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Integer Powers of Certain Symmetric Complex Pentadiagonal Matrices

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Abstract: The problem of computing integer powers of certain type matrices appears frequently in mathematical models. In general there is no computationally useful way to obtain arbitrary integer powers of matrices. Finding integer powers of an arbitrary *nxn* matrix carry a computational cost. However, this situation is different for banded symmetric matrices. In this paper, we derive a number of general expressions for integer powers of one type of symmetric complex pentadiagonal matrices.

Keywords: Pentadiagonal matrix; Matrix powers

1 Introduction

In the [1], it is said that ”Pentadiagonal as well as tridiagonal matrices have a wide number of applications in various fields of science, like mechanics, image processing, mathematical chemistry, etc.. For example, in fluid mechanics which is a commonly used subject, the number of meshes necessary to obtain reasonably good results is at times expressible in millions. Powerful techniques were developed to solve such systems. In the most common of these methods, inverses of tridiagonal and pentadiagonal matrices are encountered”. Also, these types of matrices are come upon areas of science and engineering, for example in approximation to fourth derivatives, high order approximations to second derivatives, and as intermediate steps in Given’s and Householder’s method for determining eigenvalues. In many applications, the problem of computing integer powers of such matrices arise ([2]-[6]).

In here, an explicit general expression for powers $q^{th}\,(q \in \mathbb{Z})$ of one type of the symmetric complex pentadiagonal matrices S_n are obtained, where $S_n = \left[s_{kj}\right]_{k,j=1,\ldots,n}$ is given by

$$s_{kj} = \begin{cases} c_1 - c_3, & k = j = 1 \text{ or } k = j = n \\ c_1, & k = j \text{ and } 1 < k,j < n \\ c_2, & |k-j| = 1 \\ c_3, & |k-j| = 2 \\ 0, & otherwise, \end{cases} \qquad (1)$$

where c_1, c_2 and c_3 are complex numbers. For this purpose, we should use same eigenvalue decomposition for the real matrix S_n already establish in ([3],[4]):

$$S_n = V_n D_n V_n^{-1},$$

where the eigenvector matrix

$$V_n = \left[\sin \tfrac{kj\pi}{n+1}\right]_{k,j=1,\ldots,n},$$

the inverse of matrix V_n

$$V_n^{-1} = \tfrac{2}{n+1} V_n,$$

and $D_n = diag\,(\mu_1, \mu_2, \ldots, \mu_n)$ with

$$\mu_j = c_1 + 2c_2 \cos \tfrac{j\pi}{n+1} + 2c_3 \cos \tfrac{2j\pi}{n+1}, \; j = 1, 2, \ldots n$$

(see Theorem 2 of [3] and Lemma 1 of [4]). Because, the main results of [3] are also valid for $c_1, c_2, c_3 \in \mathbb{C}$. Thus, we will give the main results of [3] for $c_1, c_2, c_3 \in \mathbb{C}$ without providing any new proofs as following;

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Theorem 1.*(Theorem 2 of [3]-complex case) Let c_1,c_2,c_3 be complex numbers and be $n\geq 3$ natural numbers. Then $V_nD_nV_n^{-1}$ are eigenvalue decompositions of the symmetric complex pentadiagonal matrices S_n as in (1), where the entries of the eigenvector matrix V_n are given by*

$$v_{kj}=\sin\tfrac{kj\pi}{n+1},\ k,\ j=1,2,\dots,n,$$

and the inverse of the eigenvector matrix V_n is given with $V_n^{-1}=\frac{2}{n+1}V_n$, and $D_n=diag(\mu_1,\mu_2,...,\mu_n)$ is eigenvalue matrix, where μ_j are eigenvalues given by

$$\mu_j=c_1+2c_2\cos\tfrac{j\pi}{n+1}+2c_3\cos\tfrac{2j\pi}{n+1},\ j=1,2,...n.$$

We present a general expression for the entries of the positive integer powers q^{th} of the matrices S_n in terms of the Chebyshev polynomials. When the matrices S_n is invertible, the generalization to arbitrary negative integer powers is immediate: The eigenvalues of S_n^{-q}, where q is a positive integer, are μ_j^{-q}.

Let $T_h(x)$ be the h^{th} degree Chebyshev polynomial of the first kind with $h\in\mathbb{N}\cup\{0\}$

$$T_h(x)=\cos(h(\arccos x)),\ -1\leq x\leq 1(\text{see [7]}).$$

Theorem 2.*(Theorem 3 of [3]-complex case) Let S_n be symmetric complex pentadiagonal matrices as in the form of (1), $c_1,c_2,c_3\in\mathbb{C}$ and $\lambda_m=2\cos\frac{m\pi}{n+1}$, $1\leq m\leq n$, all natural numbers $n\geq 3$. Then,*

$$[S_n^q]_{kj}=\frac{1}{n+1}\sum_{m=1}^{n}(c_1-2c_3+(c_2+c_3\lambda_m)\lambda_m)^q \times\left(T_{|k-j|}\left(\tfrac{\lambda_m}{2}\right)-T_{k+j}\left(\tfrac{\lambda_m}{2}\right)\right)$$

for all numbers $q\in\mathbb{Z}^+$ and $1\leq j,k\leq n$.

Corollary 1.*(Corollary 4 of [3]-complex case) Let S_n be invertible symmetric complex pentadiagonal matrices in the form (1), $c_1,c_2,c_3\in\mathbb{C}$ and $\lambda_m=2\cos\frac{m\pi}{n+1}$, $1\leq m\leq n$, all natural numbers $n\geq 3$. Then,*

$$\left[S_n^{-1}\right]_{kj}=\frac{1}{n+1}\sum_{m=1}^{n}(c_1-2c_3+(c_2+c_3\lambda_m)\lambda_m)^{-1} \times\left(T_{|k-j|}\left(\tfrac{\lambda_m}{2}\right)-T_{k+j}\left(\tfrac{\lambda_m}{2}\right)\right),\ 1\leq j,k\leq n.$$

To reduce the number of operations for $q\in\mathbb{Z}^+$ powers of S_n given within the Theorem 2, we noticed that the upper limit of the sum in the general expression can be reduced to half. Therefore, the following theorem and corollary provide a closed-form expressions for the entries of S_n^q and S_n^{-q} in terms of the Chebyshev polynomial, respectively.

Theorem 3.*(Theorem 5 of [3]-complex case) Let S_n be symmetric complex pentadiagonal matrices in the form (1), $c_1,c_2,c_3\in\mathbb{C}$ and $\lambda_m=-2\cos\frac{m\pi}{n+1}$, $1\leq m\leq n$, for all even natural numbers $n\geq 4$. Then,*

$$[S_n^q]_{kj}=\tfrac{1}{n+1}\sum_{m=1}^{\frac{n}{2}}\Big[(c_1-2c_3+(c_2+c_3\lambda_{n-m+1})\lambda_{n-m+1})^q+ (-1)^{k-j}(c_1-2c_3-(c_2-c_3\lambda_{n-m+1})\lambda_{n-m+1})^q\Big] \times\left(T_{|k-j|}\left(\tfrac{\lambda_{n-m+1}}{2}\right)-T_{k+j}\left(\tfrac{\lambda_{n-m+1}}{2}\right)\right)$$

for all numbers $q\in\mathbb{Z}^+$ and $1\leq j,k\leq n$.

Corollary 2.*Let S_n be invertible symmetric complex pentadiagonal matrices in the form (1), $c_1,c_2,c_3\in\mathbb{C}$ and $\lambda_m=-2\cos\frac{m\pi}{n+1}$, $1\leq m\leq n$, for all even natural numbers $n\geq 4$. Then,*

$$\left[S_n^{-q}\right]_{kj}=\tfrac{1}{n+1}\sum_{m=1}^{\frac{n}{2}}\Big[(c_1-2c_3+(c_2+c_3\lambda_{n-m+1})\lambda_{n-m+1})^{-q}+ (-1)^{k-j}(c_1-2c_3-(c_2-c_3\lambda_{n-m+1})\lambda_{n-m+1})^{-q}\Big] \times\left(T_{|k-j|}\left(\tfrac{\lambda_{n-m+1}}{2}\right)-T_{k+j}\left(\tfrac{\lambda_{n-m+1}}{2}\right)\right)$$

for all numbers $q\in\mathbb{Z}^+$ and $1\leq j,k\leq n$.

2 General expressions for the entries of S_n^q

We derived above eigenvalues and eigenvectors according to the Chebyshev polynomials, and we presented a closed-form expression for the arbitrary positive (or negative when the matrix is invertible) integer powers of the pentadiagonal matrix as given in (1) by using the eigenvalue decomposition. Now, to decrease the number of operations for positive integer powers (or negative when the matrix is invertible) of the odd order matrices S_n $(n\geq 3)$ given in the Theorem 2, we give the following theorem that results to reduce from n to $n/2$ the upper limit of the sum in the general expression.

Theorem 4.*Let S_n be symmetric complex pentadiagonal matrix of all odd order $n\geq 3$ in the form (1), $\lambda_m=-2\cos\frac{m\pi}{n+1}$, $1\leq m\leq n$ and $c_1,c_2,c_3\in\mathbb{C}$. Then,*

$$[S_n^q]_{kj}=\tfrac{1}{n+1}\Big\{(c_1-2c_3)^q\left(T_{|k-j|}(0)-T_{k+j}(0)\right)+ \sum_{m=1}^{\frac{n-1}{2}}\Big[(c_1-2c_3+(c_2+c_3\lambda_{n-m+1})\lambda_{n-m+1})^q +(-1)^{k-j}(c_1-2c_3-(c_2-c_3\lambda_{n-m+1})\lambda_{n-m+1})^q \times\left(T_{|k-j|}(\tfrac{\lambda_{n-m+1}}{2})-T_{k+j}(\tfrac{\lambda_{n-m+1}}{2})\right)\Big]\Big\},$$

for all $q\in\mathbb{Z}^+$ and $1\leq j,k\leq n$.

*Proof.*From the Theorem 2, the general expression for the entries of the matrix S_n^q of all order $n\geq 3$ can be noted that

$$[S_n^q]_{kj}=\tfrac{1}{n+1}\sum_{m=1}^{n}\left(c_1+2c_2\cos\tfrac{m\pi}{n+1}+2c_3\cos\tfrac{2m\pi}{n+1}\right)^q \times\left(\cos\tfrac{m(k-j)\pi}{n+1}-\cos\tfrac{m(k+j)\pi}{n+1}\right).$$

Since $\lambda_m = -2\cos\frac{m\pi}{n+1}$, for $1 \le m \le n$ the following expresion can be given as:

$$\lambda_{n-m+1} = -2\cos\tfrac{(n+1-m)\pi}{n+1} = 2\cos\tfrac{m\pi}{n+1} = -\lambda_m,$$

and

$$T_{|k-j|}\left(\cos\tfrac{m\pi}{n+1}\right) = \cos\tfrac{\pi m|k-j|}{n+1} = T_{|k-j|}\left(\tfrac{\lambda_{n-m+1}}{2}\right).$$

Consequently, if these expressions are used, an entries of expression S_n^q is obtained

$$\begin{aligned}[S_n^q]_{kj} &= \tfrac{1}{n+1}\sum_{m=1}^{n}\left(c_1-2c_3+\left(c_2+c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q \\ &\times\left(T_{|k-j|}\left(\tfrac{\lambda_{n-m+1}}{2}\right)-T_{k+j}\left(\tfrac{\lambda_{n-m+1}}{2}\right)\right).\end{aligned} \tag{2}$$

To reduce the upper limit of the sum expression, the equation (2) can be rewritten as alike

$$\begin{aligned}[S_n^q]_{kj} = \frac{1}{n+1}\Bigg\{&\sum_{m=1}^{\frac{n-1}{2}}\left[\left(c_1-2c_3+\left(c_2+c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q\right. \\ &\left.\times\left(T_{|k-j|}\left(\tfrac{\lambda_{n-m+1}}{2}\right)-T_{k+j}\left(\tfrac{\lambda_{n-m+1}}{2}\right)\right)\right] \\ &+\left(c_1-2c_3+\left(c_2+c_3\lambda_{\frac{n+1}{2}}\right)\lambda_{\frac{n+1}{2}}\right)^q \\ &\times\left(T_{|k-j|}\left(\tfrac{\lambda_{(n+1)/2}}{2}\right)-T_{k+j}\left(\tfrac{\lambda_{(n+1)/2}}{2}\right)\right) \\ &+\sum_{m=\frac{n+3}{2}}^{n}\left[\left(c_1-2c_3+\left(c_2+c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q\right. \\ &\left.\times\left(T_{|k-j|}\left(\tfrac{\lambda_{n-m+1}}{2}\right)-T_{k+j}\left(\tfrac{\lambda_{n-m+1}}{2}\right)\right)\right]\Bigg\}\end{aligned}$$

and also, the last equation is arranged as follows

$$[S_n^q]_{kj} = \tfrac{1}{n+1}\left[A_1+A_2+\left(c_1-2c_3\right)^q\left(T_{|k-j|}(0)-T_{k+j}(0)\right)\right], \tag{3}$$

where

$$\begin{aligned}A_1 = &\sum_{m=1}^{\frac{n-1}{2}}\left(c_1-2c_3+\left(c_2+c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q \\ &\times\left(T_{|k-j|}(\tfrac{\lambda_{n-m+1}}{2})-T_{k+j}(\tfrac{\lambda_{n-m+1}}{2})\right),\end{aligned} \tag{4}$$

$$\begin{aligned}A_2 = &\sum_{m=1}^{\frac{n-1}{2}}\left(c_1-2c_3-\left(c_2-c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q \\ &\times\left(T_{|k-j|}(-\tfrac{\lambda_{n-m+1}}{2})-T_{k+j}(-\tfrac{\lambda_{n-m+1}}{2})\right).\end{aligned} \tag{5}$$

Since they are known that the m^{th} degree Chebychev polynomial of the first kind is an odd or even function, when m is even or odd, respectively (that is, $T_m(-x) = (-1)^m T_m(x)$), and owing to the positive integer numbers k and j, both $k+j$ and $k-j$ are even or both of them are odd. Then, the equality (5) is arranged as following

$$\begin{aligned}A_2 = (-1)^{k-j}&\sum_{m=1}^{\frac{n-1}{2}}\left(c_1-2c_3-\left(c_2-c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q \\ &\times\left(T_{|k-j|}(\tfrac{\lambda_{n-m+1}}{2})-T_{k+j}(\tfrac{\lambda_{n-m+1}}{2})\right).\end{aligned} \tag{6}$$

The proof is finished by substituting expressions (4) and (6) into (3)

$$\begin{aligned}[S_n^q]_{kj} = \tfrac{1}{n+1}\Big\{&\left(c_1-2c_3\right)^q\left(T_{|k-j|}(0)-T_{k+j}(0)\right)+ \\ &\sum_{m=1}^{\frac{n-1}{2}}\Big[\left(c_1-2c_3+\left(c_2+c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q \\ &+(-1)^{k-j}\left(c_1-2c_3-\left(c_2-c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^q \\ &\times\left(T_{|k-j|}(\tfrac{\lambda_{n-m+1}}{2})-T_{k+j}(\tfrac{\lambda_{n-m+1}}{2})\right)\Big]\Big\}.\end{aligned}$$

Corollary 3. *Let S_n be invertible symmetric complex pentadiagonal matrix of all odd order $n \ge 3$ in the form (1), $\lambda_m = -2\cos\frac{m\pi}{n+1}$, $1 \le m \le n$ and $c_1, c_2, c_3 \in \mathbb{C}$. Then,*

$$\begin{aligned}\left[S_n^{-q}\right]_{kj} = \tfrac{1}{n+1}\Big\{&\left(c_1-2c_3\right)^{-q}\left(T_{|k-j|}(0)-T_{k+j}(0)\right)+ \\ &\sum_{m=1}^{\frac{n-1}{2}}\Big[\left(c_1-2c_3+\left(c_2+c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^{-q} \\ &+(-1)^{k-j}\left(c_1-2c_3-\left(c_2-c_3\lambda_{n-m+1}\right)\lambda_{n-m+1}\right)^{-q} \\ &\times\left(T_{|k-j|}(\tfrac{\lambda_{n-m+1}}{2})-T_{k+j}(\tfrac{\lambda_{n-m+1}}{2})\right)\Big]\Big\},\end{aligned}$$

for all $q \in \mathbb{Z}^+$ and $1 \le j,k \le n$.

Corollary 4. *Suppose that $c_1 = 2c_3$ in the symmetric complex pentadiagonal matrix S_n of all odd order $n \ge 3$ given in (1), $\lambda_m = -2\cos\frac{m\pi}{n+1}$, $1 \le m \le n$, and $c_1, c_2, c_3 \in \mathbb{C}$. Then,*

$$\begin{aligned}[S_n^q]_{kj} = \tfrac{1}{n+1}&\sum_{m=1}^{\frac{n-1}{2}}\lambda_{n-m+1}^q\Big[\left(c_2+c_3\lambda_{n-m+1}\right)^q \\ &+(-1)^{q+k-j}\left(c_2-c_3\lambda_{n-m+1}\right)^q\Big] \\ &\times\left(T_{|k-j|}(\tfrac{\lambda_{n-m+1}}{2})-T_{k+j}(\tfrac{\lambda_{n-m+1}}{2})\right),\end{aligned}$$

for all $q \in \mathbb{Z}^+$(or negative powers when the matrix S_n is invertible) and $1 \le j,k \le n$.

3 Numerical Example

Let analyze an example of matrices S_5 and S_5^q. This example has the following the general form

$$S_5 = \begin{bmatrix} c_1 - c_3 & c_2 & c_3 & 0 & 0 \\ c_2 & c_1 & c_2 & c_3 & 0 \\ c_3 & c_2 & c_1 & c_2 & c_3 \\ 0 & c_3 & c_2 & c_1 & c_2 \\ 0 & 0 & c_3 & c_2 & c_1 - c_3 \end{bmatrix},$$

$$S_5^q = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\ \gamma_2 & \gamma_8 & \gamma_6 & \gamma_7 & \gamma_4 \\ \gamma_3 & \gamma_6 & \gamma_9 & \gamma_6 & \gamma_3 \\ \gamma_4 & \gamma_7 & \gamma_6 & \gamma_8 & \gamma_2 \\ \gamma_5 & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \end{bmatrix}.$$

As it is seen that S_5^q is symmetric because matrix S_5 is symmetric matrix, thus, the powers of matrix S_5 are symmetric matrices. By using a general expression of matrix S_n^q for all odd order, the entries of matrix S_5^q are

$$\gamma_1 = \frac{1}{3}(c_1 - 2c_3)^q + \frac{1}{12}\left(w_1^q + w_2^q\right) + \frac{1}{4}\left(w_3^q + w_4^q\right),$$
$$\gamma_2 = \frac{\sqrt{3}}{12}\left(w_1^q - w_2^q\right) + \frac{1}{4}\left(w_3^q - w_4^q\right),$$
$$\gamma_3 = \frac{-1}{3}(c_1 - 2c_3)^q + \frac{1}{6}\left(w_1^q + w_2^q\right),$$
$$\gamma_4 = \frac{\sqrt{3}}{12}\left(w_1^q - w_2^q\right) - \frac{1}{4}\left(w_3^q - w_4^q\right),$$
$$\gamma_5 = \frac{1}{3}(c_1 - 2c_3)^q + \frac{1}{12}\left(w_1^q + w_2^q\right) - \frac{1}{4}\left(w_3^q + w_4^q\right),$$
$$\gamma_6 = \frac{\sqrt{3}}{6}\left(w_1^q - w_2^q\right)$$
$$\gamma_7 = \frac{1}{4}\left(w_1^q + w_2^q - w_3^q - w_4^q\right),$$
$$\gamma_8 = \frac{1}{4}\left(w_1^q + w_2^q + w_3^q + w_4^q\right),$$
$$\gamma_9 = \frac{1}{3}\left[(c_1 - 2c_3)^q + w_1^q + w_2^q\right],$$

where

$$w_1 = c_1 + c_3 + \sqrt{3}c_2,\ w_2 = c_1 + c_3 - \sqrt{3}c_2,$$
$$w_3 = c_1 - c_3 + c_2,\ w_4 = c_1 - c_3 - c_2.$$

4 Conclusion

Firstly, the main results within [3] were given for $c_1, c_2, c_3 \in \mathbb{C}$ and we have established an eigenvalue decomposition for the one type of symmetric complex pentadiagonal matrices. Then we derive a number of general expressions for integer powers of certain symmetric complex pentadiagonal matrices. Furthermore, the closed-form was given to decrease the number of operations given for positive integer powers (or negative when the matrix is invertible) of the certain matrices in terms of the Chebyshev polynomials. Lastly, a numerical example was given in the last section.

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