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Numerical Solutions of the Combined KdV-MKdV Equation by a Quintic B-spline Collocation Method

N. Murat Yagmurlu[1,*], *Orkun Tasbozan*[2], *Yusuf Ucar*[1] *and Alaattin Esen*[1]

[1] Department of Mathematics, İnönü University, Malatya, 44280, Turkey.
[2] Department of Mathematics, Mustafa Kemal University, Hatay, 31000, Turkey.

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Abstract: In this paper, a numerical solution of the combined KdV-MKdV equation is obtained by a quintic B-spline collocation finite element method. In the solution process, a linearization technique has been applied to deal with the non-linear term appearing in the equation. The computed results are compared with those given in the literature. The error norms L_2 and L_∞ are computed and found to be sufficiently small. The Fourier stability analysis of the method is also investigated and found unconditionally stable.

Keywords: Combined KdV-MKdV equation, finite element method, collocation method, B-spline, Fourier stability analysis.

1 Introduction

When many phenomena in the nature are mathematically modelled, some of them usually result in the combined KdV-MKdV equation which is modeling the wave propagation in a one dimensional nonlinear lattice [1,2]. Hence it is of great interest for many scientists and mathematicians. Therefore, its analytical and numerical solutions are found by many authors using various methods. In this paper, we will deal with the combined KdV-mKdV equation given in the form

$$u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx} = 0, \beta > 0, -30 \le x \le 70 \quad (1)$$

where u is the dependent variable, and t and x are the independent time and space parameters, respectively. The numerical solutions of Eq. (1) will be sought with the boundary conditions and initial condition obtained from its analytical solution given [3]

$$\begin{aligned} u(x,t) = \lambda / \{ & C\cosh^2(\tfrac{1}{2}\sqrt{\lambda}(x-\lambda t-\xi_0)) \\ & + D\sinh^2(\tfrac{1}{2}\sqrt{\lambda}(x-\lambda t-\xi_0))\} \end{aligned} \quad (2)$$

where ξ_0 is the integration constant and

$$C = \sqrt{\alpha^2+\beta\lambda}+\alpha,$$

$$D = \sqrt{\alpha^2+\beta\lambda}-\alpha.$$

If we take $\alpha = 1$, $\beta = 1$, $\lambda = 1$ and $\xi_0 = 0$ at $t = 0$, we obtain the initial condition. Similarly if we take $x = -30$ and $x = 70$, we easily obtain the left and right hand boundary conditions, respectively.

The main purpose of this study is to apply the quintic B-spline collocation finite element method to develop a numerical technique for solving the combined KdV-MKdV equation. Eq. (1) has been solved by few authors using various methods and techniques. In 1984, Taha and Ablowitz derived differential-difference equations that have as limiting forms the KdV, and MKdV equations [3]. Huang and Zhang [4] have have obtained new exact travelling waves solutions to the combined KdV-MKdV and generalized Zakharov equations. Lu and Shi [5] have established exact solutions for the combined KdV-MKdV equation, constructing four new types of Jacobi elliptic functions solutions and extending the Jacobi eliptic functions expansion method. Yan and et al [6] have studied the soliton perturbations for a combined KdV-MKdV equation and derived the first-order effects of perturbation on a soltion through constructing the appropriate Green's function. Naher and Abdullah [7] have applied the improved $(G\prime/G)$-expansion method to contruct some new exact traveling solutions including soliton and periodic solutions of the combined KdV-MKdV equation involving parameters.

* Corresponding author e-mail: murat.yagmurlu@inonu.edu.tr

In this paper, we have used a linearization technique to obtain the numerical solution of the combined KdV-MKdV equation. The performance of the method has been tested on a numerical example, and the stability analysis of the numerical scheme has also been investigated and found to be unconditionally stable.

2 The Finite Element Solution

Before starting to solve Eq. (1) with the boundary conditions and the initial conditions obtained from the exact solution given by Eq.(2) using quintic collocation finite element method, we firstly define quintic B-spline functions. Let us consider the interval $[a,b]$ is partitioned into M finite elements of uniformly equal length by the knots x_m, $m = 0,1,2,...,M$ such that $a = x_0 < x_1 \cdots < x_M = b$ and $h = x_{m+1} - x_m$. The quintic B-splines $\phi_m(x)$, $(m = -1(1)M)$, at the knots x_m are defined over the interval $[a,b]$ by [8]

$$\phi_m(x) = \frac{1}{h^5}\begin{cases} (x-x_{m-3})^5, & [x_{m-3},x_{m-2}] \\ (x-x_{m-3})^5-6(x-x_{m-2})^5, & [x_{m-2},x_{m-1}] \\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5, & [x_{m-1},x_m] \\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5- \\ 20(x-x_m)^5, & [x_m,x_{m+1}] \\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5- \\ 20(x-x_m)^5+15(x-x_{m+1})^5, & [x_{m+1},x_{m+2}] \\ (x-x_{m-3})^5-6(x-x_{m-2})^5+15(x-x_{m-1})^5- \\ 20(x-x_m)^5+15(x-x_{m+1})^5-6(x-x_{m+2})^5, & [x_{m+2},x_{m+3}] \\ 0, & otherwise. \end{cases} \quad (3)$$

The set of splines $\{\phi_{-2}(x), \phi_{-1}(x),\ldots,\phi_{M+1}(x),\phi_{M+2}(x)\}$ constitutes a base for the functions defined over $[a,b]$. Therefore, an approximation solution $U_M(x,t)$ can be written in terms of the quadratic B-splines trial functions as follows

$$U_M(x,t) = \sum_{j=-2}^{M+2} \delta_j(t)\phi_j(x) \quad (4)$$

where $\delta_m(t)$'s are unknown, time dependent parameters to be determined from the boundary and weighted residual conditions. Each quintic B-spline functions covers six elements so that each element $[x_m,x_{m+1}]$ is covered by six quintic B-spline functions. For this problem, the finite elements are identified with the interval $[x_m,x_{m+1}]$ and the elements knots x_m, x_{m+1}. Using the nodal values U_m, U_m' and U_m''' given in terms of the following element parameters $\delta_m(t)$

$$\begin{aligned} U_M(x_m,t) = U_m &= \delta_{m-2}+26\delta_{m-1}+66\delta_m+26\delta_{m+1}+\delta_{m+2}, \\ U_m' &= \tfrac{5}{h}(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2}), \\ U_m'' &= \tfrac{20}{h^2}(\delta_{m-2}+2\delta_{m-1}-6\delta_m+2\delta_{m+1}+\delta_{m+2}), \\ U_m''' &= \tfrac{60}{h^3}(-\delta_{m-2}+2\delta_{m-1}-2\delta_{m+1}+\delta_{m+2}), \end{aligned} \quad (5)$$

the variation of $U_M(x,t)$ over the typical element $[x_m,x_{m+1}]$ is given by

$$U_M = \sum_{j=m-2}^{m+3} \delta_j(t)\phi_j(x). \quad (6)$$

For the linearization, we suppose that the quantity U to be locally constant. This is equal to assuming that in Eq. (1) all Us are equal to a local constant Z_m. If we put the nodal values given by Eq. (5) into Eq. (1), and take $u = Z_m$ we obtain the following system of equations:

$$\dot{\delta}_{m-2}+26\dot{\delta}_{m-1}+66\dot{\delta}_m+26\dot{\delta}_{m+1}+\dot{\delta}_{m+2} \quad (7)$$

$$+6Z_m\left[\frac{5}{h}(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2})\right] \quad (8)$$

$$+\ 6Z_m^2\left[\frac{5}{h}(-\delta_{m-2}-10\delta_{m-1}+10\delta_{m+1}+\delta_{m+2})\right] \quad (9)$$

$$+\left[\frac{60}{h^3}(-\delta_{m-2}+2\delta_{m-1}-2\delta_{m+1}+\delta_{m+2})\right] = 0.$$

In Eq. (8), if we take the

$$\dot{\delta} = \frac{\delta^{n+1}-\delta^n}{\Delta t}$$

and

$$\delta = \frac{\delta^{n+1}+\delta^n}{2}$$

and put them in their places, we obtain

$$\begin{aligned} &\delta_{m-2}^{n+1}(\alpha_1-\alpha_2-\alpha_3-\alpha_4)+\delta_{m-1}^{n+1}(26\alpha_1-10\alpha_2-10\alpha_3+2\alpha_4)+ \\ &\delta_m^{n+1}(66\alpha_1)+\delta_{m+1}^{n+1}(26\alpha_1+10\alpha_2+10\alpha_3-2\alpha_4)+ \\ &\delta_{m+2}^{n+1}(\alpha_1+\alpha_2+\alpha_3+\alpha_4) = \delta_{m-2}^n(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+ \\ &\delta_{m-1}^n(26\alpha_1+10\alpha_2+10\alpha_3-2\alpha_4)+\delta_m^n(66\alpha_1)+ \\ &\delta_{m+1}^n(26\alpha_1-10\alpha_2-10\alpha_3+2\alpha_4)+\delta_{m+2}^n(\alpha_1-\alpha_2-\alpha_3-\alpha_4), \\ &\quad m = 0(1)M \end{aligned} \quad (10)$$

where

$$\begin{aligned} \alpha_1 &= \tfrac{1}{\Delta t}, \\ \alpha_2 &= \tfrac{15Z_m}{h}, \\ \alpha_3 &= \tfrac{15Z_m^2}{h}, \\ \alpha_4 &= \tfrac{30}{h^3}. \end{aligned}$$

This iterative system (10) consists of $M+1$ equations and $M+5$ unknown parameters $(\delta_{-2},\delta_{-1},\delta_0,\ldots,\delta_M,\delta_{M+1},\delta_{M+2})^T$. In order for this system to have a unique solution, we need four additional constraints. These four additional constraints are obtained from the boundary conditions $u(a,t) = u(b,t) = 0$ and their first derivatives $u'(a,t) = u'(b,t) = 0$ and then are used to eliminate $\delta_{-2},\delta_{-1},\delta_{M+1}$ and δ_{M+2} from the system (10) as follows

$$\begin{aligned} \delta_{-2} &= \tfrac{165}{4}\delta_0+\tfrac{65}{2}\delta_1+\tfrac{9}{4}\delta_2, \\ \delta_{-1} &= -\tfrac{33}{8}\delta_0-\tfrac{9}{4}\delta_1-\tfrac{1}{8}\delta_2, \\ \delta_{M+1} &= -\tfrac{33}{8}\delta_M-\tfrac{9}{4}\delta_{M-1}-\tfrac{1}{8}\delta_{M-2}, \\ \delta_{M+2} &= \tfrac{165}{4}\delta_M+\tfrac{65}{2}\delta_{M-1}+\tfrac{9}{4}\delta_{M-2}, \end{aligned}$$

Then, this system of equations becomes a matrix equation with the $M+1$ unknowns $\mathbf{d} = (\delta_0, \delta_1, \ldots, \delta_M)^T$ in the form

$$\mathbf{A}\mathbf{d}^{\mathbf{n+1}} = \mathbf{B}\mathbf{d}^{\mathbf{n}}. \tag{11}$$

Here, both of the matrices A and B are pentagonal $(M+1)\times(M+1)$ matrices and therefore are easily solved using a variant of Thomas algorithm.

2.1 Initial state

To be able to proceed with the newly obtained iterative formula (10), first of all, we do need the initial vector d^0 which is going to be determined from the initial and boundary conditions. In order to achieve this, the approximation (4) ought to be rewritten particularly for the initial condition as

$$U_M(x,t_0) = \sum_{m=-2}^{M+2} \delta_m(t_0)\phi_m(x)$$

where the δ_m's are unknown element parameters. Now, if we force the initial numerical approximation $U_M(x,t_0)$ comply with the following boundary conditions to discard $\delta_{-2}, \delta_{-1}, \delta_{M+1}$ and δ_{M+2}

$$\begin{array}{ll} U_M(x,t_0) = u(x_m,t_0), & m = 0,1,\ldots,M \\ (U_M)_x(a,t_0) = 0, & (U_M)_x(b,t_0) = 0, \\ (U_M)_{xx}(a,t_0) = 0, & (U_M)_{xx}(b,t_0) = 0, \end{array}$$

we obtain the matrix form for the initial vector $\mathbf{d}^0$ as follows

$$\mathbf{W}\mathbf{d}^0 = \mathbf{b}$$

where

$$\mathbf{W} = \begin{bmatrix} 54 & 60 & 6 & & & & & \\ 25.25 & 67.50 & 26.25 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & 1 & 26 & 66 & 26 & 1 & & \\ & & & \ddots & & & & \\ & & & 1 & 26 & 66 & 26 & 1 \\ & & & & 1 & 26.25 & 67.50 & 25.25 \\ & & & & & 6 & 60 & 54 \end{bmatrix}$$

$$\mathbf{d}^0 = (\delta_0, \delta_1, \delta_2, \ldots, \delta_{M-2}, \delta_{M-1}, \delta_M)^T$$

and

$$\mathbf{b} = (u(x_0,t_0), u(x_1,t_0), \ldots, u(x_{M-1},t_0), u(x_M,t_0))^T.$$

2.2 Stability analysis

To investigate the stability analysis of the scheme, it is convenient to use the von Neumann method in which the growth factor of a typical Fourier mode is defined as

$$\delta_j^n = \hat{\delta}^n e^{ij\phi}, \tag{12}$$

where ϕ is a real number ($i = \sqrt{-1}$). To apply the von Neumann stability analysis, the nonlinear term u^2u_x in Eq. (1) needs to be linearized by making the quantity u a local constant so that the nonlinear term u^2u_x becomes $Z_m^2 u_x$. Therefore, the generalized m^{th} row of Eq. (10) remains the same.

Substituting the Fourier mode (12) into the iterative formula (10) and writing $\hat{\delta}^{n+1} = g\hat{\delta}^n$, the linearized recurrence relationship results in the growth factor g as follows:

$$g = \frac{a-ib}{a+ib}$$

where

$$\begin{array}{l} a = h^3(33+26\cos\phi+\cos 2\phi), \\ b = 30\Delta t(-2+5h^2Z_m(1+Z_m)+(2+h^2Z_m(1+Z_m))\cos\phi)\sin\phi, \end{array}$$

Since the stability condition $|g| \le 1$ is satisfied. Therefore the linearized scheme is unconditionally stable.

3 Numerical examples and results

In this section, numerical results of the test problem considered in the below have been obtained and all computations have been executed on a Pentium i7 PC in the Fortran code using double precision arithmetic. The accuracy of the method is measured by the error norms L_2 and L_∞ defined as

$$L_2 = \left\|U^{exact} - U_N\right\|_2 = \sqrt{h\sum_{j=0}^{N}\left|U_j^{exact} - (U_N)_j\right|^2},$$

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty = \max_j \left|U_j^{exact} - (U_N)_j\right|$$

respectively. In this study, to implement the performance of the scheme, as a test problem we consider the combined KdV-MKdV equation (1) equation with the boundary conditions and the initial condition taken from the exact solution given in Eq. (2).

During the solution process, various time steps and space steps have been taken over the problem domain $[-30,70]$. The program has been run for different time and space values. Then error norms L_2 and L_∞ are computed and compared with those available in the literature.

In Table 1, we have compared the error norms L_2 and L_∞ with those of [3] computed by inverse scattering

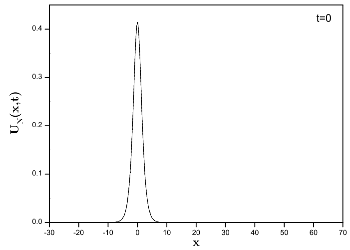

Fig. 1: The graph of numerical solutions at $t = 0$

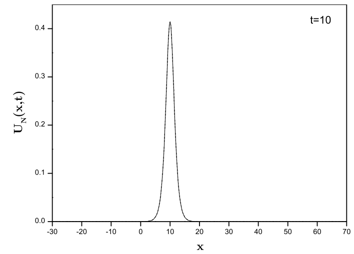

Fig. 2: The graph of numerical solutions at $t = 10$

transform (IST) and combination IST for $N = 200$, $\Delta t = 0.01$. It is obvious from the table that the present results are better than the compared ones. Table 2 shows a comparison of the error norms L_2 and L_∞ with those of [3] computed by IST and combination IST for $N = 400$, $\Delta t = 0.01$. Again, it is easily seen from the table that the newly obtained results are better than the compared ones. In table 3, we have tabulated the values of the error norms L_2 and L_∞ for $N = 400$ at various values of Δt. The table clearly shows that the error norms are at acceptable level. Moreover, it is seen from the table that as the values of Δt decrease so the error norms. In Table 4, we present the values of the error norms L_2 and L_∞ for $\Delta t = 0.001$ at various values of N. From the table, it is clear that as the number of partitions of the solution domain increase, the error norms decrease. In the figure 1-5, we have shown the graphs of the numerical solutions obtained in the present article at various values of t. In figure 6, we have shown the graph error at $t = 35$. If we consider the fact that the present method uses quintic B-splines, we can say that the present method yields much better results.

Table 1: A comparison of the error norms L_2 and L_∞ for $N = 200$, $\Delta t = 0.01$.

t	Present	Present	IST[3]	IST[3]	Com. IST[3]	Com. IST[3]
	L_2	L_∞	L_2	L_∞	L_2	L_∞
5	0.000829	0.000451	0.00229	0.01234	0.00751	0.04293
35	0.003331	0.001868	0.00563	0.03237	0.03792	0.20920

Table 2: A comparison of the error norms L_2 and L_∞ for $N = 400$, $\Delta t = 0.01$.

t	Present	Present	IST[3]	IST[3]	Com. IST[3]	Com. IST[3]
	L_2	L_∞	L_2	L_∞	L_2	L_∞
5	0.000052	0.000025	0.00051	0.00313	0.00215	0.01263
35	0.000100	0.000065	0.00124	0.00701	0.01164	0.06360

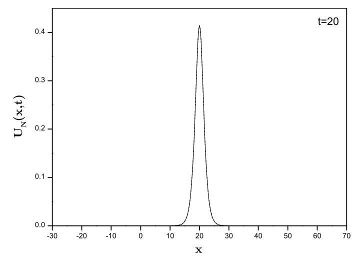

Fig. 3: The graph of numerical solutions at $t = 20$

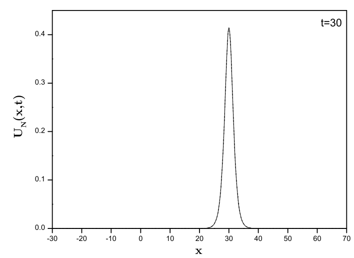

Fig. 4: The graph of numerical solutions at $t = 30$

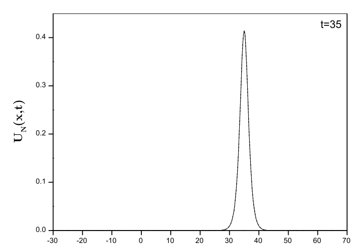

Fig. 5: The graph of numerical solutions at $t = 35$

4 Conclusions

In this paper, numerical solutions of the combined KdV-MKdV equation based on the quintic B-spline finite element method have been calculated and presented. A test problem is worked out to examine the performance of the present algorithm. The performance and efficiency of the method are shown by calculating the error norms L_2 and L_∞. The obtained results show that the error norms are sufficiently small during all computer runs. The obtained results indicate that the present method is a particularly successful numerical scheme to solve the combined KdV-MKdV equation. As a conclusion, the method can be efficiently applied to this type of

Table 3: The values of the error norms L_2 and L_∞ for $N = 400$ at various values of Δt.

t	$\Delta t = 0.5$	$\Delta t = 0.5$	$\Delta t = 0.1$	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.05$	$\Delta t = 0.01$	$\Delta t = 0.01$
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
5	0.086799	0.057534	0.002153	0.001354	0.000420	0.000261	0.000052	0.000025
10	0.243529	0.146236	0.005701	0.003366	0.001008	0.000590	0.000046	0.000029
15	0.423256	0.241175	0.010697	0.006192	0.001793	0.001030	0.000058	0.000040
20	0.586533	0.318655	0.017129	0.009782	0.002771	0.001575	0.000073	0.000044
25	0.708646	0.368562	0.024980	0.014150	0.003941	0.002217	0.000075	0.000045
30	0.784421	0.394173	0.034232	0.019278	0.005297	0.002978	0.000098	0.000060
35	0.824944	0.407603	0.044877	0.025181	0.006846	0.003826	0.000100	0.000065

Table 4: The values of the error norms L_2 and L_∞ for $\Delta t = 0.001$ at various values of N.

t	$N = 200$	$N = 200$	$N = 400$	$N = 400$	$N = 800$	$N = 800$	$N = 1000$	$N = 1000$
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
5	0.000834	0.000449	0.000058	0.000031	0.000005	0.000002	0.000003	0.000002
10	0.001159	0.000689	0.000071	0.000044	0.000004	0.000002	0.000004	0.000002
15	0.001591	0.000930	0.000098	0.000061	0.000006	0.000003	0.000004	0.000002
20	0.002030	0.001135	0.000124	0.000074	0.000009	0.000006	0.000006	0.000003
25	0.002521	0.001529	0.000141	0.000082	0.000015	0.000009	0.000007	0.000003
30	0.002961	0.001699	0.000168	0.000104	0.000025	0.000015	0.000009	0.000005
35	0.003420	0.001922	0.000187	0.000114	0.000037	0.000022	0.000030	0.000020

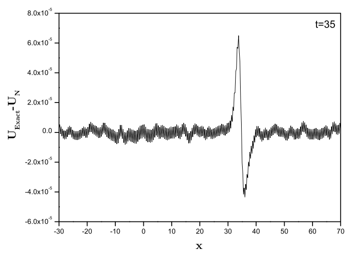

Fig. 6: The graph of errors at $t = 35$

non-linear problems arising in physics and mathematics with success.

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N. Murat Yagmurlu received his diploma in computer engineering from the Middle East Technical University. Then he has completed his Pd.D. degree in applied mathematics. He is currently studying about the approximate solutions of both ordinary and partial twodimensional differential equations. He contributed national and international talks and conferences. He is primarily interested in numerical methods, computer science and combination of the both.

Orkun Tasbozan graduated from the department of Mathematics of the Afyon Kocatepe University. He has completed his M.Sc. degree and Pd.D. thesis in applied mathematics. His main interest areas include analytical solutions, fractional differential equations, FEM etc.

Yusuf Ucar received his diploma in computer engineering from the Karadeniz Technical University. Then he has completed his Msc. and Pd.D. degrees in applied mathematics. He is currently studying about the approximate solutions of both ordinary and partial coupled differential equations. He has attended to national and international talks and conferences. He is primarily interested in numerical methods, computer science.

Alaattin Esen received his diploma in mathematics from the nonu University in 1994. He has completed his M.Sc. and Ph.D. degrees in applied mathematics. He is currently studying about the numerical solutions of a wide range of partial differential equations. He has many research papers published in various national and international journals. He has given talks and conferences. His research interests include finite difference methods, finite element methods, computational methods and algorithms.