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On Solving Single-Objective Fuzzy Integer Linear Fractional Programs

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A suggested program with fuzzy linear fractional objective and integer decision variables (FILFP) is considered. The fuzzy coefficients are involved in the numerator of the linear objective function and can be characterized by trapezoidal fuzzy numbers. The purpose of this paper is to outline an algorithm available to solve (FILFP). In addition, an illustrative example is included to demonstrate the correctness of the proposed solution algorithm.

Keywords: Fractional programming, fuzzy programming, integer programming.

1 Introduction

Fractional programs arise in management decision making as well as outside of it. They also occur sometimes indirectly in modeling where initially no ration is involved. Earlier applications and a more recent comprehensive survey can be found in [16]. The efficiency of a system is sometimes characterized by a ratio of technical and/or economical terms. The survey in [16] includes examples from information theory, applied mathematics and physics, among others.

A number of studies have been done by authors in the field of linear, nonlinear and integer fractional programming problems, some of them under fuzziness [6, 12-15, 18, 19]. For interesting review of the work in the field of fuzzy mathematical programming and its applications, we refer to [2, 3].

According to our experience, it is believed that the solution method in fuzzy integer linear fractional programming has not been given comprehensive attention in the literature before.

In the present paper we consider a single-objective integer linear fractional program involving fuzzy parameters in the numerator of the objective. In this setting, use of Charnes and Cooper transformation [4] seems inhibitive.

Section 2 contains the mathematical formulation of fuzzy integer linear fractional program. In addition, a nonfuzzy version of the formulated model is stated along with the solution concept. Also, an equivalent fuzzy linear fractional program is constructed corresponding to the problem under consideration via Gomory' cutting-plane algorithm. Some basic fuzzy concepts are reported in Section 3. An algorithm is described to solve the problem of concern in Section 4. In section 5 we demonstrate the proposed solution algorithm with a numerical example. Finally, the paper is concluded in Section 6.

2 Problem Motivation and the Solution Concept

The problem studied in this paper is the following integer linear fractional program involving fuzzy parameters in the objective function. The problem of concern is formulated as follows:

$$\max \quad \frac{(c^T x + \tilde{\theta}^T x) + c_0}{d^T x + d_0} \tag{2.1}$$

subject to

$$x \in M.$$

In problem (2.1), $c, d \in R^n$ and $c_0, d_0 \in R$. The set M is defined as the feasible region and might be, for example, of the form

$$M = \{x \in R^n | Ax \leq b, x \geq 0 \text{ and integer}\} \tag{2.2}$$

where A is an $m \times n$ matrix, x is an n-vector of the integer decision variables, b is an m-vector of the constraint right-hand sides, R^n is the n-dimensional Euclidean space and T denotes the transpose.

It is assumed that $\tilde{\theta}$ is an n-vector of fuzzy parameters and for simplicity, let $\tilde{\theta} = [\theta_1^{\sim}, \theta_2^{\sim}, \ldots, \theta_n^{\sim}]$. Moreover, we suppose that the feasible region M is compact set and that $d^T x + d_0 \succ 0$ for all $x \in M$, where M is a nonconvex polyhedron in general.

The set of constraints $Ax \leq b, x \geq 0$ will be denoted throughout this paper by M_R and can be obtained by dropping the integer requirement on the decision variables x_j for all $j = 1, 2, \ldots, n$ in (2.2).

In what follows, an equivalent fuzzy linear fractional program associated with program (2.1) can be stated with the help of the cutting-plane technique [5, 7, 17]. This equivalent program can be written in the form

$$\max \quad \frac{(c^T x + \tilde{\theta}^T x) + c_0}{d^T x + d_0} \tag{2.3}$$

subject to

$$x \in [M],$$

where $[M]$ is defined as the convex hull of the set of feasible solutions M defined by (2.2). The point to be noted here is that the optimal solution of program (2.1) is the same the optimal solution of program (2.3), see [9, 10].

To find the convex hull $[M]$, the Gomory's cutting plane algorithm will be used [5, 7, 17] and for this, we consider the equivalent fuzzy linear fractional program (2.3) in the form

$$\max \quad \frac{(c^T x + \tilde{\theta}^T x) + c_0}{d^T x + d_0} \tag{2.4}$$

subject to

$$x \in M_R^{(s)},$$

where $M_R^{(s)}$ is defined as

$$M_R^{(s)} = \{x \in R^n | A^{(s)} x \leq b^{(s)}, x \geq 0\}. \tag{2.5}$$

In addition,

$$A^{(s)} = \begin{bmatrix} A \\ \cdots \\ a_1 \\ . \\ . \\ a_s \end{bmatrix} \qquad \text{and} \qquad b^{(s)} = \begin{bmatrix} b \\ \cdots \\ b_1 \\ . \\ . \\ b_s \end{bmatrix} \tag{2.6}$$

are the original constraint matrix A and the right-hand side vector b, respectively, with s-additional constraints each corresponding to an efficient cut in the form $a_i x \leq b_i$ [5, 7]. By an efficient cut, we mean that a cut which is not redundant. Therefore, the feasible solution set $M_R^{(s)} = [M]$. For more details, the reader is referred to [9].

Now, by introducing the transformation

$$\rho = \frac{1}{d^T x + d_0} \tag{2.7}$$

with the following additional variable change [4]

$$y_j = x_j \rho \qquad \text{for all } j = 1, 2, \ldots, n, \tag{2.8}$$

program (2.4) can be reduced to

$$\max \quad (c^T y + \tilde{\theta}^T y) + c_0 \rho, \tag{2.9}$$

subject to

$$A^{(s)}y - b^{(s)}\rho \leq 0,$$

$$d^T y + d_0\rho = 1,$$

$$\rho \succ 0, y \geq 0.$$

It should be noted here that problem (2.9) above is a parametric single-objective fuzzy nonlinear programming problem.

The next section reports some fuzzy concept introduced in [11] needed later to construct the solution algorithm.

3 Theoretical Fuzzy Foundations

Fuzzy set theory has been developed for solving problems in which descriptions of activities and observations are imprecise, vague and uncertain. The term "fuzzy" refers to the situation in which there are no well-defined boundaries of the set of activities or observations to which the descriptions apply.

A fuzzy set is a class of objects with membership grades. A membership function, which assigns to each object a grade of membership, is associated with each fuzzy set. Usually the membership grades are in $[0, 1]$. When the grade of membership for an object in a set is one, this object is absolutely in that set; when the grade of membership is zero, the object is absolutely not in that set. Borderline cases are assigned numbers between zero and one.

A fuzzy number is defined differently by many authors. The most frequently used definition belongs to a trapezoidal fuzzy type.

Definition 3.1 ([11])**.** It is appropriate to recall that a real fuzzy number $\tilde{a}$ is a continuous fuzzy subset from the real line R whose membership function $\mu_{\tilde{a}}(a)$ is defined by

(1) A continuous mapping from R to the closed interval $[0, 1]$,
(2) $\mu_{\tilde{a}}(a) = 0$ for all $a \in (-\infty, a_1]$,
(3) $\mu_{\tilde{a}}(a)$ is strictly increasing on $[a_1, a_2]$,
(4) $\mu_{\tilde{a}}(a) = 1$ for all $a \in [a_2, a_3]$,
(5) $\mu_{\tilde{a}}(a)$ is strictly decreasing on $[a_3, a_4]$,
(6) $\mu_{\tilde{a}}(a) = 0$ for all $a \in [a_4, +\infty)$.

Figure 3.1 illustrates the graph of a possible shape of a membership function of fuzzy number $\tilde{a}$.

Here the vector of fuzzy parameters $\tilde{\theta}$ involved in problems (2.1) and in its equivalent one (2.9) is a vector of fuzzy numbers whose membership function is $\mu_{\tilde{\theta}}(\theta)$.

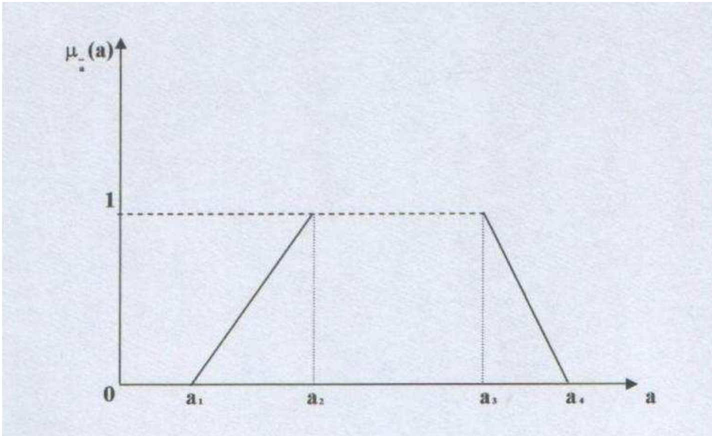

Figure 3.1: Membership function of a fuzzy number $\tilde{a}$.

Throughout this paper, a membership function in the following form will be elicited

$$\mu_{\tilde{a}}(a) = \begin{cases} 0, & a \le a_1, \\ 1 - \left(\frac{a-a_2}{a_1-a_2}\right)^2, & a_1 \le a \le a_2, \\ 1, & a_2 \le a \le a_3, \\ 1 - \left(\frac{a-a_3}{a_4-a_3}\right)^2, & a_3 \le a \le a_4, \\ 0, & \text{otherwise.} \end{cases}$$

In what follows, we give the definition of the α-level set or α-cut of the fuzzy vector $\tilde{\theta} = [\theta_1^{\sim}, \theta_2^{\sim}, \ldots, \theta_n^{\sim}]$.

Definition 3.2 ([11]). The α-level set of the vector of fuzzy parameters $\tilde{\theta}$ in problems (2.1) and (2.9) is defined as the ordinary set $L_\alpha(\tilde{\theta})$ for which the degree of its membership function exceeds the level $\alpha \in [0, 1]$, where

$$L_\alpha(\tilde{\theta}) = \{\theta \in R \mid \mu_{\tilde{\theta}}(\theta) \ge \alpha\}. \tag{3.1}$$

For a certain degree $\alpha = \alpha^* \in [0, 1]$, estimated by the decision maker, the problem (2.9) can be understood as the following nonfuzzy α-nonlinear programming problem (α-NLPP):

$$\max \quad (c^T y + \theta^T y) + c_0 \rho, \tag{3.2}$$

subject to

$$A^{(s)} y - b^{(s)} \rho \le 0,$$

$$d^T y + d_0 \rho = 1,$$

$$\theta \in L_\alpha(\tilde{\theta}),$$

$$\rho \succ 0, y \geq 0.$$

It should be emphasized that in the α-nonlinear programming problem (α-NLPP) (3.2) the vector of parameters θ is treated as a vector of decision variable rather than constants.

Based on the concept of the α-cut of the fuzzy vector $\tilde{\theta} = [\theta_1^{\sim}, \theta_2^{\sim}, \ldots, \theta_n^{\sim}]$ defined earlier, problem (α-NLPP) (3.2) can be rewritten as follows:

$$\max \quad (c^T y + \theta^T y) + c_0 \rho, \tag{3.3}$$

subject to

$$A^{(s)} y - b^{(s)} \rho \leq 0,$$

$$d^T y + d_0 \rho = 1,$$

$$\ell_j \leq \theta_j \leq u_j, \; j = 1, 2, \ldots, n,$$

$$\rho \succ 0, y \geq 0,$$

where ℓ_j and u_j are lower and upper bounds on the variables $\theta_j, j = 1, 2, \ldots, n$, respectively.

It should be noted that constraint $\theta \in L_\alpha(\tilde{\theta})$ in problem (3.2) has been replaced by the equivalent one $\ell_j \leq \theta_j \leq u_j, j = 1, 2, \ldots, n$ in problem (3.3).

Let

$$f(y, \theta, \rho) = (c^T y + \theta^T y) + c_0 \rho$$

and Y denote the set of the feasible solutions of problem (3.3). The following definition can be introduced.

Definition 3.3. A point $(y_j^*, \theta_j^*, \rho_j^*) \in Y$ is said to be an α-optimal solution of problem (α-NLPP) (3.3) if and only if there is no other point $(y_j, \theta_j, \rho_j) \in Y$ such that $f(y_j, \theta_j, \rho_j) \prec f(y_j^*, \theta_j^*, \rho_j^*)$, where θ_j^*, $j = 1, 2, \ldots, n$, are called α-level optimal parameters.

In concluding this section, if (y_j^*, ρ_j^*) is an α-optimal solution of problem (3.3) with the corresponding α-level optimal parameters θ_j^*, $j = 1, 2, \ldots, n$, then $x_j^* = y_j^* / \rho_j^*, j = 1, 2, \ldots, n$, is an α-optimal solution of problem (2.4), which is also an α-optimal solution of the formulated fuzzy integer linear fractional problem (2.1).

It is clear that a systematic variation of the α-level set of the vector of fuzzy parameters $\tilde{\theta}$ will yield another α-optimal solution of fuzzy integer linear fractional problem (2.1).

4 Solution Algorithm

In this section, a solution algorithm to solve fuzzy integer linear fractional programming problem (2.1) is described in two main phases. For the sake of completeness, the suggested algorithm can be summarized in the following manner.

Phase I

Step 1: Characterize the set $M_R^{(s)} = [M]$ and this can be done in the following substeps:

(a) Use Balinski's algorithm [1] to find all the vertices of the feasible region M_R.

(b) Select one of the non-integer vertices $x^1 = (x_1^1, x_2^2, \dots, x_n^1)$ of the solution space. In the tableau of this vertex, choose the row vector where the basic variable has the largest fractional part and construct its corresponding cut in the form $a_1 x \leq b_1$, see [5,7].

(c) Add the first cut $a_1 x \leq b_1$ to the original set of constraints M_R. This will yield a new feasible region $M_R^{(1)}$.

(d) Repeat again the steps (a) to (c) until, some step r; the obtained vertices of the solution space become all integers.

(e) Eliminate (drop) all the redundant constraints of the applied cuts.

(f) Add all the constraints of the applied s-efficient cuts to the original set of constraints M_R to get $M_R^{(s)}$, where $M_R^{(s)} =]M]$.

Step 2: Formulate the equivalent fuzzy linear fractional program in the form of problem (2.4).

Phase II

Step 1: Make the transformations given by (2.7) and (2.8) to transform problem (2.4) to a fuzzy nonlinear program in the form of problem (2.9).

Step 2: Start with an initial level set $\alpha = \alpha^* = 0$.

Step 3: Determine points $(\theta_1, \theta_2, \theta_3, \theta_4)$ for the vector of fuzzy parameters $\tilde{\theta}$ in problem (2.9) to elicit a membership function $\mu_{\tilde{\theta}}(\theta)$ satisfying assumptions (1)-(6) in Definition 3.1.

Step 4: Convert problem (2.9) into its nofuzzy version (α-NLPP) (3.2) or (3.3).

Step 5: Solve problem (3.3) using the usual parametric nonlinear programming approach to find its optimal integer solution $(y_j^*, \theta_j^*, \rho_j^*)$. Then, the optimal solution of problem (2.9) can be obtained directly and therefore the optimal integer solution of the problem of concern (FILPP) (2.1) is found explicitly as $x_j^* = y_j^*/\rho_j^*, j = 1, 2, \dots, n$, with the corresponding α-level optimal parameters $\theta_j^*, j = 1, 2, \dots, n$.

Step 6: Set $\alpha = (\alpha^* + step) \in [0, 1]$ and go to step 1 of Phase II.

Step 7: Repeat again the above procedure of Phase II until the interval $[0, 1]$ is fully exhausted. Then, stop.

The following numerical example demonstrates the steps of the suggested solution algorithm described above.

5 An Illustrative Example

The problem under consideration is the following integer linear fractional program involving fuzzy parameter $\tilde{\theta}$ in the objective function

$$P_1: \qquad \max\ z(x,\tilde{\theta}) = \frac{(1+2\tilde{\theta})x_1 - 4}{-x_2+3}$$

subject to

$$x \in M,$$

where

$$M = \{x \in R^2 | -x_1 + 4x_2 \leq 0, 2x_1 - x_2 \leq 8; x_1, x_2 \geq 0 \text{ and integer}\}.$$

The convex hull $M_R^{(s)} = [M]$ is given by

$$[M] = \{x \in R^2 | -x_1 + 4x_2 \leq 0, 2x_1 - x_2 \leq 8; x_1 \leq 4; x_1, x_2 \geq 0\},$$

where $s = 1$ an efficient Gomory cut: $x_1 \leq 4$. Hence, the fuzzy linear fractional program equivalent to program P_1 can be formulated as

$$P_2: \qquad \max\ z(x,\tilde{\theta}) = \frac{(1+2\tilde{\theta})x_1 - 4}{-x_2+3}$$

subject to

$$x \in [M].$$

Now, using Charnes-Cooper Transformation method [4], then problem P_2 above can be understood as the following fuzzy nonlinear problem in the form

$$P_3: \qquad \max\ z(x,\rho,\tilde{\theta}) = (1+2\tilde{\theta})y_1 - \frac{4}{3}y_2 - \frac{4}{3}$$

subject to

$$\begin{aligned} -y_1 + 4y_2 &\leq 0, \\ 2y_1 - \frac{11}{3}y_2 &\leq \frac{8}{3}, \\ y_1 - \frac{4}{3}y_2 &\leq \frac{4}{3}, \\ y_1, y_2 &\geq 0. \end{aligned}$$

Let the fuzzy parameter $\tilde{\theta}$ be defined by the fuzzy numbers $\tilde{\theta} = (0, 3, 5, 7)$ and its membership function be elicited as

$$\mu_{\tilde{\theta}}(\theta) = \begin{cases} 0 & \theta \le \theta_1, \\ 1 - \left(\frac{\theta-\theta_2}{\theta_1-\theta_2}\right)^2, & \theta_1 \le \theta \le \theta_2, \\ 1, & \theta_2 \le \theta \le \theta_3, \\ 1 - \left(\frac{\theta-\theta_3}{\theta_4-\theta_3}\right)^2, & \theta_3 \le \theta \le \theta_4, \\ 0, & \text{otherwise.} \end{cases}$$

Now, the nonfuzzy nonlinear problem associated with the original problem P_1 can be written as

$$P_4: \qquad \max\ z(x, \rho, \theta) = (1 + 2\theta)y_1 - \frac{4}{3}y_2 - \frac{4}{3}$$

subject to

$$-y_1 + 4y_2 \le 0,$$
$$2y_1 - \frac{11}{3}y_2 \le \frac{8}{3},$$
$$y_1 - \frac{4}{3}y_2 \le \frac{4}{3},$$
$$0 \le \theta \le 7,$$
$$y_1, y_2 \ge 0.$$

The above problem P_4 has been solved using Gino software package [8] and the results are reported in Table 5.1.

α-level set	Range of θ	The optimal integer solution
$\alpha = 0$	$0 \le \theta \le 7$	$\theta = 0, \quad (x_1^*, x_2^*) = (4, 0)$
		$\theta \in (0, 7], (x_1^*, x_2^*) = (4, 1)$
$\alpha = 0.36$	$0.6 \le \theta \le 6.6$	$(x_1^*, x_2^*) = (4, 1)$
$\alpha = 0.84$	$1.8 \le \theta \le 5.8$	$(x_1^*, x_2^*) = (4, 1)$
$\alpha = 1$	$3 \le \theta \le 5$	$(x_1^*, x_2^*) = (4, 1)$

Table 5.1: Results of the Illustrative Example.

6 Conclusion

In the presented paper a solution algorithm has been proposed to solve fuzzy integer linear fractional programs (FILFPP). Some fuzzy concepts have been given to convert problem (FILFPP) to a nofuzzy version and the Charnes & Cooper transformations have been used to complete the solution process.

Summarizing, many aspects and general questions remain to be studied and explored in the area of fuzzy integer linear fractional programming. Despite the limitations, we believe that this paper is an attempt to establish underlying results which hopefully will help others to answer of these questions.

There are however several open points for future research in the area of (FILFPP), in our opinion, to be studied. Some of these points of interest are stated in the following:

(i) An algorithm is required for solving single-objective integer linear fractional programs with fuzzy parameters in the constraint function.

(ii) Stability of the integer optimal solution should be investigated for (FILFPP).

(iii) An algorithm is required for solving multiobjective integer linear fractional programs with fuzzy parameters in (i) in the objective functions and (ii) in the constraint function.

(iv) Stability of the integer efficient solution should be investigated for fuzzy multiobjective integer linear fractional programming problems.

(v) Computer codes are needed to be constructed to solve the problems recommended above.

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