Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/030104

Some Common Fixed Point Results for $(\alpha-\psi-\varphi)$-Contractive Mappings in Metric Spaces

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Received: 9 Jan. 2015, Revised: 10 Oct. 2015, Accepted: 15 Oct. 2015
Published online: 1 Jan. 2016

Abstract: In this paper, we consider a new class of pairs of generalized contractive type mappings defined in partial metric spaces. Some coincidence and common fixed point results for these mapping are presented.

Keywords: common fixed point, coincidence point, $g-\alpha-$admissible mapping, $\alpha-$regular, triangular $\alpha-$admissible.

1 Introduction

Fixed-point theory is one of the most intriguing research fields in nonlinear analysis. It is well known that the Banach contraction principle [6] is a very useful and classical tool in nonlinear analysis. There are many generalizations of the Banach's contraction mapping principle in the literature. These generalization were made either by using the contractive condition or by imposing some additional conditions on an ambient space. There have been a number of generalizations of metric spaces such as, fuzzy metric spaces, cone metric spaces, $G-$metric spaces, partial metric spaces, $b-$metric spaces(see[1,3,4,5,7,16,18]). It is also known that common fixed point theorems are generalizations of fixed point theorems. Thus, over the past few decades, there have been many researchers who have interested in generalizing fixed point theorems to coincidence point theorems and common fixed point theorems(see[2,12,13,14,15]). One of the most interesting results was given by Samet et al. [19] by defining $\alpha-\psi-$contractive mappings via admissible mappings, see also [10]. In this paper, we introduce a generalized $(\alpha-\psi-\varphi)$-contractive mappings in the setting of complete metric spaces via a $g-\alpha-$admissible and triangular $\alpha-$admissible mapping. We prove the existence and uniqueness of a common fixed point of such a mapping. Throughout this paper, the letters $\mathbb{R}_+$ and $\mathbb{N}$ will denote the sets of all non negative real numbers and positive integers.

Definition 11[9] *Let X be a non-empty set and T,g be given self maps on X. The pair $\{T,g\}$ is said to be weakly compatible if $Tgx=gTx$, whenever $Tx=gx$ for some x in X.*

Samet et al. [19] defined the notion of $\alpha-$admissible mappings as follows.

Definition 12*Let $T:X\to X$ be a map and $\alpha:X\times X\to\mathbb{R}$ be a function. Then T is said to be $\alpha-$admissible if*

$$\alpha(x,y)\geq 1 \Longrightarrow \alpha(Tx,Ty)\geq 1.$$

Recently, Rosa et al. [17] introduced the following new notions of $g-\alpha-$admissible mapping.

Definition 13*Let $T,g:X\to X$ and $\alpha:X\times X\to\mathbb{R}$. The mapping T is $g-\alpha-$admissible if, for all $x,y\in X$ such that $\alpha(gx,gy)\geq 1$, we have $\alpha(Tx,Ty)\geq 1$. If g is the identity mapping, then T is called $\alpha-$admissible.*

Definition 14[11] *An $\alpha-$admissible map T is said to be triangular $\alpha-$admissible if*

$$\alpha(x,z)\geq 1 \ \text{and} \ \alpha(z,y)\geq 1 \ \Longrightarrow \alpha(x,y)\geq 1.$$

Definition 15[8] *Let S denote the class of those functions $\beta:[0,+\infty)\to[0,1)$ which satisfies the condition $\beta(t_n)\to 1$ implies $t_n\to 0$.*

2 Maim Results

In this section, we prove some common fixed point results for two self-mappings satisfying a generalized (α,ψ,φ)-

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Geraghty contraction type map. For the notion of $\alpha-\psi$-contractive type mappings, see Samet et al.[19].
Next, we introduce the novel notion of generalized $(\alpha-\psi-\varphi)$-contractive mapping as follows:

Definition 21 *Let (X,d) be a metric space and T,g be self-mappings on X. We say that the pair (T,g) is a generalized $(\alpha-\psi-\varphi)$-contractive pair of mappings if there exists $\alpha: X\times X\to\mathbb{R}$ and two continuous and nondecreasing functions $\psi,\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<\psi(t)$ for each $t>0$, $\varphi(0)=\psi(0)=0$ such that for all $x,y\in X$, we have*

$$\alpha(x,y)\psi(d(Tx,Ty))\le\varphi(M(x,y)), \tag{1}$$

where

$$M(x,y)=\max\{d(gx,gy),\frac{d(gx,Tx)+d(gy,Ty)}{2},\frac{d(gx,Ty)+d(gy,Tx)}{2}\}.$$

Definition 22 *Let (X,d) be a metric space, $g:X\to X$ and $\alpha:X\times X\to\mathbb{R}$. X is $\alpha-$regular with respect to g if, for every sequence $\{x_n\}\subseteq X$ such that $\alpha(gx_n,gx_{n+1})\ge1$ for all $n\in\mathbb{N}$ and $gx_n\to gx\in gX$ as $n\to\infty$, then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that for all $k\in\mathbb{N}$, $\alpha(gx_{n(k)},gx)\ge1$. If g is the identity mapping, then X is called $\alpha-$regular.*

Lemma 21 *Let $T,g:X\to X$ and $\alpha:X\times X\to\mathbb{R}$. Suppose T be a $g-\alpha-$admissible and triangular $\alpha-$admissible. Assume that there exists $x_0\in X$ such that $\alpha(gx_0,Tx_0)\ge1$. Then*

$$\alpha(gx_m,gx_n)\ge1 \text{ for all } m,n\in\mathbb{N} \text{ with } m<n,$$

where

$$gx_{n+1}=Tx_n.$$

Proof 21 *Since there exists $x_0\in X$ such that $\alpha(gx_0,Tx_0)\ge1$ and T is a $g-\alpha-$admissible, we deduce that*

$$\alpha(gx_0,gx_1)=\alpha(gx_0,Tx_0)\ge1$$
$$\Longrightarrow\alpha(gx_1,gx_2)=\alpha(Tx_0,Tx_1)\ge1,$$
$$\alpha(gx_1,gx_2)\ge1\Longrightarrow\alpha(gx_2,gx_3)=\alpha(Tx_1,Tx_2)\ge1.$$

By continuing this process, we get

$$\alpha(gx_n,gx_{n+1})\ge1,\ n=0,1,2,\cdots.$$

Suppose that $m<n$. Since $\alpha(gx_m,gx_{m+1})\ge1$, $\alpha(gx_{m+1},gx_{m+2})\ge1$ and T is triangular $\alpha-$admissible, we have $\alpha(gx_m,gx_{m+2})\ge1$. Again, since $\alpha(gx_m,gx_{m+2})\ge1$ and $\alpha(gx_{m+2},gx_{m+3})\ge1$, we have $\alpha(gx_m,gx_{m+3})\ge1$. Continuing this process inductively, we obtain

$$\alpha(gx_m,gx_n)\ge1.$$

We start this section with the first of our main theorems.

Theorem 22 *Let (X,d) be a complete metric space, $T,g: X\to X$ be such that $TX\subseteq gX$ and suppose gX is closed. Assume that the pair (T,g) is a generalized $(\alpha-\psi-\varphi)$-contractive pair of mappings and the following conditions hold:*

(i) T is $g-\alpha-$admissible and triangular;
(ii) there exists $x_0\in X$ such that $\alpha(gx_0,Tx_0)\ge1$;
(iii) X is $\alpha-$regular with respect to g.

Then T and g have a coincidence point.
Moreover, suppose that the following conditions hold:

(a) The pair $\{T,g\}$ is weakly compatible;
(b) either $\alpha(u,v)\ge1$ or $\alpha(v,u)\ge1$ whenever $Tu=gu$ and $Tv=gv$.

Then T and g have a unique common fixed point.

Proof 22 *Let $x_0\in X$ be such that $\alpha(gx_0,Tx_0)\ge1$ (such a point exists from the condition (ii)). Since $TX\subseteq gX$ we can choose a point $x_1\in X$ such that $Tx_0=gx_1$. Also, there exists $x_2\in X$ such that $Tx_1=gx_2$, this can done, since $TX\subseteq gX$. Continuing this process having chosen $x_1,x_2,\dots,x_n\in X$, we have $x_{n+1}\in X$ such that*

$$gx_{n+1}=Tx_n,\ n=0,1,2,\cdots. \tag{2}$$

By Lemma 21, we have

$$\alpha(gx_n,gx_{n+1})\ge1,\ n=0,1,2,\cdots. \tag{3}$$

If $Tx_{n_0}=Tx_{n_0+1}$ for some n_0, then by (2), we have

$$gx_{n_0}=Tx_{n_0+1}=Tx_{n_0},$$

that is, T and g have a coincidence point at $x=x_{n_0}$, and so we have finished the proof. For this, we suppose that for all $n\in\mathbb{N}$, $Tx_n\neq Tx_{n+1}$. Since the pair (T,g) is a generalized $(\alpha-\psi-\varphi)$-contractive pair of mappings and using (3), we obtain

$$\begin{aligned}\psi(d(gx_{n+1},gx_{n+2}))&=\psi(d(Tx_n,Tx_{n+1}))\\&\le\alpha(gx_n,gx_{n+1})\psi(d(Tx_n,Tx_{n+1}))\\&\le\varphi(M(x_n,x_{n+1})),\end{aligned} \tag{4}$$

for all $n\in\mathbb{N}$, where

$$\begin{aligned}&M(x_n,x_{n+1})\\&=\max\{d(gx_n,gx_{n+1}),\frac{d(gx_n,Tx_n)+d(gx_{n+1},Tx_{n+1})}{2},\frac{d(gx_n,Tx_{n+1})+d(gx_{n+1},Tx_n)}{2}\}\\&=\max\{d(gx_n,gx_{n+1}),\frac{d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})}{2},\frac{d(gx_n,gx_{n+2})+d(gx_{n+1},gx_{n+1})}{2}\}\\&=\max\{d(gx_n,gx_{n+1}),\frac{d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})}{2},\frac{d(gx_n,gx_{n+2})}{2}\}.\end{aligned}$$

Since

$$\frac{d(gx_n,gx_{n+2})}{2} \le \frac{d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})}{2} \le \max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\},$$

then we get

$$M(x_n,x_{n+1}) \le \max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\}. \quad (5)$$

By (4) and (5), we have

$$\psi(d(gx_{n+1},gx_{n+2})) \le \varphi(\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\}). \quad (6)$$

If for some $n \in \mathbb{N}$, $\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\} = d(gx_{n+1},gx_{n+2})$, *then by (6) and using the properties of the function* φ, *we get*

$$\begin{aligned}&\psi(d(gx_{n+1},gx_{n+2}))\\ &\le \varphi(\max\{d(gx_n,gx_{n+1}),d(gx_{n+1},gx_{n+2})\})\\ &= \varphi(d(gx_{n+1},gx_{n+2}))\\ &< \psi(d(gx_{n+1},gx_{n+2})),\end{aligned}$$

which is a contradiction. So

$$\begin{aligned}\psi(d(gx_{n+1},gx_{n+2})) &\le \varphi(d(gx_n,gx_{n+1})\\ &< \psi(d(gx_n,gx_{n+1})),\\ &for\ each\ n \in \mathbb{N}.\end{aligned} \quad (7)$$

From (7), we deduce that $\{\psi(d(gx_{n+1},gx_{n+2}))\}$ *is a non-negative non-increasing sequence and* ψ *is increasing, we get that the sequence* $\{d(gx_{n+1},gx_{n+2})\}$ *is non-increasing and consequently there exists* $\delta \ge 0$ *such that*

$$\lim_{n\to\infty} d(gx_{n+1},gx_{n+2}) = \delta.$$

We claim that $\delta = 0$. *On the contrary, assume that*

$$\lim_{n\to\infty} d(gx_{n+1},gx_{n+2}) = \delta > 0. \quad (8)$$

Since ψ *and* φ *are continuous then from (7) and (8), we have*

$$\begin{aligned}\psi(\delta) &= \lim_{n\to\infty} \psi(d(gx_{n+1},gx_{n+2}))\\ &= \lim_{n\to\infty} \varphi(d(gx_{n+1},gx_{n+2}))\\ &= \varphi(\delta),\end{aligned}$$

and so $\delta = 0$, *a contradiction. Thus*

$$\lim_{n\to\infty} d(gx_{n+1},gx_{n+2}) = 0. \quad (9)$$

Now, we claim that

$$\lim_{n,m\longrightarrow\infty} d(gx_n,gx_m) = 0. \quad (10)$$

Assume on the contrary that there exists $\varepsilon > 0$ *and subsequences* $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ *of* $\{gx_n\}$ *with* $n(k) > m(k) \ge k$ *such that*

$$d(gx_{m(k)},gx_{n(k)}) \ge \varepsilon. \quad (11)$$

Additionally, corresponding to $m(k)$, *we may choose* $n(k)$ *such that it is the smallest integer satisfying (11) and* $n(k) > m(k) \ge k$. *Thus,*

$$d(gx_{m(k)},gx_{n(k)-1}) < \varepsilon. \quad (12)$$

Using the triangle inequality in metric space and (11) and (12) we obtain that

$$\begin{aligned}\varepsilon \le d(gx_{n(k)},gx_{m(k)}) &\le d(gx_{n(k)},gx_{n(k)-1}) + d(gx_{n(k)-1},gx_{m(k)})\\ &< d(gx_{n(k)},gx_{n(k)-1}) + \varepsilon.\end{aligned}$$

Taking the limit as $k \longrightarrow \infty$ *and using (9) we obtain*

$$\lim_{k\longrightarrow\infty} d(gx_{m(k)},gx_{n(k)}) = \varepsilon. \quad (13)$$

Also

$$\begin{aligned}\varepsilon \le d(gx_{m(k)},gx_{n(k)}) &\le d(gx_{m(k)},gx_{n(k)+1}) + d(gx_{n(k)+1},gx_{n(k)})\\ &\le d(gx_{m(k)},gx_{n(k)}) + d(gx_{n(k)},gx_{n(k)+1}) + d(gx_{n(k)+1},gx_{n(k)})\\ &\le d(gx_{m(k)},gx_{n(k)}) + 2d(gx_{n(k)},gx_{n(k)+1}).\end{aligned}$$

So from (9) and (13), we have

$$\lim_{k\longrightarrow\infty} d(gx_{m(k)},gx_{n(k)+1}) = \varepsilon. \quad (14)$$

Also

$$\begin{aligned}\varepsilon \le d(gx_{n(k)},gx_{m(k)}) &\le d(gx_{n(k)},gx_{m(k)+1}) + d(gx_{m(k)+1},gx_{m(k)})\\ &\le d(gx_{n(k)},gx_{m(k)}) + d(gx_{m(k)},gx_{m(k)+1}) + d(gx_{m(k)+1},gx_{m(k)})\\ &\le d(gx_{n(k)},gx_{m(k)}) + 2d(gx_{m(k)},gx_{m(k)+1}).\end{aligned}$$

So from (9) and (13), we have

$$\lim_{k\longrightarrow\infty} d(gx_{n(k)},gx_{m(k)+1}) = \varepsilon. \quad (15)$$

Now using inequality (1) and Lemma 21, we have

$$\begin{aligned}\psi(\varepsilon) &\le \psi(d(Tx_{m(k)},Tx_{n(k)}))\\ &\le \alpha(gx_{m(k)},gx_{n(k)})\psi(d(Tx_{m(k)},Tx_{n(k)}))\\ &\le \varphi(M(x_{m(k)},x_{n(k)})).\end{aligned} \quad (16)$$

where

$$\begin{aligned}&M(x_{m(k)},x_{n(k)})\\ &= \max\{d(gx_{m(k)},gx_{n(k)}), \frac{d(gx_{m(k)},gx_{m(k)+1}) + d(gx_{n(k)},gx_{n(k)+1})}{2},\\ &\qquad \frac{d(gx_{m(k)},gx_{n(k)+1}) + d(gx_{n(k)},gx_{m(k)+1})}{2}\}.\end{aligned}$$

Letting $k \longrightarrow \infty$ *in the above equality and using (9),(13),(14) and (15), we obtain*

$$\lim_{k\longrightarrow\infty} M(x_{m(k)},x_{n(k)}) = \varepsilon.$$

As $k \longrightarrow \infty$, *inequality (16) becomes,*

$$\psi(\varepsilon) \le \varphi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. So, we conclude that $\{gx_n\}$ *is a Cauchy sequence in* (X,d). *Since by (2) we have* $\{Tx_n\} = \{gx_{n+1}\} \subseteq gX$ *and* gX *is closed, there exists* $x \in X$ *such that*

$$\lim_{n\to\infty} gx_n = gx. \tag{17}$$

Now, we claim that x *is a coincidence point of* T *and* g. *On contrary, assume that* $d(Tx,gx) > 0$. *Since* X *is* $\alpha-$*regular with respect to* g *and (17), we have*

$$\alpha(gx_{n(k)+1},gx) \geq 1 \ for\ all\ k \in \mathbb{N}. \tag{18}$$

Also by the use of triangle inequality, we have

$$\begin{aligned} d(gx,Tx) &\leq d(gx,gx_{n(k)+1}) + d(gx_{n(k)+1},Tx) \\ &= d(gx,gx_{n(k)+1}) + d(Tx_{n(k)},Tx). \end{aligned}$$

On taking limit as $k \to \infty$ *in the above inequality, we have*

$$d(gx,Tx) \leq \lim_{k\to\infty} d(Tx_{n(k)},Tx). \tag{19}$$

By property of ψ, *(18) and (19), we have*

$$\begin{aligned} \psi(d(gx,Tx)) &\leq \lim_{k\to\infty} \psi(d(Tx_{n(k)},Tx)) \\ &\leq \lim_{k\to\infty} \alpha(gx_{n(k)+1},gx)\psi(d(Tx_{n(k)},Tx)) \\ &\leq \lim_{k\to\infty} \varphi(M(x_{n(k)},x)) = \varphi(\lim_{k\to\infty} M(x_{n(k)},x)) \\ &= \varphi(\frac{d(gx,Tx)}{2}) \\ &< \psi(\frac{d(gx,Tx)}{2}), \end{aligned}$$

which is a contradiction. Indeed,

$$\begin{aligned} &M(x_{n(k)},x) \\ &= \max\{d(gx_{n(k)},gx), \frac{d(gx_{n(k)},Tx_{n(k)}) + d(gx,Tx)}{2}, \\ &\qquad \frac{d(gx_{n(k)},Tx) + d(gx,Tx_{n(k)})}{2}\}. \end{aligned}$$

We deduce, taking limit as $n \to \infty$, *that*

$$\lim_{k\to\infty} M(x_{n(k)},x) = \frac{d(gx,Tx)}{2}.$$

Hence, $d(gx,Tx) = 0$, *that is,* $gx = Tx$ *and* x *is a coincidence point of* T *and* g. *We claim that, if* $Tu = gu$ *and* $Tv = gv$, *then* $gu = gv$. *By hypotheses,* $\alpha(u,v) \geq 1$ *or* $\alpha(v,u) \geq 1$. *Suppose that* $\alpha(u,v) \geq 1$, *then*

$$\begin{aligned} \psi(d(gu,gv)) &= \psi(d(Tu,Tv)) \leq \alpha(u,v)\psi(d(Tu,Tv)) \\ &\leq \varphi(M(u,v)), \end{aligned}$$

where

$$\begin{aligned} M(u,v) &= \max\{d(gu,gv), \frac{d(gu,Tu)+d(gv,Tv)}{2}, \\ &\qquad \frac{d(gu,Tv)+d(gv,Tu)}{2}\} \\ &= \max\{d(gu,gv), \frac{d(gu,gu)+d(gv,gv)}{2}, \\ &\qquad \frac{d(gu,gv)+d(gv,gu)}{2}\} \\ &= d(gu,gv). \end{aligned}$$

So,

$$\psi(d(gu,gv)) \leq \varphi(d(gu,gv)) < \psi(d(gu,gv)),$$

which is a contradiction. Thus we deduce that $gu = gv$. *Similarly, if* $\alpha(v,u) \geq 1$ *we can prove that* $gu = gv$. *Now, we show that* T *and* g *have a common fixed point. Indeed, if* $w = Tu = gu$, *owing to the weakly compatible of* T *and* g, *we get* $Tw = T(gu) = g(Tu) = gw$. *Thus* w *is a coincidence point of* T *and* g, *then* $gu = gw = w = Tw$. *Therefore,* w *is a common fixed point of* T *and* g. *The uniqueness of common fixed point of* T *and* g *is a consequence of the conditions (1) and* (b), *and so we omit the details.*

From Theorem 22, if we choose $g = I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 23*Let* (X,d) *be a complete metric space,* $T : X \to X$ *be a self-mapping on* X *and* $\alpha : X \times X \to \mathbb{R}$. *Assume that the following condition holds:*

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \varphi(M(x,y)),$$

for all $x,y \in X$, *where* $\psi,\varphi : [0,\infty) \to [0,\infty)$ *are continuous and nondecreasing with* $\varphi(t) < \psi(t)$ *for each* $t > 0$, $\varphi(0) = \psi(0) = 0$ *and*

$$\begin{aligned} M(x,y) = \max\{d(x,y), &\frac{d(x,Tx)+d(y,Ty)}{2}, \\ &\frac{d(x,Ty)+d(y,Tx)}{2}\}. \end{aligned}$$

Also that the following conditions hold:

(i)T *is* $\alpha-$*admissible and triangular;*
(ii)there exists $x_0 \in X$ *such that* $\alpha(x_0,Tx_0) \geq 1$;
(iii)X *is* $\alpha-$*regular;*
(iv)either $\alpha(u,v) \geq 1$ *or* $\alpha(v,u) \geq 1$ *whenever* $Tu = u$ *and* $Tv = v$.

Then T *has a unique fixed point.*

From Theorem 22, if the function $\alpha : X \times X \to \mathbb{R}$ is such that $\alpha(x,y) = 1$ for all $x,y \in X$, we deduce the following theorem.

Theorem 24*Let* (X,d) *be a complete metric space,* $T,g : X \to X$ *be such that* $TX \subseteq gX$. *Assume that* gX *is closed and that the following conditions hold:*

$$\psi(d(Tx,Ty)) \leq \varphi(M(x,y)),$$

for all $x,y \in X$, *where* $\psi,\varphi : [0,\infty) \to [0,\infty)$ *are continuous and nondecreasing with* $\varphi(t) < \psi(t)$ *for each* $t > 0$, $\varphi(0) = \psi(0) = 0$ *and*

$$\begin{aligned} M(x,y) = \max\{d(gx,gy), &\frac{d(gx,Tx)+d(gy,Ty)}{2}, \\ &\frac{d(gx,Ty)+d(gy,Tx)}{2}\}. \end{aligned}$$

Then T *and* g *have a coincidence point. Moreover, if* T *and gare weakly compatible, then* T *and* g *have a unique common fixed point.*

From Theorem 22, if $\psi(t)=\psi_1(t)$ and $\varphi(t)=\psi_1(t)-\varphi_1(t)$ for each $t\in\mathbb{R}_+$ where $\psi_1,\varphi_1:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ are continuous functions such that $\psi_1(t)>\varphi_1(t)>0$ for $t>0$, $\psi_1(0)=\varphi_1(0)=0$, φ_1 is nonincreasing and ψ_1 is increasing, we deduce the following theorem.

Theorem 25*Let (X,d) be a complete metric space, $T,g:X\to X$ be such that $TX\subseteq gX$ and $\alpha:X\times X\to\mathbb{R}$. Assume that gX is closed and that the following conditions hold:*

$$\alpha(x,y)\psi_1(d(Tx,Ty))\le\psi_1(M(x,y))-\varphi_1(M(x,y)),$$

for all $x,y\in X$, where $\psi_1,\varphi_1:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ are continuous functions such that $\psi_1(t)>\varphi_1(t)>0$ for $t>0$, $\psi_1(0)=\varphi_1(0)=0$, φ_1 is nonincreasing, ψ_1 is increasing and

$$M(x,y)=\max\{d(gx,gy),\frac{d(gx,Tx)+d(gy,Ty)}{2},\frac{d(gx,Ty)+d(gy,Tx)}{2}\}.$$

Assume also that the following conditions hold:

(i)T is $g-\alpha-$admissible and triangular;
(ii)there exists $x_0\in X$ such that $\alpha(gx_0,Tx_0)\ge 1$;
(iii)X is $\alpha-$regular with respect to g.

Then T and g have a coincidence point.
Moreover, the following conditions hold:

(a)The pair $\{T,g\}$ is weakly compatible;
(b)either $\alpha(u,v)\ge 1$ or $\alpha(v,u)\ge 1$ whenever $Tu=gu$ and $Tv=gv$.

Then T and g have a unique common fixed point.

From Theorem 25, if we choose $g=I_X$ the identity mapping on X, we deduce the following corollary.

Corollary 26*Let (X,d) be a complete metric space, $T:X\to X$ be a self-mapping on X and $\alpha:X\times X\to\mathbb{R}$. Assume that the following condition holds:*

$$\alpha(x,y)\psi_1(d(Tx,Ty))\le\psi_1(M(x,y))-\varphi_1(M(x,y)),$$

for all $x,y\in X$, where $\psi_1,\varphi_1:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ are continuous functions such that $\psi_1(t)>\varphi_1(t)>0$ for $t>0$, $\psi_1(0)=\varphi_1(0)=0$, φ_1 is nonincreasing, ψ_1 is increasing and

$$M(x,y)=\max\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

Assume also that the following conditions hold:

(i)T is $\alpha-$admissible and triangular;
(ii)there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$;
(iii)X is $\alpha-$regular;
(iv)either $\alpha(u,v)\ge 1$ or $\alpha(v,u)\ge 1$ whenever $Tu=u$ and $Tv=v$.

Then T has a unique fixed point.

From Theorem 25, if the function $\alpha:X\times X\to\mathbb{R}$ is such that $\alpha(x,y)=1$ for all $x,y\in X$ and $g=I$, we deduce the following corollary.

Corollary 27*Let (X,d) be a complete metric space, $T:X\to X$ be a self-mapping on X. Assume that the following condition holds:*

$$\psi_1(d(Tx,Ty))\le\psi_1(M(x,y))-\varphi_1(M(x,y)),\qquad(20)$$

for all $x,y\in X$, where $\psi_1,\varphi_1:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ are continuous functions such that $\psi_1(t)>\varphi_1(t)>0$ for $t>0$, $\psi_1(0)=\varphi_1(0)=0$, φ_1 is nonincreasing, ψ_1 is increasing and

$$M(x,y)=\max\{d(x,y),\frac{d(x,Tx)+d(y,Ty)}{2},\frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

Then T has a unique fixed point.

3 Application to integral equations

Here, in this section, we wish to study the existence of a unique solution to an integral equation. Consider the integral equation

$$x(t)=h(t)+\lambda\int_0^1 k(t,s)f(s,x(s))ds,\ t\in I=[0,1],\lambda\ge 0.\qquad(21)$$

We consider the space $C(I)$ of real continuous functions defined on $I=[0,1]$. Obviously, the space $C(I)$ with the metric given by

$$d(x,y)=\sup_{t\in I}|x(t)-y(t)|,\quad for\ \ x,y\in C(I).$$

is a complete metric space. We will analyze Eq. (21) under the following assumptions:

$(a_1)$$h:I\longrightarrow\mathbb{R}$ is a continuous function.
$(a_2)$$f:I\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous function, $f(t,x)\ge 0$ and there exist a constant $0\le L<1$ and a nondecreasing and continuous function $\gamma:[0,+\infty)\longrightarrow[0,+\infty)$ with $\gamma(t)<t$ for all $t>0$ and $\gamma(0)=0$ such that

$$|f(t,x)-f(t,y)|\le L\sqrt{\gamma(|x-y|^2)}\ for\ each\ t\in I,x,y\in\mathbb{R}.$$

$(a_3)$$k:I\times I\longrightarrow\mathbb{R}$ is continuous in $t\in I$ for every $s\in I$ and measurable in $s\in I$ for all $t\in I$ such that $k(t,x)\ge 0$ and $\sup_{t\in I}\int_0^1 k(t,s)ds\le K$.
$(a_4)$$\lambda KL\le 1$.

Now, we formulate the main result of this section.

Theorem 31*Under assumptions $(a_1)-(a_4)$, Eq. (21) has a unique solution in $X=C(I)$.*

Proof 31*We consider the operator $T:X\longrightarrow X$ defined by*

$$T(x)(t)=h(t)+\lambda\int_0^1 k(t,s)f(s,x(s))ds,\ \ for\ t\in I.$$

By virtue of our assumptions, T is well defined (this means that if $x \in X$ then $Tx \in X$). Also, for $x,y \in X$, we have

$$\begin{aligned}|T(x)(t)-T(y)(t)| &= |h(t)+\lambda\int_0^1 k(t,s)f(s,x(s))ds-h(t)\\ &\quad -\lambda\int_0^1 k(t,s)f(s,y(s))ds|\\ &\le \lambda\int_0^1 k(t,s)|f(s,x(s))-f(s,y(s))|ds\\ &\le \lambda\int_0^1 k(t,s)L\sqrt{\gamma(|x(s)-y(s)|^2)}ds.\end{aligned}$$

Since the function γ is non-decreasing, we have

$$\begin{aligned}\sqrt{\gamma(|x(s)-y(s)|^2)} &\le \sqrt{\gamma[(\sup_{t\in I}|x(s)-y(s)|)^2]}\\ &= \sqrt{\gamma[d^2(x,y)]} \le \sqrt{\gamma[M^2(x,y)]},\end{aligned}$$

hence

$$|T(x)(t)-T(y)(t)| \le \lambda KL\sqrt{\gamma[dM^2(x,y)]} \le \sqrt{\gamma[M^2(x,y)]}.$$

Then, we can obtain

$$d(Tx,Ty) = \sup_{t\in I}|T(x)(t)-T(y)(t)| \le \sqrt{\gamma[M^2(x,y)]},$$

which gives us that

$$\begin{aligned}d^2(Tx,Ty) &\le \gamma[M^2(x,y)]\\ &= M^2(x,y)-[M^2(x,y)-\gamma(M^2(x,y))].\end{aligned}$$

Now, by considering the functions $\psi_1,\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ defined by:

$$\psi_1(t) = t^2 \text{ and } \varphi_1(t) = t^2-\gamma(t^2),$$

we get

$$\psi_1(d(Tx,Ty)) \le \psi_1(M(x,y))-\varphi_1(M(x,y)).$$

This proves that the operator T satisfies the contractive condition (20) appearing in Corollary 27. So Eq. (21) has a unique solution in $C(I)$ and the proof is complete.

Conclusion

Throughout this paper, we have dealt with the $(\alpha-\psi-\varphi)$-contractive mappings in complete metric spaces via a $g-\alpha-$admissible and triangular $\alpha-$admissible mapping. Some coincidence and common fixed point results for these mapping are presented.

Acknowledgements

The Authors are very thankful to the referees for their useful suggestions in this paper.

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