi} ds, \tag{44}$$

where $\varphi_4 = \left(\frac{(n+1)\theta}{\delta}\right)^{\frac{1}{2n}}$.
Completing above integrals we obtain

$$\varphi = \pm\left(\sqrt{\frac{(n+1)\theta}{\delta}} \csc n\sqrt{-\theta}\xi\right)^{\frac{1}{n}}. \tag{45}$$

$$\varphi = \pm\left(\sqrt{\frac{(n+1)\theta}{\delta}} \sec n\sqrt{-\theta}\xi\right)^{\frac{1}{n}}. \tag{46}$$

Noting that (21) and (23), we get the following periodic blow-up wave solutions

$$\begin{cases} q_3(x,y,t) = \pm e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}} \csc n\sqrt{-\theta}\xi\right)^{\frac{1}{n}}, \\ r_3(x,y,t) = \frac{-(n+1)\beta\theta(\csc n\sqrt{-\theta}\xi)^2}{\delta(p^2+m^2)} + g, \end{cases} \tag{47}$$

and

$$\begin{cases} q_4(x,y,t) = \pm e^{i\eta}\left(\sqrt{\frac{(n+1)\theta}{\delta}} \sec n\sqrt{-\theta}\xi\right)^{\frac{1}{n}}, \\ r_4(x,y,t) = \frac{-(n+1)\beta\theta(\sec n\sqrt{-\theta}\xi)^2}{\delta(p^2+m^2)} + g, \end{cases} \tag{48}$$

where δ is given by (25), θ is given by (26), $\eta = kx + ly + wt$ and $\xi = px + my - ct$.

4 Conclusion

This paper studied the DS equation with power law nonlinearity in a fairly detailed fashion. First of all, the ansatz method was applied to the DS equation and thus the topological soliton or rather, topological defect of the DS equation was obtained. In this context, the conclusion is that the DS equation with power law nonlinearity supports topological solitons provided the power law nonlinearity parameter n collapses to $n = 1$. This means that the power law nonlinearity must compress to cubic nonlinearity in order for the DS equation to support topological soliton solution. Additionally, this paper also carried out the bifurcation analysis of this equation. This analysis lead to the study of the various fixed points and thus, in turn, other traveling wave solutions were obtained. These are solitary wave solutions and singular periodic solutions.
These results are going to be very useful in further future studies where time-dependent coefficients of dispersion and nonlinearity terms are gong to be considered. Additionally, the stochastic perturbation terms are to be considered where the mean free velocity of the soliton will be determined by the aid of Langevin equation. These are just the preliminary pictures of profound future works.

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Ming Song is Professor of mathematics at Shaoxing University, China. He received the PhD degree in applied mathematics from South China University of Technology. His research interests are in the areas of applied mathematics. He has published extensively in internationally refereed journals.

Anjan Biswas earned his BS degree in Mathematics from St. Xavier's College in Calcutta, India. Subsequently he obtained his M.Sc and M.Phil degrees in Applied Mathematics from the University of Calcutta, India. After that, he earned his MA and Ph.D. degrees in Applied Mathematics from the University of New Mexico in Albuquerque, NM, USA. Currently, he is an Associate Professor of Mathematics in the Department of Mathematical Sciences at Delaware State University in Dover, DE, USA. His current research area of interest is Theory of Solitons.