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Totally Equivalent H-J Matrices

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Abstract: In 1927 W. A. Hurwitz showed that a row finite matrix is totally regular if and only if it has at most a finite number of diagonals with negative entries. He also proved that a regular Hausdorff matrix is totally regular if and only if it has all nonnegative entries. In 1921 Hausdorff proved that the Hölder and Cesáro matrices are equivalent for each $\alpha > -1$. Basu, in 1949, compared these matrices totally. In this paper we investigate these theorems of Hurwitz, Hausdorff, and Basu for the E-J and H-J generalized Hausdorff matrices.

Keywords: E-J matrices, H-J matrices, totally equivalent, totally monotone, totally regular

Let A be an infinite matrix. Then the convergence domain of A, c_A, is the set of all real or complex sequences $\{x_n\}$ for which $A_n(x) := \sum_k a_{nk}x_k$ converges. A matrix A is called conservative if its convergence domain contains c, the space of convergence sequences. Necessary and sufficient conditions for a matrix A to be conservative are the well-known Silverman-Toeplitz conditions:

(i) $\|A\|_\infty = \sup_n \sum_k |a_{nk}| < \infty$,

(ii) $t := \lim_n \sum_k a_{nk}$ exists,

(iii) $a_k := \lim_n a_{nk}$ exists for each k.

A matrix A is called regular if it preserves the limit of every convergence sequence. Necessary and sufficient conditions for regularity are

(i) $\|A\|_\infty < \infty$,

(ii) $t = 1$,

(iii) $a_k = 0$ for each k.

Considering only real sequences, a matrix A is said to be totally regular if it is regular and preserves the points at $\pm\infty$; i.e., if $x_n \to +\infty$ then $A_n(x) \to +\infty$ (and $x_n \to -\infty$ implies that $A_n(x) \to -\infty$).

A matrix A is said to be row finite if each row of A as only a finite number of nonzero entries. In 1927 W. A. Hurwitz [8] proved that a row finite regular matrix A is totally regular if and only if it has a finite number of columns containing negative elements; i.e., there exists a $k_0 \geq 0$ such that $a_{nk} \geq 0$ for each n and each $k \geq k_0$.

A Hausdorff matrix $H = (h_{nk})$ is a lower triangular matrix with entries

$$h_{nk} = \binom{n}{k} \Delta^{n-k}\mu_k,$$

where $\{\mu_n\}$ is any sequence and Δ is the forward difference operator defined by $\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\Delta^{n+1}\mu_k = \Delta(\Delta^n\mu_k)$. Every Hausdorff matrix has row sums equal to μ_0.

They were defined by F. Hausdorff [6], who showed, among other things, that a Hausdorff matrix is conservative if and only if μ_n has the representation

$$\mu_n = \int_0^1 x^n d\chi(x), \tag{1}$$

where $\chi(x)$ is a function of bounded variation over $[0,1]$. The function χ is called the mass function associated with the μ_n and $\{\mu_n\}$ is called the moment generating sequence for the corresponding Hausdorff matrix. The same terminology is used for generalizations of Hausdorff matrices.

If a Hausdorff matrix has finite norm, then it is conservative and all of the column limits, except possibly the first, are zero.

Hurwitz [8] also proved that a regular Hausdorff matrix is totally regular if and only if it has all nonnegative entries.

A lower triangular matrix A with each $a_{nn} \neq 0$ is called a triangle. Two regular triangles A and B are said to

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be totally equivalent if both AB^{-1} and BA^{1-} are totally regular. Basu [3] showed that two regular Hausdorff triangles are totally equivalent if and only if they are identical. Since it is known that C^α and H^α, the Cesàro and Hölder matrices of order $\alpha > -1$, respectively, are equivalent, Basu's result motivates the study of the following question. Given two equivalent regular Hausdorff matrices, which is totally stronger? Basu [3] answered this question for the Hölder and Cesàro matrices.

Theorem 1.(a) *For* $-1 < \alpha < 0$, H^α *is totally stronger* (t.s.) *than* C^α.
(b) *For* $0 < \alpha < 1$, *is t.s.* H^α.
(c) *For* $1 < \alpha < \infty$ H^α *is t.s.* C^α.

There are two well-known generalizations of Hausdorff matrices. One of these was defined independently by Endl [4] and Jakimovski [9], and are called the E-J generalized Hausdorff matrices. For any $\alpha \geq 0$, the entries of an E-J matrix $H_{(\alpha)} = (h_{nk}^{(\alpha)})$ are defined by

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k}\Delta^{n-k}\mu_k^{(\alpha)}, \quad 0 \leq k \leq n.$$

The case $\alpha = 0$ reduces to the ordinary Hausdorff matrices.

For a conservative E-J matrix, the $\mu_n^{(\alpha)}$ satisfy the condition

$$\mu_n^{(\alpha)} = \int_0^1 x^{n+\alpha} d\chi(x),$$

where $\chi \in BV[0,1]$. Let $\{\lambda_n\}$ be defined by

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots,$$

such that

$$\sum_k \frac{1}{\lambda_k} = \infty.$$

Hausdorff [7] defined another class of generalized Hausdorff matrices $H(\mu;\lambda) = h(\mu;\lambda)_{nk}$ with nonzero entries

$$h(\mu;\lambda)_{nk} = \lambda_{k+1}\cdots\lambda_n[\mu_k,\ldots,\mu_n], \quad 0 \leq k \leq n,$$

where $[\cdot]$ is the divided difference operator defined by

$$[\mu_k,\mu_{k+1}] = \frac{1}{\lambda_{k+1}-\lambda_k}(\mu_k-\mu_{k+1}),$$

and

$$[\mu_k,\ldots,\mu_{n+1}] = \frac{1}{\lambda_{n+1}-\lambda_k}([\mu_k,\ldots,\mu_n]-[\mu_{k+1},\ldots,\mu_{n+1}]).$$

An H-J matrix is conservative if and only if the μ_n have the representation

$$\mu_n = \int_0^1 x^{\lambda_n} d\chi(x), \tag{2}$$

where $\chi \in BV[0,1]$. Jakimovski [9] extended this class to consider the cases in which $\lambda_0 > 0$. Therefore it is appropriate to call these generalized Hausdorff matrices the H-J matrices.

The H-J generalization reduces to the E-J class by choosing $\lambda_n = n+\alpha$, and to ordinary Hausdorff matrices by choosing $\lambda_n = n$.

One of the important programs has been to extend known properties for Hausdorff matrices to the E-J or H-J generalizations. In this paper we continue that study by extending some of the results of Hurwitz and Basu to E-J and H-J matrices.

A sequence is called totally monotone if all of the successive forward differences are nonnegative; i.e., $\Delta^n\mu_k \geq 0$ for all n and k. The corresponding statement for H-J sequences is that $[\mu_k,\ldots,\mu_n] \geq 0$ for all n and k.

Our first result is to show that every totally regular H-J matrix has all nonnegative entries. The method of proof is different from that of Hurwitz.

Theorem 2.*Let H be a regular H-J matrix. Then H is totally regular if and only if H has all nonnegative entries.*

*Proof.*Let H be a totally regular H-J matrix. Then, from Hurwitz [8], there exists an integer $k_0 \geq 0$ such that $h_{nk} \geq 0$ for all n and all $k \geq k_0$. Assuming that k_0 is positive and the smallest such integer, choose N to be the smallest value of n such that $h_{N,k_0-1} < 0$. Then, from the definition of the entries of an H-J matrix,

$$\begin{aligned}
&h_{N+1,k_0-1} - h_{N,k_0-1}\\
&= \lambda_{k_0}\cdots\lambda_{N+1}[\mu_{k_0-1},\ldots,\mu_{N+1}] - \lambda_{k_0}\cdots\lambda_N[\mu_{k_0-1},\ldots,\mu_N]\\
&= \lambda_{k_0}\cdots\lambda_N\Big\{\lambda_{N+1}([\mu_{k_0-1},\ldots,\mu_{N+1}] - [\mu_{k_0-1},\ldots,\mu_N]\Big\}\\
&= \lambda_{k_0}\cdots\lambda_N\Big\{\frac{\lambda_{N+1}[\mu_{k_0-1},\ldots,\mu_N]-[\mu_{k_0},\ldots,\mu_{N+1}]}{\lambda_{N+1}-\lambda_{k_0-1}}\\
&\qquad - [\mu_{k_0-1},\ldots,\mu_N]\Big\}\\
&= \frac{\lambda_{k_0}\cdots\lambda_N}{\lambda_{N+1}-\lambda_{k_0-1}}\Big\{\lambda_{k_0-1}[\mu_{k_0-1},\ldots,\mu_N] - \lambda_{N+1}[\mu_{k_0},\ldots,\mu_{N+1}]\Big\}.
\end{aligned}$$

Since $h_{N,k_0-1} < 0, [\mu_{k_0-1},\ldots,\mu_N] < 0$. Since $h_{N+1,k_0} > 0, [\mu_{k_0},\ldots,\mu_{N+1}] > 0$. Therefore $h_{N+1,k_0-1} - h_{N,k_0-1} < 0$ i.e., $h_{N+1,k_0-1} < h_{N,k_0-1}$.

Assuming that $h_{N+n,k_0-1} < 0$ it can be shown, in the same manner, that $h_{N+n+1,k_0-1} < 0$, and therefore h_{N+n,k_0-1} is a monotone decreasing negative sequence. Hence

$$lim_n h_{N+n,k_0-1} \leq h_{N,k_0-1} < 0,$$

contradicting the fact that, since H is regular, every column limit must be zero. Consequently column k_0-1 also has all nonnegative terms. Continuing this process it follows that H has all nonnegative entries.

The converse is trivial.

Corollary 1.*Let* $H^{(\alpha)}$ *be a regular E-J matrix. Then* $H^{(\alpha)}$ *is totally regular if and only if* $H^{(\alpha)}$ *has all nonnegative entries.*

Proof. In Theorem 1 set $\lambda_n = n+\alpha$.

Corollary 2. *Let H be a regular Hausdorff matrix. Then H is totally regular if and only if H has all nonnegative entries.*

Proof. In Corollary 2 set $\alpha = 0$.

Corollary 2 is Theorem 6 of [8].

We shall now extend the result of Basu [3] to H-J matrices.

Theorem 3. *Two H-J triangles are totally equivalent if and only if one is a positive scalar multiple of the other.*

Proof. For brevity of notation, let H_α and H_β denote two totally equivalent H-J triangles with entries

$$h_{nk} = \lambda_{k+1}\cdots\lambda_n[\alpha_k,\ldots,\alpha_n],$$

and

$$h'_{nk} = \lambda_{k+1}\cdots\lambda_n[\beta_k,\ldots,\beta_n],$$

respectively.

From the definition of total equivalence, $H_\alpha H_\beta^{-1}$ and $H_\beta H_\alpha^{-1}$ are both totally regular. By Theorem 1 they have all nonnegative entries. In particular, for each $n\in\mathbb{N}$,

$$\begin{aligned}(H_\alpha H_\beta^{-1})_{n,n-1} &= \lambda_n\left[\frac{\alpha_{n-1}}{\beta_{n-1}},\frac{\alpha_n}{\beta_n}\right]\\ &= \frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\left(\frac{\alpha_{n-1}}{\beta_{n-1}}-\frac{\alpha_n}{\beta_n}\right)\geq 0,\end{aligned}$$

which implies that

$$\frac{\alpha_{n-1}}{\beta_{n-1}} \geq \frac{\alpha_n}{\beta_n}. \tag{3}$$

Similarly,

$$(H_\beta H_\alpha^{-1})_{n,n-1} = \lambda_n\left[\frac{\beta_{n-1}}{\alpha_{n-1}},\frac{\beta_n}{\alpha_n}\right]\geq 0,$$

which implies that

$$\frac{\beta_{n-1}}{\alpha_{n-1}}-\frac{\beta_n}{\alpha_n}\geq 0. \tag{4}$$

Combining (3) and (4) gives

$$\frac{\alpha_n}{\beta_n}\geq\frac{\alpha_{n-1}}{\beta_{n-1}}\geq\frac{\alpha_n}{\beta_n},$$

which implies that

$$\frac{\alpha_n}{\beta_n}=\frac{\alpha_{n-1}}{\beta_{n-1}}.$$

In particular,

$$\frac{\alpha_n}{\beta_n}=\frac{\alpha_0}{\beta_0}. \tag{5}$$

Since α_0 and β_0 are positive, H_α is a positive scalar multiple of H_β.

The converse is trivial.

Corollary 3. *Let H_μ and H_γ be two totally regular H-J triangles with $\lambda_0 = 0$. Then they are totally equivalent if and only if they are identical.*

Proof. Since $\lambda_0 = 0$, as shown by Hausdorff [7], the row sums of H_α and H_β are α_0 and β_0, respectively. Since the matrices are regular, $\alpha_0=\beta_0=1$, and the result follows from equation (5).

Corollary 4. *Let H_1 and H_2 be two totally regular Hausdorff triangles. Then they are totally equivalent if and only if they are identical.*

Proof. In Corollary 3 set $\lambda_n = n$ for each n.

Corollary 4 is Lemma 2 of Basu [3].

Corollary 5. *Two E-J triangles are totally equivalent if and only if one is a positive scalar multiple of the other.*

Proof. In Theorem 3 set $\lambda_n = n+\alpha$.

It is well known that, for every $\alpha>\beta>-1$, C_α is totally stronger than C_β, and that, for $\alpha>\beta\geq 0$, $H_\alpha t.s.H_\beta$, where C_α and H_α denote the Cesàro and Holder matrices, respectively. We shall now show that the same conclusion holds for the H-J analogues.

Theorem 4. *(a) Let $\alpha>\beta>-1$. Then $C(\lambda;\alpha)t.s.C(\lambda;\beta)$.*
(b) Let $\alpha>\beta>0$. Then $H(\lambda;\alpha)t.s.H(\lambda;\beta)$.

Proof. The moment generating sequence for $C(\lambda;\alpha)$ is

$$\mu_n=\int_0^1 t^{\lambda_n}\alpha(1-\alpha)^{\alpha-1}dt=\frac{\Gamma(\lambda_n+1)\alpha\Gamma(\alpha)}{\Gamma(\lambda_n+\alpha+1)}.$$

Therefore the generating sequence for $C(\lambda;\alpha)(C(\lambda;\beta))^{-1}$ is

$$\begin{aligned}\rho_n &= \frac{\Gamma(\lambda_n+1)\Gamma(\alpha+1)}{\Gamma(\lambda_n+\alpha+1)}\frac{\Gamma(\lambda_n+\beta+1)}{\Gamma(\lambda_n+1)\Gamma(\beta+1)}\\ &=\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(1-\beta)}\frac{\Gamma(\lambda_n+\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\lambda_n+\alpha+1)}\\ &=\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(1-\beta)}\int_0^1 t^{\lambda_n+\beta}(1-t)^{\alpha-\beta-1}dt\\ &=\int_0^1 t^{\lambda_n}d\chi(t),\end{aligned}$$

where

$$d\chi(t)=\frac{\Gamma(\alpha+1)t^\beta(1-t)^{\alpha-\beta-1}}{\Gamma(\beta+1)\Gamma(\alpha-\beta)},$$

and χ is increasing over [0, 1]. Therefore $\{\rho_n\}$ is totally monotone.

(b). Let ε_n denote the moment generating sequence for $H(\lambda;\alpha)$. Then

$$\varepsilon_n=\int_0^1 t^{\lambda_n}\left(\log\left(\frac{1}{t}\right)\right)^{\alpha-1}dt=\frac{\Gamma(\alpha)}{(\lambda_n+1)^\alpha},$$

and the moment generating sequence for $H(\lambda;\alpha)(H(\lambda;\beta)^{-1}$ is

$$\tau_n = \frac{\Gamma(\alpha)}{(\lambda_n+1)^{\alpha}}\frac{(\lambda_n+1)^{\beta}}{\Gamma(\beta)} = \frac{\Gamma(\alpha)}{\Gamma(\beta)(\lambda_n+1)^{\alpha-\beta}},$$

which is clearly totally monotone.

Basu ([3]) compared totally the Cesàro and Hölder matrices of the same order α for $\alpha > -1$. We shall now extend this result to E-J matrices.

Theorem 5. *Let $\alpha \geq 0$. (i) For $-1 < \beta < 0$, $H_\beta^{(\alpha)}$ t.s. $C_\beta^{(\alpha)}$.*

(ii) For $0 \leq \beta < 1$, $C_\beta^{(\alpha)}$ t.s. $H_\beta^{(\alpha)}$.

(iii) For $1 \leq \beta$, $H_\beta^{(\alpha)}$ t.s. $C_\beta^{(\alpha)}$.

The method of proof will make use of Theorem 1 of [10], which reads as follows:

Lemma 1. *Let $\mu = \{\mu_n\}$ be a real positive sequence, $f(t)$ a function of class C^∞ for $t > 0$ such that $f(k) = \mu_k, k = 0,1,2,\ldots, g(t) = f(t)/f(t+1), h(t) = (-1)g'(t)/g(t)$. If $\lim_{t\to\infty} g(t) \geq 1, g(t) > 0$, and $h(t)$ is totally monotone for $t > 0$, then μ is a totally monotone sequence.*

Let μ_n denote the moment generating sequence for $H_\beta^{(\alpha)}(C_\beta^{(\alpha)})^{-1}$. Then

$$\mu_n = \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)(n+\alpha+1)^{\beta}\Gamma(n+\alpha+1)}.$$

Let

$$g(t) = \frac{(t+\alpha+2)^{\beta}}{(t+\alpha+1)^{\beta-1}(t+\alpha+\beta+1)}.$$

Then

$$h(t) := (-1)\frac{g'(t)}{g(t)} = \frac{(\beta-1)}{(t+\alpha+1)} + \frac{1}{(t+\alpha+\beta+1)} - \frac{\beta}{t+\alpha+2}. \tag{6}$$

The corresponding mass function is

$$\phi(u) = (\beta-1)u + \frac{u^{\beta+1}}{\beta+1} - \frac{\beta u^2}{2}.$$

which satisfies

$$\frac{d\phi}{du} = \beta - 1 + u^{\beta} - \beta u,$$

and

$$\frac{d^2(\phi)}{du^2} = \beta u^{\beta-1} - \beta = \beta(u^{\beta-1}-1).$$

Case I. $-1 < \beta < 0$. Then $d^2\phi/du^2 < 0$ and $d\phi/du$ is monotone decreasing in u. Since $d\phi/du(1) = 0$, ϕ is totally monotone.

Case 2. $1 \leq \beta$. (Assume that $\beta > 1$, since the two matrices are identical for $\beta = 1$.) Again $d^2\phi/du^2 < 0$ and $d\phi/du(1) = 0$, so again ϕ is totally monotone.

Case 3. $0 \leq \beta < 1$. (Assume that $\beta > 0$ since, for $\beta = 0$, both matrices reduce to the identity matrix.) In analyzing $1/\mu_n$, the corresponding h-function is equal to $-h$. Therefore, from (6),

$$\frac{d(-\phi)}{du} = 1 - \beta - u^{\beta} + \beta u,$$

and

$$\frac{d^2(-\phi)}{du^2} = -\beta u^{\beta-1} + \beta = \beta(1-u^{\beta-1}) > 0.$$

Then $d(-\phi)/du$ is monotone increasing in u. Since $d(-\phi/du)(0) = 1-\beta > 0$, $-\phi(u)$ is totally monotone.

Theorem 6. *For each $\alpha,\beta,\delta > 0$ $C_\delta^{(\alpha)}$ and $C_\delta^{(\beta)}$ are totally equivalent.*

Proof. From Theorem 1 of [2] the matrices are equivalent. They are also totally regular. With C_δ denoting the ordinary Hausdorff Cesáro matrix of order δ, it was shown in the proof of Theorem 1 that $C_\delta^{(\alpha)}$ and C_δ are equivalent. That fact was proved by showing that $C_\delta^{(\alpha)}(C_\delta)^{-1}$ is a diagonal matrix with positive entries, with the limit of the diagonal entries one. Therefore $C_\delta(C_\delta^{(\alpha)})^{-1}$ is a diagonal matrix with positive entries with diagonal limit one.

For any $\alpha,\beta > 0$, we may write

$$C_\delta^{(\alpha)}(C_\delta^{(\beta)})^{-1} = (C(\alpha)_\delta(C_{\delta)})^{-1})(C_{(\delta)}(C_\delta^{(\beta)})^{-1}),$$

the associativity of multiplication being guaranteed since the matrices are triangles. It then follows that $C_\delta^{(\alpha)}(C_\delta^{(\beta)})^{-1}$ and its inverse are each diagonal matrices with positive entries and limit one. Therefore the two matrices are totally equivalent.

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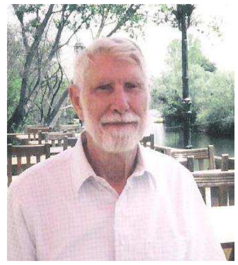

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