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Periodic and Secular Solutions in the Restricted Three–Body Problem under the Effect of Zonal Harmonic Parameters

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Abstract: The aim of the present work is to study the periodic structure of the restricted three–body problem considering the effect of the zonal harmonics J_2 and J_4 for the more massive body. We show that the triangular points in the restricted three–body problem have long or short periodic orbits in the range $0 \leq \mu < \mu_c$. We also present a graphical analysis for the variations of the angular frequencies for the long and short periodic orbits computing explicitly the expressions of the lengths of the semi–major and semi–minor axes and determining the orientations of the principal axes for the ellipses that represent periodic orbits. Moreover, the secular solution when $\mu = \mu_c$ is stated and it is proved that the triangular points have periodic orbits in this case too. This model has special significance in space missions either to place telescopes or for dispatching satellites and exploring vehicles.

Keywords: Restricted three–body problem, Periodic solution, Secular solution, Oblateness coefficients, Zonal harmonics J_2 and J_4

1 Introduction

There are many motivations for studying periodic orbits in the framework of the restricted–three body problem, some of them are the following: For this model it is always possible to find a periodic solution for any particular solution. Also asymptotic and periodic solutions can be obtained from linearized solutions of the motion in the proximity of the libration points. It is clear that it is not possible to describe completely all the solutions for this problem as a consequence of the non–integrability. But, the study of the periodic orbits is considered as a matter of great interest and a starting point for attacking the problem of classifying solutions. Furthermore, the study of periodic orbits provides us interesting information on spin–orbits and orbital resonances. Therefore many researchers have devoted their efforts for studying the existence of libration points and their stability as well as the periodic orbits in the framework of the restricted problem under the effects of oblateness, triaxiality, radiation pressure force, small perturbations in centrifugal and Coriolis forces.

Some of these works will be stated in the sequel and provide a photography of the “state of the art” for this problem. Sharma [20] studied the stationary solutions of the planar restricted three–body problem when the bigger primary is radiating as well as the smaller primary is an oblate spheroid with its equatorial plane coinciding with the plane of motion. It was stated that the collinear points have conditional retrograde elliptical periodic orbits in the linear sense, while the triangular points have long or short periodic retrograde elliptical orbits when the parameter of mass in the range $0 \leq \mu < \mu_{crit}$.

Elipe and Lara [8] studied the periodic orbits when both the primaries are radiating in the restricted problem. Several families of periodic orbits in two and three dimensions were found. In addition, Ishwar and Elipe [10] found the secular solutions at the triangular points in the generalized photogravitational restricted three–body

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problem. [10] generalizes the problem considering that the bigger primary is a source of radiation and the smaller primary is an oblate spheroid. It was observed that the triangular points have long or short–period retrograde elliptical orbits.

The motion around the collinear equilibrium points of restricted three–body problem was studied when the larger primary is a source of radiation and the second primary is an oblate spheroid by Tsirogiannis et al. [24]. A third–fourth order Lindstedt–Poincaré local analysis type and predictor–corrector algorithms are use for the computation of the Liapunov families of two and three dimensional periodic orbits. Also the stability of these families is studied.

Mittal et. al. [14] studied periodic orbits generated by Lagrangian solutions of the restricted three–body problem when one of the primaries is an oblate spheroid. They used mobile coordinates to determine the periodic orbits for different values of μ, h and A where these parameters represent the mass ratio of the two primaries, the energy constant and oblateness factor respectively. Using the predictor–corrector method such orbits are represented and, by taking some fixed values of three previous parameters, the effect of the oblateness is studied.

Singh and Begha [22] studied the existence of periodic orbits around the triangular points in the restricted three–body problem when the bigger primary is a triaxial, the smaller primary is considered as an oblate spheroid, working in the range of linear stability with the perturbed forces of Coriolis and centrifugal. It is deduced that long and short periodic orbits exist around these points and their periods, orientation and eccentricities are affected by the non sphericity and the perturbation in Coriolis and centrifugal forces.

Abouelmagd and El–Shaboury [4] studied periodic orbits around the triangular points when the three participating bodies are oblate spheroids and the primaries are radiating. It was found that these orbits are elliptical, the frequencies of long and short orbits of the periodic motion are affected by the terms which involve the parameters that characterize the oblateness and radiation repulsive forces. It was proved that the period of long periodic orbits adjusts with the change in its frequency while the period of short periodic orbit decreases. Furthermore, Abouelmagd and Sharaf [6] studied and found the previous orbits around the libration points when the more massive primary is radiating and the smaller is an oblate spheroid. Their study included the effects of oblateness up to 10^{-6} of the main term.

On the other hand, some contributions exploring the families of asymmetric periodic orbits are Papadakis [15, 16, 17], Henrard and Navarro [9], Papadakis and Rodi [18] , Shibayama and Yagasaki [21]. Also, there are some interesting papers recently published devoted to the study of periodic orbits, see for instance Margheri et al. [13], Perdiou et al. [19], Lü et al. [12], Lei and Xu [11].

Abouelmagd [1] studies the effects of the zonal harmonics J_2 and J_4 for the more massive primary in the restricted three–body problem on the locations of the triangular points and their linear stability. It is proved that these locations are affected by the coefficients of oblateness. Also is showed that the triangular points are stable for $0 < \mu < \mu_c$ and unstable when $\mu_c \leq \mu \leq 1/2$, where μ_c is the critical mass parameter which depends on the coefficients J_2 and J_4. Furthermore, Planets–Moons systems are used to produce some numerical values for the positions of the triangular points and mass ratio μ as well as the values of the critical mass μ_c . A numerical study of the range of stability is presented. Also, some examples non affected by the influence of J_4 are studied on the range of stability for some planetary systems as in Earth–Moon, Saturn–Phoebe and Uranus–Caliban systems.

In addition there are many contributions for studying the effects of the non-spherecity and radiation pressure on the existence locations of librations points and their stability as well as the periodic orbits around these points, see Abouelmagd et al. [2,3,5,7]." Inspired in [1] we continue with the study of the periodic structure of the restricted three–body problem considering the effect of the zonal harmonics J_2 and J_4 for the more massive body. We prove that the triangular points $L_{4,5}$ have periodic orbits in the range $0 < \mu < \mu_c$, where μ_c is the critical mass ratio and belongs to the open interval $(0, 1/2)$. This fact depends on expressions that include the factors of the zonal harmonics J_2 and J_4. We show that the angular frequency of the long periodic orbits is an increasing function with respect to the mass ratio μ. While the angular frequency of the short periodic orbits is a decreasing function due to the same parameter for specified values of oblateness factors. In addition, the variations of the angular frequencies for the long and short periodic orbits, s_1 and s_2, are graphically investigated for distinct values of the oblateness parameters. It is also proved that the trajectories of the infinitesimal body are represented by ellipses. The orientation of principal axes is determined, the expressions that represent the lengths of semi–major and semi–minor axes, the eccentricities as well as the eccentricity of the curves for zero velocity are also found. On the other hand the secular solution is constructed. Moreover we prove that the secular solution can be reduced to a periodic solution when the initial conditions are selected properly.

We emphasize that our study is significantly different from the others previously stated in the literature, in fact it is more general, because of the consideration of the oblateness effect up to 10^{-6}. Note that the inclusion of this fact is worthy from the applications point of view. This model has special importance in astrodynamics in order to send satellites or explorations vehicles to stable regions to move in gravitational fields for some planetary systems. Note that the literature on astrodynamics has been enhanced by a greet numbers of papers dealing with some aspects of the classical restricted problem, constructed on the basis that the celestial systems are composed by bodies that are actually considered not point like. The mass of planet or a satellite is distributed in a way which we cannot reduce it to a material point. In fact, these objects are extended bodies which in most cases are far from being considered as spheres. So, the replacement of these bodies by a triaxial bodies or oblate bodies which are symmetric is considered an appropriate approximation for studying them. Actually, the coefficients of zonal harmonic are considered one of the most important perturbed forces which arise from the lack of the sphericity. Consequently our study included the effects of zonal harmonic coefficients.

The structure of the work is as follows: In section 1 a complete review of previous works around related studies on periodic orbits of the three–body problem is presented. In section 2, we recall the equations of motion for the infinitesimal body in a synodic coordinates reference system that were originally stated in [1]. In section 3, the characteristic equation and their roots are computed. Thereafter, graphically explorations due to the variations of the angular frequencies for the long and short periodic orbits are presented. In section 4 the periodic solution and periodic orbits are deduced as well as are characterized the elliptical trajectories of the infinitesimal body around the triangular points. In section 5, we find the secular solution and show that the triangular points have also periodic orbits in this case. Finally, in section 6 we summarize and recall all the results presented in this work.

2 Equations of motion

Let m_1, m_2 and m be the masses of the more massive, the smaller primary and the infinitesimal body respectively. Furthermore, m_1 and m_2 respectively have a circular orbit around their common center of mass and m moves in their plane under their mutual gravitational fields without affecting their motion. We consider, as in [1], that $m_1+m_2=1$ and the distance between them does not change and also is consider as one. The unit of time is chosen to make the constant of gravitation and the unperturbed mean motion equal to the unity. Let the origin of the sidereal and the synodic coordinates be the common center of mass of the primaries and the synodic coordinates rotate with angular velocity n in positive direction. Hence, we can write $m_1 = 1-\mu$ and $m_2=\mu \leq 1/2$ and the coordinates of m_1, m_2 and m in a synodic frame are $(\mu,0)$, $(-1+\mu,0)$ and (x,y) respectively where μ is the mass ratio.

We assume that the coordinates of these masses in an inertial reference frame are (X_1,Y_1), (X_2,Y_2) and (X,Y) respectively.

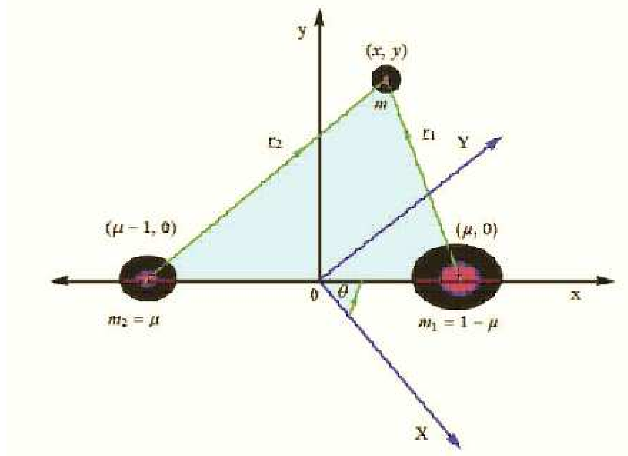

Fig. 1: Configuration of inertial and rotating frames for the restricted three–body problem

The inertial and rotating frames are related in the form

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt \\ \sin nt & \cos nt \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1}$$

We assume that the orbital plane of m_1 and m_2 occur in XY plane. The equations of the motion of the infinitesimal body m in the inertial frame by using the Lagrangian function L are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = 0, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) - \frac{\partial L}{\partial Y} = 0, \end{aligned} \tag{2}$$

where $L=T-V$ is the Lagrangian function, T is the kinetic energy of the infinitesimal mass and V is the potential experienced by the mass m due to m_1 and m_2. The values of these quantities are

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{X}^2+\dot{Y}^2), \\ V &= -Gmm_1\left(\frac{1}{r_1}+\frac{A_1}{2r_1^3}-\frac{3A_2}{8r_1^5}\right)-\frac{Gmm_2}{r_2}, \\ L &= \frac{1}{2}m(\dot{X}^2+\dot{Y}^2)+Gmm_1\left(\frac{1}{r_1}+\frac{A_1}{2r_1^3}-\frac{3A_2}{8r_1^5}\right)+\frac{Gmm_2}{r_2}, \end{aligned} \tag{3}$$

where $A_i = J_{2i}R_1^{2i}$ for $i=1,2$, R_1 is the mean radius of the more massive body and J_i for $i=1,2$ are the dimensionless coefficients of the zonal harmonic. Note that r_1 and r_2 are the magnitudes of the position vectors of the infinitesimal body with respect to m_1 and m_2 respectively given by

$$\begin{aligned} r_1^2 &= (X-X_1)^2+(Y-Y_1)^2, \\ r_2^2 &= (X-X_2)^2+(Y-Y_2)^2, \end{aligned}$$

see Figure 1.

Making two times derivatives in (1) we get

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \end{pmatrix} = \begin{pmatrix} \cos nt & -\sin nt \\ \sin nt & \cos nt \end{pmatrix} \begin{pmatrix} \ddot{x} - 2n\dot{y} - n^2 x \\ \ddot{y} + 2n\dot{x} - n^2 y \end{pmatrix}. \quad (4)$$

Substituting equations (3) into (2) and inserting the outcome in (4) we obtain that the equations of motion for this problem for the infinitesimal body in a synodic frame xy are

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \end{aligned} \quad (5)$$

where

$$\Omega = \frac{1}{2}n^2[(1-\mu)r_1^2 + \mu r_2^2] + (1-\mu)[\frac{1}{r_1} + \frac{\mathrm{A}_1}{2r_1^3} - \frac{3\mathrm{A}_2}{8r_1^5}] + \frac{\mu}{r_2}, \quad (6)$$

Here the magnitudes of the position vectors $\underline{r}_1$ and $\underline{r}_2$ are given by

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2, \\ r_2^2 &= (x-\mu+1)^2 + y^2. \end{aligned}$$

The perturbed mean of motion of the primaries is governed by

$$n^2 = [1 + \frac{3}{2}\mathrm{A}_1 - \frac{15}{8}\mathrm{A}_2]. \quad (7)$$

Furthermore, equations (5) admit a Jacobi integral in the form

$$\dot{x}^2 + \dot{y}^2 - 2\Omega + c = 0,$$

where c is the integration constant.

3 Characteristic equation and their roots

We assume that the infinitesimal body is displaced a little from one of the triangular points (x_0, y_0) to the point $(x_0 + \xi, y_0 + \eta)$ where ξ and η are the variation. The values of x_0 and y_0 are given by

$$\begin{aligned} x_0 &= -\frac{1}{2}\left[1 - 2\mu + [A_1 - \frac{5}{4}(A_2 + A_1^2)]\right], \\ y_0 &= \pm\frac{\sqrt{3}}{2}\left[1 - \frac{1}{3}[A_1 - \frac{5}{4}(A_2 + \frac{7}{15}A_1^2)]\right]. \end{aligned} \quad (8)$$

Therefore the equation of motion and its characteristic equation corresponding to our linear model are given by

$$\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \Omega_{xx}^0 \xi + \Omega_{xy}^0 \eta, \\ \ddot{\eta} + 2n\dot{\xi} &= \Omega_{xy}^0 \xi + \Omega_{yy}^0 \eta, \end{aligned} \quad (9)$$

and

$$\sigma^4 + (4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0)\sigma^2 + \Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{xy}^0)^2 = 0. \quad (10)$$

Where the subscripts x, y indicates the second partial derivatives of Ω, while the superscript 0 indicates that these derivatives are to be evaluated at one of the triangular points.

Furthermore, the expansion of the potential function Ω around the triangular points $L_{4,5}$ up to second order of (ξ, η) is given by

$$\Omega = \Omega^0 + \frac{1}{2}\Omega_{xx}^0\xi^2 + \Omega_{xy}^0\xi\eta + \frac{1}{2}\Omega_{yy}^0\eta^2, \quad (11)$$

taking account that the third or higher powers of ξ and η are ignored and the values of Ω_{xx}^0, Ω_{xy}^0, Ω_{yy}^0 and Ω^0 are given by the following expressions:

$$\begin{aligned} \Omega^0 &= \frac{3}{2}\left\{1 + \frac{5}{6}(1 - \frac{2}{5}\mu)A_1 - \frac{7}{8}(1 - \frac{2}{7}\mu)A_2 - \frac{1}{4}\mu A_1^2\right\}, \\ \Omega_{xx}^0 &= \frac{3}{4}\left\{1 + \frac{9}{2}(1 - \frac{8}{9}\mu)A_1 - \frac{55}{8}(1 - \frac{10}{11}\mu)A_2 \right. \\ &\quad \left. + \frac{7}{2}(1 - \frac{15}{14}\mu)A_1^2\right\}, \\ \Omega_{xy}^0 &= \pm\frac{3\sqrt{3}}{4}\left\{1 - 2\mu + \frac{19}{6}(1 - \frac{26}{19}\mu)A_1 \right. \\ &\quad \left. - \frac{125}{24}(1 - \frac{32}{25}\mu)A_2 + \frac{5}{18}(1 + \frac{1}{10}\mu)A_1^2\right\}, \\ \Omega_{yy}^0 &= \frac{9}{4}\left\{1 + \frac{11}{6}A_1 - \frac{85}{24}(1 - \frac{6}{17}\mu)A_2 - \frac{7}{6}(1 - \frac{15}{14}\mu)A_1^2\right\}. \end{aligned} \quad (12)$$

Furthermore we have two roots for σ^2 that we shall call $\sigma_{1,2}^2$ such that

$$\begin{aligned} \sigma_{1,2}^2 &= -\frac{1}{2}\left[C \pm \sqrt{D}\right], \\ C &= 4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0, \\ D &= \left(4n^2 - \Omega_{xx}^0 - \Omega_{yy}^0\right)^2 - 4\left(\Omega_{xx}^0\Omega_{yy}^0 - (\Omega_{xy}^0)^2\right). \end{aligned} \quad (13)$$

The roots of the characteristic equation in the region $0 < \mu < \mu_c$ could be written as

$$\sigma_{1,2}^2 = -s_{1,2}^2, \quad (14)$$

where

$$s_1 = \frac{1}{2}\left(C - \sqrt{D}\right),$$

and

$$s_2 = \frac{1}{2}\left(C + \sqrt{D}\right).$$

If we restrict ourselves to the terms $\mu, \mu^2, A_1, A_1^2, A_2, \mu A_1, \mu A_1^2$ and μA_2 the appropriate approximation for the values of angular frequencies for the long and the short periodic orbits will be respectively determined by

$$s_1 = \frac{3}{2}\sqrt{3\mu}\left\{1 + \frac{23}{8}\mu(1 - \frac{23}{16}\mu) + \frac{35}{12}(1 - \frac{23}{8}\mu)A_1 - \frac{295}{48}(1 - \frac{23}{8}\mu)A_2 + \frac{221}{288}(1 - \frac{16629}{884}\mu)A_1^2\right\},$$
$$s_2 = \left\{1 - \frac{27}{8}\mu(1 + \frac{119}{16}\mu) - \frac{3}{4}(1 + \frac{221}{8}\mu)A_1 + \frac{45}{16}(1 + \frac{403}{24}\mu)A_2 - \frac{9}{32}(1 + \frac{579}{2}\mu)A_1^2\right\}. \tag{15}$$

Moreover the variations of s_1 and s_2 due to the change in mass ratio will be investigated graphically through the following diagrams, when the parameters of oblateness take distinct values. In these diagrams the solid lines indicate the effect of J_2 uniquely considered. The dotted lines take account the effects of J_2 and J_4, while the dashed lines refer to the classical case (i.e., the effects of the zonal harmonics coefficients are ignored).

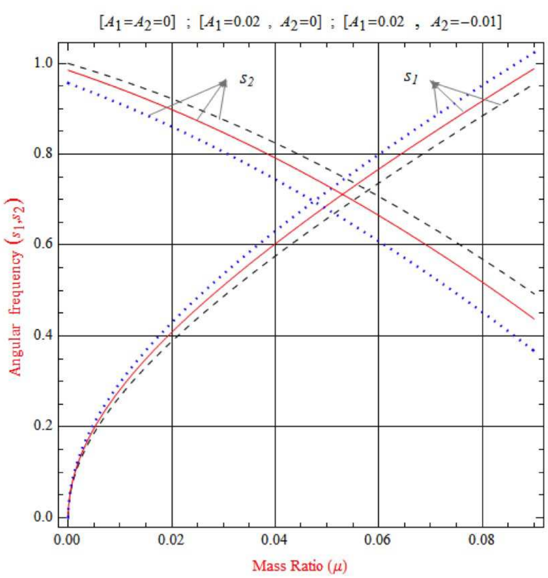

Fig. 2: s_1 and s_2 versus μ when $(A_1 = 0, A_2 = 0)$, $(A_1 = 0.02, A_2 = 0)$ and $(A_1 = 0.02, A_2 = -0.01)$

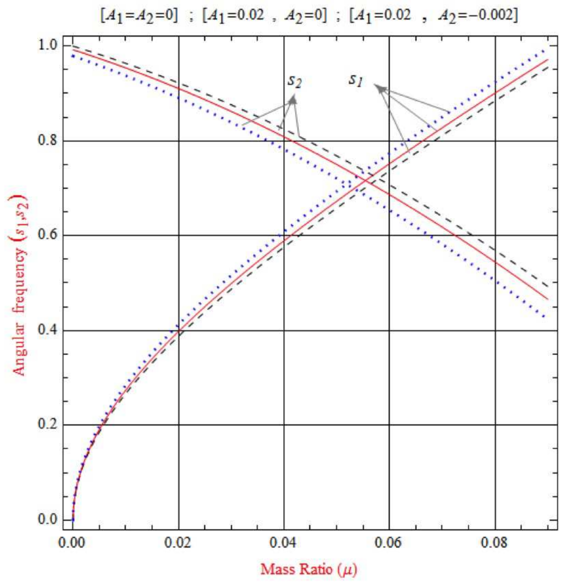

Fig. 3: s_1 and s_2 versus μ when $(A_1 = 0, A_2 = 0)$, $(A_1 = 0.02, A_2 = 0)$ and $(A_1 = 0.02, A_2 = -0.002)$

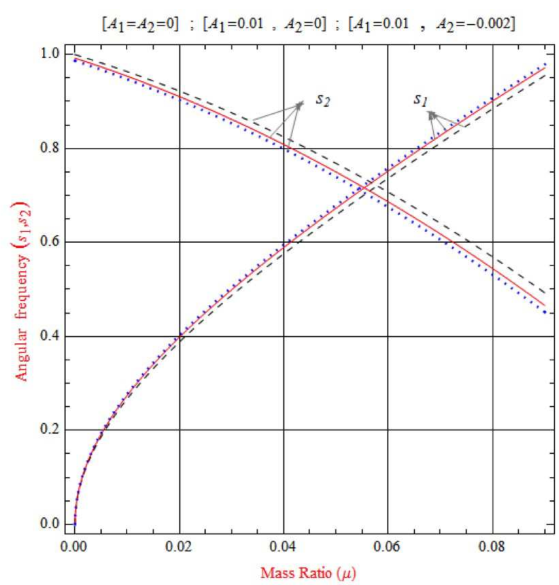

Fig. 4: s_1 and s_2 versus μ when $(A_1 = 0, A_2 = 0)$, $(A_1 = 0.01, A_2 = 0)$ and $(A_1 = 0.01, A_2 = -0.002)$

We observe that the angular frequency of the long periodic orbits s_1 is an increasing function, while the frequency of the short periodic orbits s_2 is a decreasing function, see Figures 2–7 . Furthermore, the growth of the variation for s_2 due to the influence of the zonal harmonic coefficients is bigger than the corresponding to s_1, see Figures 2 and 3. While in cases of $A_1 = 0.01$, $A_2 = 0$ and $A_1 = 0.01$, $A_2 = 0.002$ the curves s_1 and s_2 may be coincident and the changes in the curves are very small compared with the classical case, see Figures 4 and 5. The most important remark is that s_1 (s_2) might have the

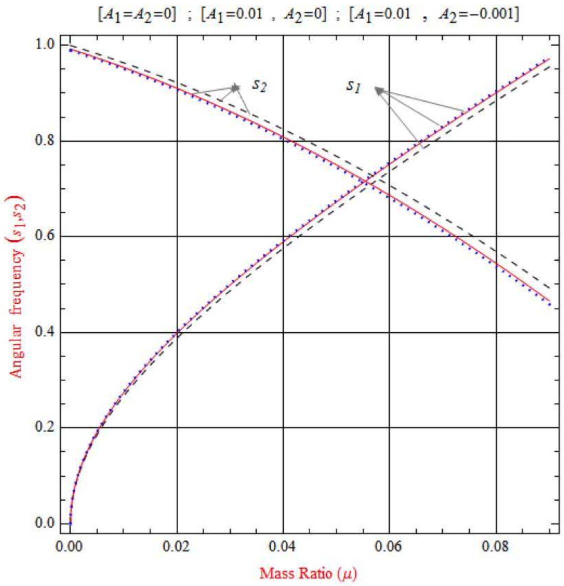

Fig. 5: s_1 and s_2 versus μ when $(A_1 = 0, A_2 = 0)$, $(A_1 = 0.01, A_2 = 0)$ and $(A_1 = 0.01, A_2 = -0.001)$

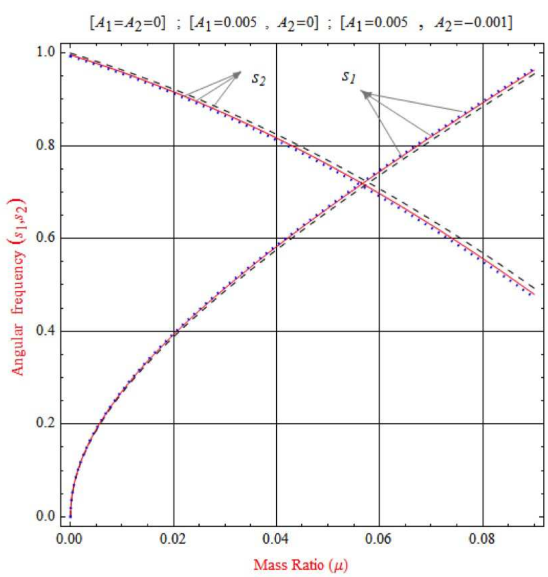

Fig. 6: s_1 and s_2 versus μ when $(A_1 = 0, A_2 = 0)$, $(A_1 = 0.005, A_2 = 0)$ and $(A_1 = 0.005, A_2 = -0.001)$

same behavior if the parameters of oblateness $A_1 \in (1 \times 10^{-6}, 1 \times 10^{-3})$ and $A_2 \in (-1 \times 10^{-6}, -1 \times 10^{-3})$, see Figures 6 and 7, i.e., neither s_1 nor s_2 are affected by A_1 and A_2 in such range.

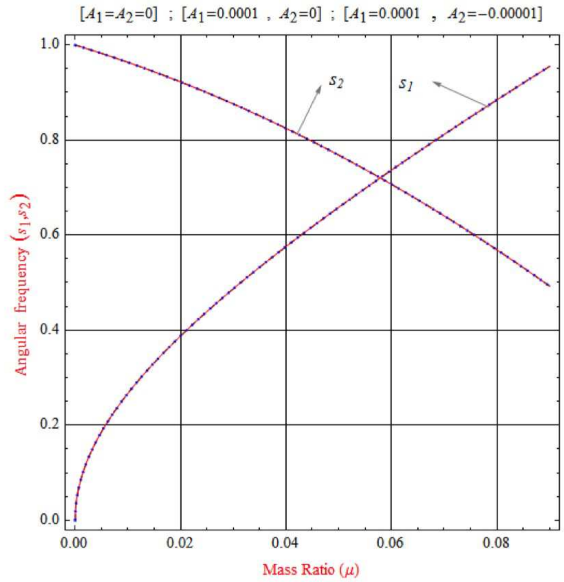

Fig. 7: s_1 and s_2 versus μ when $(A_1 = 0, A_2 = 0)$, $(A_1 = 0.0001, A_2 = 0)$ and $(A_1 = 0.0001, A_2 = -0.00001)$

4 Periodic solutions of linearized equations

4.1 Harmonic motion

By (14), it is clear that the characteristic equation has four pure imaginary roots. Consequently the motion around the triangular points is bounded and composed by two harmonic motions represented by the following equations

$$\begin{aligned} \xi &= C_1 \cos s_1 t + D_1 \sin s_1 t + C_2 \cos s_2 t + D_2 \sin s_2 t, \\ \eta &= \overline{C}_1 \cos s_1 t + \overline{D}_1 \sin s_1 t + \overline{C}_2 \cos s_2 t + \overline{D}_2 \sin s_2 t. \end{aligned} \tag{16}$$

where the coefficients $C_1, D_1, \overline{C}_1$ and $\overline{D}_1$ are associated to the long periodic terms while the coefficients $C_2, D_2, \overline{C}_2$ and $\overline{D}_2$ are the associated ones to the short periodic terms. We note that this is the linear approximation of the solutions around the triangular points.

From equations (16), we observe that if the initial conditions are properly chosen then the short or the long periodic terms can be eliminated from the solution. We assume that the coefficients of the short periodic terms are zero. Therefore, the periodic solutions of the linearized equations (9) can be written as

$$\begin{aligned} \xi &= C_1 \cos s_1 t + D_1 \sin s_1 t, \\ \eta &= \overline{C}_1 \cos s_1 t + \overline{D}_1 \sin s_1 t. \end{aligned} \tag{17}$$

4.2 The trajectory of the infinitesimal body

4.2.1 Elliptic orbits

Since the Hessian determinant $H = \left|\begin{smallmatrix} a_1 & a_2 \\ a_2 & a_3 \end{smallmatrix}\right|$ where the terms $a_1 = \frac{1}{2}\Omega^0_{xx}$, $a_2 = \frac{1}{2}\Omega^0_{yy}$, and $a_3 = \frac{1}{2}\Omega^0_{xy}$ for the quadratic expression in (11) is

$$H = \frac{27}{16}\mu(1-\mu)\left\{1+\frac{13}{3}A_1-\frac{2}{3}(10A_2)\right\} > 0,$$

it indicates that the periodic orbits around the triangular points $L_{4,5}$ are ellipses.

On the other hand, (11) includes an expression that comes out as a result of the translation of the origin of the coordinates system (x,y) to another one composed by the triangular points. This point will be considered the center of the ellipse. Furthermore, we observe that this equation contains a bilinear term $\xi\eta$ which is the responsible for the rotation of the principal axes of the ellipse regarding to the coordinates system reference frame (ξ,η) by an angle Θ, see Figure 8. This remark suggests us the introduction of a normal reference frame of coordinates system $(\overline{\xi},\overline{\eta})$ such that makes zero the bilinear term. The relations between the normal and the previous ones coordinates systems is given by by the following transformation

$$\begin{aligned}\overline{\xi} &= \xi\cos\Theta + \eta\sin\Theta,\\ \overline{\eta} &= -\xi\sin\Theta + \eta\cos\Theta,\end{aligned} \tag{18}$$

where the angle Θ is given by (30).

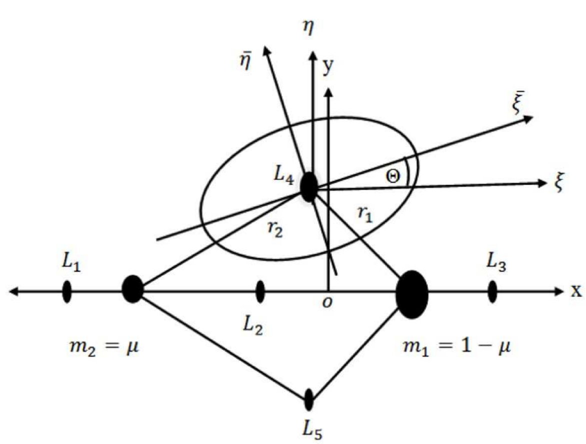

Fig. 8: Orientation of the principal axes for the periodic orbits.

Under this transformation, the equations of motion (9) can be written as

$$\begin{aligned}\ddot{\overline{\xi}} - 2n\dot{\overline{\eta}} &= \overline{\lambda}_1\overline{\xi},\\ \ddot{\overline{\eta}} + 2n\dot{\overline{\xi}} &= \overline{\lambda}_2\overline{\eta}.\end{aligned} \tag{19}$$

Therefore, the new potential function can be written in the following form

$$\overline{\Omega} = \Omega^0 + \frac{1}{2}\overline{\lambda}_1\overline{\xi}^2 + \frac{1}{2}\overline{\lambda}_2\overline{\eta}^2,$$

where

$$\begin{aligned}\overline{\lambda}_1 &= \frac{9}{4}\mu\left\{(1-\frac{1}{4}\mu)+\frac{11}{6}A_1-\frac{55}{24}A_2-\frac{223}{24}A_1^2\right\},\\ \overline{\lambda}_2 &= 3\left\{1-\frac{3}{4}\mu(1-\frac{1}{4}\mu)+\frac{15}{2}(1-\frac{19}{20}\mu)A_1\right.\\ &\quad\left.-\frac{105}{8}(1-\frac{27}{28}\mu)A_2+\frac{669}{32}\mu A_1^2\right\}.\end{aligned}$$

On the other hand, the periodic solutions of the linearized equations that are represented by (17) could be written in the new coordinates in the following form

$$\begin{aligned}\overline{\xi} &= K_1\cos s_1 t + M_1\sin s_1 t,\\ \overline{\eta} &= \overline{K}_1\cos s_1 t + \overline{M}_1\sin s_1 t.\end{aligned} \tag{20}$$

Substituting equations (20) into (19) the relations between the coefficients K_1, M_1, $\overline{K}_1$ and $\overline{M}_1$ are controlled by

$$\overline{K}_1 = \overline{\alpha}M_1, \overline{M}_1 = -\overline{\alpha}K_1 \tag{21}$$

where

$$\overline{\alpha} = \frac{1}{2}\left(s_1+\frac{\overline{\lambda}_1}{s_1}\right) = \frac{2s_1}{s_1^2+s_2^2}. \tag{22}$$

Now, if we consider $\overline{\xi}_0$, $\overline{\eta}_0$, $\dot{\overline{\xi}}_0$ and $\dot{\overline{\eta}}_0$ the initial conditions at the initial time $t=0$, substituting these quantities into (20) and after some calculations using (21) and (22) we obtain

$$\dot{\overline{\xi}}_0 = \frac{\overline{\eta}_0 s_1}{\overline{\alpha}}\,,\ \dot{\overline{\eta}}_0 = -\overline{\xi}_0\overline{\alpha}s_1 \tag{23}$$

and equations (20) become

$$\begin{aligned}\overline{\xi} &= \overline{\xi}_0\cos s_1 t + \frac{\overline{\eta}_0}{\overline{\alpha}}\sin s_1 t,\\ \overline{\eta} &= \overline{\eta}_0\cos s_1 t - \overline{\alpha}\overline{\xi}_0\sin s_1 t.\end{aligned} \tag{24}$$

(23) shows that the initial velocities components depend on the initial positions of the infinitesimal body. In addition, (20) represents a particular solution with only two arbitrary constants. Hence, these components cannot

be freely chosen. If we eliminate cosine and sine from (24), the elliptic orbits can be written as

$$\frac{\overline{\xi}^2}{\overline{\xi}_0^2+\overline{\eta}_0^2\overline{\alpha}_1^2}+\frac{\overline{\eta}^2}{\overline{\xi}_0^2\overline{\alpha}_1^2+\overline{\eta}_0^2}=1. \tag{25}$$

4.2.2 Elements of the ellipses

From (25) the lengths of the semi–major (a) and semi–minor (b) axes are given by

$$\begin{aligned} a^2&=\overline{\xi}_0^2+\overline{\eta}_0^2\overline{\alpha}_1^2,\\ b^2&=\overline{\xi}_0^2\overline{\alpha}_1^2+\overline{\eta}_0^2. \end{aligned} \tag{26}$$

While the eccentricities are given by

$$e^2=1-\overline{\alpha}^2, \tag{27}$$

where

$$\overline{\alpha}=\frac{1}{2}\left(s_1+\frac{\overline{\lambda}_1}{s_1}\right)=\frac{2s_1}{s_1^2+\overline{\lambda}_2}. \tag{28}$$

Furthermore the eccentricity of the curves of zero velocity is given by

$$\overline{e}^2=1-\frac{\overline{\lambda}_1}{\overline{\overline{\lambda}}_2}, \tag{29}$$

see [6].

Thus, we consider that the origin of coordinates of the system is one of the triangular points. Consequently $(\xi_0,\eta_0)=(-x_0,-y_0)$ where the values of x_0 and y_0 are given by equation (8).

4.2.3 The orientation of principal axes

Substituting (18) into (11) and chosen the angle Θ such that the bilinear term $\overline{\xi}\overline{\eta}$ vanishes, the direction of the semi–major axis is governed by the relation: $\tan 2\Theta=2\Omega_{xy}^0/(\Omega_{xx}^0-\Omega_{yy}^0)$, therefore

$$\begin{aligned}\tan 2\Theta=\pm\sqrt{3}\Big\{&1-2\mu+1+\frac{8}{3}(1-2\mu)A_1\\ &-\frac{10}{3}(1-\frac{5}{4}\mu)A_2+\frac{22}{9}(1-\frac{241}{44}\mu)A_1^2\Big\}.\end{aligned} \tag{30}$$

Note that the positive sign refers to the periodic motion around L_4 while negative one gives the motion around L_5.

Remark 1. We summarize that in this section, the elliptical orbits that represent the trajectory of the infinitesimal body in the vicinity of the triangular points have been determined by (25), while equations (26) give the lengths of semi–major and semi–minor axes. The eccentricities for all cases are governed by expressions (27), (28) and (29). After that the directions of the principal axes have been stated by equation (30).

5 Secular solutions of linearized equations

When $\mu=\mu_c$ (critical mass), the discriminant D of the quadratic terms in equation (10) is equal to zero and the value of the critical mass will be given by

$$\mu_c=\left\{\begin{array}{l}\frac{1}{2}\left(1-\frac{\sqrt{69}}{9}\right)-\frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)A_1\\ +\frac{5}{18}\left(1+\frac{25}{2\sqrt{69}}\right)A_2+\frac{13}{27}\left(1+\frac{13671}{1196\sqrt{69}}\right)A_1^2\end{array}\right\}.$$

Consequently, equation (14) can be written as $\sigma_{1,2}^2=-\omega^2$ and its roots are $\sigma_1=\sigma_3=i\omega$, $\sigma_2=\sigma_4=-i\omega$ where $\omega=\sqrt{\frac{1}{2}C}$ where C is given by (13). Now, substituting equations (7) and the expressions of Ω_{xx}^0 and Ω_{xy}^0 given by (12) into the expression of C given by (13) we obtain that

$$C=1-\frac{3}{2}(1-2\mu)A_1+\frac{45}{8}(1-\frac{4}{3}\mu)A_2.$$

Therefore, the value of ω can be written as

$$\omega=\frac{1}{\sqrt{2}}[1-\frac{3}{4}(1-2\mu)A_1+\frac{45}{16}(1-\frac{4}{3}\mu)A_2-\frac{9}{32}(1-4\mu)A_1^2].$$

Thus, the solution of (19) has secular terms. Since $\sigma_1=\sigma_3$ and $\sigma_2=\sigma_4$, the triangular points are unstable. In this case the equation of motion (19) can be written as

$$\begin{aligned}\ddot{\overline{\xi}}-2n\dot{\overline{\eta}}&=\overline{\lambda}_{1c}\overline{\xi},\\ \ddot{\overline{\eta}}+2n\dot{\overline{\xi}}&=\overline{\lambda}_{2c}\overline{\eta},\end{aligned} \tag{31}$$

where the subscript c means that $\overline{\lambda}_1$ and $\overline{\lambda}_2$ will be evaluated when $\mu=\mu_c$. The general solution of these equations has the form

$$\begin{aligned}\overline{\xi}&=(\alpha_1+\alpha_2 t)\cos\omega t+(\alpha_3+\alpha_4 t)\sin\omega t,\\ \overline{\eta}&=(\beta_1+\beta_2 t)\cos\omega t+(\beta_3+\beta_4 t)\sin\omega t.\end{aligned} \tag{32}$$

Now, substituting (32) into (31) and identifying the coefficients of sine and cosine respectively we obtain that the relations between the coefficients at the solution are the following

$$\begin{aligned}\beta_1&=\overline{\gamma}_1\alpha_2+\overline{\gamma}_2\alpha_3,\\ \beta_2&=\overline{\gamma}_2\alpha_4,\\ \beta_3&=\overline{\gamma}_1\alpha_4+\overline{\gamma}_2\alpha_1,\\ \beta_4&=-\overline{\gamma}_2\alpha_2\end{aligned}$$

where

$$\begin{aligned}\overline{\gamma}_1&=\frac{\omega^2-\overline{\lambda}_{1c}}{2n\omega^2},\\ \overline{\gamma}_1&=\frac{\omega^2-\overline{\lambda}_{2c}}{2n\omega^2}.\end{aligned}$$

Let $\overline{\xi}_{0c}$, $\overline{\eta}_{0c}$, $\dot{\overline{\xi}}_{0c}$ and $\dot{\overline{\eta}}_{0c}$ be the initial conditions at the initial time ($t = 0$). Substituting such quantities in (32) and after some simplifications, the coefficients α_i, $i = 1,2,3,4$ can be written as

$$\begin{aligned}
\alpha_1 &= \overline{\xi}_{0c},\\
\alpha_2 &= \frac{\omega}{\overline{\gamma}_1\omega-\overline{\gamma}_2}\overline{\eta}_{0c} - \frac{\overline{\gamma}_2}{\overline{\gamma}_1\omega-\overline{\gamma}_2}\dot{\overline{\xi}}_{0c}\\
\alpha_3 &= -\frac{1}{\overline{\gamma}_1\omega-\overline{\gamma}_2}\overline{\eta}_{0c} + \frac{\overline{\gamma}_1}{\overline{\gamma}_1\omega-\overline{\gamma}_2}\dot{\overline{\xi}}_{0c},\\
\alpha_4 &= \frac{\overline{\gamma}_2\omega}{\overline{\gamma}_1\omega+\overline{\gamma}_2}\overline{\xi}_{0c} + \frac{1}{\overline{\gamma}_1\omega+\overline{\gamma}_2}\dot{\overline{\eta}}_{0c}
\end{aligned} \tag{33}$$

Equtions (33) suggest us that for special values of the initial velocities $\dot{\overline{\xi}}_{0c}$ and $\dot{\overline{\eta}}_{0c}$ the secular terms can be eliminated. Therefore, if we choose $\dot{\overline{\xi}}_{0c} = \omega\overline{\eta}_{0c}/\overline{\gamma}_2$ and $\dot{\overline{\eta}}_{0c} = -\omega\overline{\xi}_{0c}\overline{\gamma}_2$, then (33) is reduced to $\alpha_1 = \overline{\xi}_{0c}$, $\alpha_2 = \alpha_4 = 0$ and $\alpha_3 = -\frac{1}{\overline{\gamma}_2}\overline{\eta}_{0c}$.

Hence, the solution can be written as

$$\begin{aligned}
\overline{\xi} &= \overline{\xi}_{0c}\cos\omega t + \frac{\overline{\eta}_{0c}}{\overline{\gamma}_2}\sin\omega t,\\
\overline{\eta} &= \overline{\eta}_{0c}\cos\omega t - \overline{\gamma}_2\overline{\xi}_{0c}\sin\omega t.
\end{aligned} \tag{34}$$

Equations (34) prove that it is possible to find periodic orbits around the triangular points. However, these points are unstable when the solution contain secular terms.

6 Conclusions

As a summary we recall that we have proved that the triangular points $L_{4,5}$ have periodic orbits in the range $0 < \mu < \mu_c$, where μ_c is the critical mass ratio and belongs to the open interval $(0,1/2)$. This fact depends on expressions that include the factors of zonal harmonics J_2 and J_4. It is observed that the angular frequency of the long periodic orbits are increasing functions with respect to the mass ratio μ. While the angular frequency of the short periodic ones are decreasing functions due to the same parameter for specified values of the oblateness factors. In addition the variations of s_1 and s_2 are graphically investigated for distinct values of the oblateness parameters.

It was also proved that the trajectories of the infinitesimal body are represented by ellipses. The orientation of the principal axes, the expressions that represent the lengths of semi–major and semi–minor axes, the eccentricities as well as the eccentricity of the curves for zero velocity are determined. Moreover, the secular solution is constructed. In addition, it is showed that the triangular points have also periodic orbits at this solution when $\mu = \mu_c$. The results that we have obtained include the effect of zonal harmonics J_2 and J_4 with respect to the more massive body.

We emphasize that our study is significantly different from the previous ones stated in the literature, since our results are more general because of the consideration of the oblateness effect that we consider up to 10^{-6}. Finally, we remark that this model has special importance in astrodynamics to send satellites or explorations vehicles to stable regions to move in gravitational fields for some planetary systems.

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