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Common Fixed Point Results using Weakly Compatible Maps along with (CLRg) Property in G-Metric Spaces

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Abstract: In this paper, first we prove common fixed point theorems for a pair of weakly compatible maps. Secondly, we prove a fixed point theorem for weakly compatible maps along with (CLRg) property. In fact, our results generalize the results of Z. Mustafa et al. [10].

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1 Introduction

In metric fixed point theory, Banach proved an important result which is the mile stone in fixed point theory and its applications. In 1922, Banach gives classical theorem known as Banach Contraction Principle. This principle gives under appropriate conditions, the existence and uniqueness of fixed points and provides methods for obtaining approximate fixed points.

In 1963, Gahler [2] introduced the concept of 2-metric, which denotes the area of a triangle. In 1992, Dhage [1] introduced the concept of D-metric spaces and it denotes the perimeter of the triangle with vertices x, y, z in R^2. However, Hsiao [3] showed that, in 2-metric spaces, all the maps are either reduces to a single map or maps are constant.

In 2003, Mustafa and Sims [7] pointed out the most of the results claimed by Dhage and others are invalid and they introduced a new structure of generalized metric space and called it G-metric space. For more details on G-metric spaces one can refer to the papers [7]-[11].

Now we give preliminaries and basic definitions which are used throughout the paper.

In 2006, Mustafa and Sims [8] introduced the concept of G-metric spaces as follows:

Definition 1.1. Let X be a nonempty set, $G : X \times X \times X \to R^+$ a function satisfying the following axioms:

(G1) $G(x,y,z) = 0$ if $x = y = z$,
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$,
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$,
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \dots$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, (rectangle inequality).

The function G is called a generalized metric or, more specifically, a G-metric on X and the pair (X,G) is called a G-metric space.

Let (X,G) be a G-metric space, $\{x_n\}$ a sequence of points in X. we say that $\{x_n\}$ is G-convergent to x if $\lim\limits_{m,n\to\infty} G(x,x_n,x_m) = 0$; i.e., for each $\varepsilon > 0$ there exists an N such that $G(x,x_n,x_m) < \varepsilon$ for all $m,n \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

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Proposition 1.1. *Let (X,G) be a G-metric space. Then the following are equivalent:*

(i) $\{x_n\}$ *is G convergent to* x,
(ii) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$,
(iii) $G(x_n,x,x) \to 0$ *as* $n \to \infty$,
(iv) $G(x_m,x_n,x) \to 0$ *as* $m,n \to \infty$.

Definition 1.2. Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\varepsilon > 0$ there exists an N such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$.

Proposition 1.2. *In a G-metric space (X,G) the following are equivalent:*

(i) *The sequence* $\{x_n\}$ *is G-Cauchy,*
(ii) *For each* $\varepsilon > 0$ *there exists an* N *such that* $G(x_n,x_m,x_l) < \varepsilon$ *for all* $n,m,l \geq N$.

Proposition 1.3. *Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.3. A G-metric space (X,G) is called a symmetric G-metric space if

$$G(x,y,y) = G(y,x,x) \quad \text{for all } x,y \in X.$$

Proposition 1.4. *Every G-metric space (X,G) defines a metric space (X,d_G) by*

(i) $d_G(x,y) = G(x,y,y) + G(y,x,x)$ *for all x, y in* X.

If (X,G) is a symmetric G-metric space, then

(ii) $d_G(x,y) = 2G(x,y,y)$ *for all x, y in* X.

However, if (X,G) is not symmetric, then it follows from the G-metric properties that

(iii) $3/2G(x,y,y) \leq d_G(x,y) \leq 3G(x,y,y)$ *for all x, y in* X.

Proposition 1.5. *A G-metric space (X,G) is G-complete if and only if (X,d_G) is a complete metric space.*

Proposition 1.6. *Let (X,G) be a G-metric space. Then, for any x, y, z, a in X it follows that:*

(i) *if* $G(x,y,z) = 0$, *then* $x = y = z$,
(ii) $G(x,y,z) \leq G(x,x,y) + G(x,x,z)$,
(iii) $G(x,y,y) \leq 2G(y,x,x)$,
(iv) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$,
(v) $G(x,y,z) \leq 2/3(G(x,a,a) + G(y,a,a) + G(z,a,a))$.

There has been a considerable interest to study common fixed point for a pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. It was the turning point in the "fixed point arena" when the notion of commutativity was used by Jungck [4] to obtain common fixed point theorems. This result was further generalized and extended in various ways by many authors. In particular, now we look in the context of common fixed point theorem in G-metric spaces. Start with the following contraction conditions:

Let T be a mapping from a complete metric space (X,G) into itself and consider the following conditions:

$$G(Tx,Ty,Tz) \leq \alpha G(x,y,z) \ \text{ for all } x,y,z \in X, \quad \text{where } 0 \leq \alpha < 1. \tag{1.1}$$

It is clear that every self mapping T of X satisfying condition (1.1) is continuous. Now we focus to generalize the condition (1.1) for a pair of self maps S and T of X in the following way:

$$G(Sx,Sy,Sz) \leq \alpha G(Tx,Ty,Tz) \ \text{ for all } x,y,z \in X, \quad \text{where } 0 \leq \alpha < 1. \tag{1.2}$$

To prove the existence of common fixed points for (1.2), it is necessary to add additional assumptions such as the construction of the sequence $\{x_n\}$ and making some mechanism to obtain common fixed point and this problem was overcomed by imposing additional hypothesis of commutative pair $\{S,T\}$.

Most of the theorems followed the following pattern:

(i) Contraction
(ii) Continuity of functions (either one or both)
(iii) Commuting pair of mappings.

In some cases condition (ii) can be relaxed by imposing some certain conditions but conditions (i) and (iii) are unavoidable.

In 2011, Z. Mustafa et al. [10] have proved the following results:

Theorem 1.1. *Let (X,G) be a complete G-metric space and let $T : X \to X$, be a mapping which satisfies the following condition, for all $x,y \in X$.*

$$\begin{aligned} &G(Tx,Ty,Ty) \\ &\leq \max\{(aG(x,y,y), b[G(x,Tx,Tx) + 2G(y,Ty,Ty)], \\ &\qquad b[G(x,Ty,Ty) + G(y,Ty,Ty) + G(y,Tx,Tx)])\}, \end{aligned}$$

where $0 \leq a < 1$ and $0 \leq b < 1/3$. Then T has a unique fixed point, say u and T is G-continuous at u.

Theorem 1.2. *Let (X,G) be a complete G-metric space and let $T : X \to X$, be a mapping which satisfies the* *following condition, for all $x, y \in X$.*

$$
\begin{aligned}
&G(Tx, Ty, Ty) \\
&\le k \max\{([G(x, Tx, Tx) + 2G(y, Ty, Ty)], \\
&\quad [G(x, Ty, Ty) + G(y, Ty, Ty) + G(y, Tx, Tx)], \\
&\quad [G(y, y, Tx) + G(y, y, Ty) + G(x, x, Ty)])\},
\end{aligned}
$$

where $0 \le k < 1/4$. Then T has a unique fixed point, say u and T is G-continuous at u.

2 Weakly compatible Maps

In 1996, Jungck [6] introduced the notion of weakly compatible mappings as follows:

Definition 2.1. Two maps f and g are said to be weakly compatible if they commute at coincidence points.

Example 2.1. Let $X = [0, 3]$. Define $f, g : [0, 3] \to [0, 3]$ by

$$
f(x) = \begin{cases} x & \text{if } x \in [0, 1), \\ 3 & \text{if } x \in [1, 3] \end{cases}
$$

and

$$
g(x) = \begin{cases} 3 - x & \text{if } x \in [0, 1), \\ 3 & \text{if } x \in [1, 3]. \end{cases}
$$

Then for any $x \in [1, 3]$, x is a coincidence point and $fgx = gfx$, showing that f, g are weakly compatible maps on $[0, 3]$.

We start our work with the following theorem:

Theorem 2.1. *Let f and g be weakly compatible self maps of a G-metric space (X, G) satisfying the following conditions:*

(2.1) $f(X) \subseteq g(X)$,
(2.2) *any one of the subspace $f(X)$ or $g(X)$ is complete,*
(2.3) $G(fx, fy, fy)$

$$
\begin{aligned}
&\le \max\{(aG(gx, gy, gy), b[G(gx, fx, fx) + 2G(gy, fy, fy)], \\
&c[G(gx, fy, fy) + G(gy, fy, fy) + G(gy, fx, fx)])\}
\end{aligned}
$$

for all $x, y \in X$ and $0 \le a < 1$, $0 \le b < 1/3$ and $0 \le c < 1/5$.

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (2.1), one can choose a point x_1 in X such that $fx_0 = gx_1$. In general one can choose x_{n+1} such that $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$.

From (2.3), we have

$$
\begin{aligned}
&G(fx_n, fx_{n+1}, fx_{n+1}) \\
&\le \max\{(aG(gx_n, gx_{n+1}, gx_{n+1}), \\
&\quad b[G(gx_n, fx_n, fx_n) + 2G(gx_{n+1}, fx_{n+1}, fx_{n+1})], \\
&\quad c[G(gx_n, fx_{n+1}, fx_{n+1}) + G(gx_{n+1}, fx_{n+1}, fx_{n+1}) \\
&\quad + G(gx_{n+1}, fx_n, fx_n)])\}
\end{aligned}
$$

or

$$
\begin{aligned}
&G(y_n, y_{n+1}, y_{n+1}) \\
&\le \max\{(aG(y_{n-1}, y_n, y_n), \\
&\quad b[G(y_{n-1}, y_n, y_n) + 2G(y_n, y_{n+1}, y_{n+1})], \\
&\quad c[G(y_{n-1}, y_{n+1}, y_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \\
&\quad + G(y_n, y_n, y_n)])\}.
\end{aligned}
$$

Case 1. If,

$$
\begin{aligned}
&\max\{(aG(y_{n-1}, y_n, y_n), \\
&\quad b[G(y_{n-1}, y_n, y_n) + 2G(y_n, y_{n+1}, y_{n+1})], \\
&\quad c[G(y_{n-1}, y_{n+1}, y_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \\
&\quad + G(y_n, y_n, y_n)])\} \\
&\quad = aG(y_{n-1}, y_n, y_n),
\end{aligned}
$$

then, we get

$$
G(y_n, y_{n+1}, y_{n+1}) \le aG(y_{n-1}, y_n, y_n).
$$

Continuing in the same way, we have

$$
G(y_n, y_{n+1}, y_{n+1}) \le a^n G(y_0, y_1, y_1).
$$

Therefore, for all $n, m \in N$, $n < m$, we have by rectangle inequality

$$
\begin{aligned}
&G(y_n, y_m, y_m) \\
&\le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\
&\quad + G(y_{n+2}, y_{n+3}, y_{n+3}) + \ldots + G(y_{m-1}, y_m, y_m) \\
&\le (a^n + a^{n+1} + \ldots + a^{m-1}) G(y_0, y_1, y_1) \\
&\le \frac{a^n}{1-a} G(y_0, y_1, y_1). \qquad (2.4)
\end{aligned}
$$

Letting as $n, m \to \infty$, we have $\lim_{n\to\infty} G(y_n, y_m, y_m) = 0$, as $0 \le a < 1$.

Thus $\{y_n\}$ is a G-Cauchy sequence in X.

Case 2. If

$$
\begin{aligned}
&\max\{(aG(y_{n-1}, y_n, y_n), \\
&\quad b[G(y_{n-1}, y_n, y_n) + 2G(y_n, y_{n+1}, y_{n+1})], \\
&\quad c[G(y_{n-1}, y_{n+1}, y_{n+1}) + G(y_n, y_{n+1}, y_{n+1}) \\
&\quad + G(y_n, y_n, y_n)])\} \\
&\quad = b[G(y_{n-1}, y_n, y_n) + 2G(y_n, y_{n+1}, y_{n+1})].
\end{aligned}
$$

Then, we get $(1-2b)G(y_n,y_{n+1},y_{n+1}) \leq bG(y_{n-1},y_n,y_n)$.

Gives $G(y_n,y_{n+1},y_{n+1}) \leq \frac{b}{1-2b}G(y_{n-1},y_n,y_n)$

i.e.,

$$G(y_n,y_{n+1},y_{n+1}) \leq qG(y_{n-1},y_n,y_n),$$

where $q = \frac{b}{1-2b}$, $q < 1$ as $0 \leq b < \frac{1}{3}$.

Now in view of (2.4), we conclude that the sequence $\{y_n\}$ is a G-Cauchy sequence in X.

Case 3. Finally, If

$$\begin{aligned}
&\max\{(aG(y_{n-1},y_n,y_n),\\
&\quad b[G(y_{n-1},y_n,y_n)+2G(y_n,y_{n+1},y_{n+1})],\\
&\quad c[G(y_{n-1},y_{n+1},y_{n+1})+G(y_n,y_{n+1},y_{n+1})\\
&\quad +G(y_n,y_n,y_n)])\}\\
&\qquad = c[G(y_{n-1},y_{n+1},y_{n+1})+G(y_n,y_{n+1},y_{n+1})]\\
&\qquad \leq c[G(y_{n-1},y_n,y_n)+G(y_n,y_{n+1},y_{n+1})\\
&\qquad\quad +G(y_n,y_{n+1},y_{n+1})].
\end{aligned}$$

Then, we get

$$(1-2c)G(y_n,y_{n+1},y_{n+1}) \leq cG(y_{n-1},y_n,y_n)$$

or

$$G(y_n,y_{n+1},y_{n+1}) \leq \frac{c}{1-2c}G(y_{n-1},y_n,y_n)$$

i.e.,

$$G(y_n,y_{n+1},y_{n+1}) \leq qG(y_{n-1},y_n,y_n),$$

where $q = \frac{c}{1-2c}$, $q < 1$ as $0 \leq c < \frac{1}{5}$.

Consequently in view of (2.4), we conclude that the sequence $\{y_n\}$ is a G-Cauchy sequence in X. Hence in all cases the sequence $\{y_n\}$ is a G-Cauchy sequence in X.

Since either $f(X)$ or $g(X)$ is complete, for definiteness assume that $g(X)$ is complete subspace of X then the subsequence of $\{y_n\}$ must get a limit in $g(X)$. Call it be t. Let $u \in g^{-1}t$. Then $gu = t$, as $\{y_n\}$ is a G-Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also convergent implying thereby the convergence of subsequence of the convergent sequence. Now we show that $fu = t$.

On setting $x = u$, $y = x_n$ in (2.3), we have

$$\begin{aligned}
&G(fu, fx_n, fx_n)\\
&\leq \max\{(aG(gu,gx_n,gx_n),\\
&\quad b[G(gu, fu, fu)+2G(gx_n, fx_n, fx_n)],\\
&\quad c[G(gu, fx_n, fx_n)+G(gx_n, fx_n, fx_n)\\
&\qquad +G(gx_n, fu, fu)])\}.
\end{aligned}$$

Proceeding limit as $n \to \infty$ and in view of Proposition 1.6, we have

$$G(fu,t,t) \leq (b+c)G(t, fu, fu) \leq 2(b+c)G(fu,t,t)$$

which is a contradiction, as $2(b+c) < 1$. Hence $fu = gu = t$. Thus u is a coincident point of f and g. Since f and g are weakly compatible, it follows that $fgu = gfu$, i.e., $ft = gt$.

We now show that $ft = t$. Suppose that $ft \neq t$, therefore $G(ft,t,t) > 0$.

From (2.3), on setting $x = t$, $y = u$, we have

$$\begin{aligned}
&G(ft, fu, fu)\\
&\leq \max\{(aG(gt,gu,gu), b[G(gt, ft, ft)+2G(gu, fu, fu)],\\
&\quad c[G(gt, fu, fu)+G(gu, fu, fu)+G(gu, ft, ft)])\}.
\end{aligned}$$

i.e.,

$G(ft,t,t) \leq \max\{aG(ft,t,t), c(G(ft,t,t)+G(t, ft, ft))\}$.

Now in both cases $G(ft,t,t) \leq qG(ft,t,t)$, where $0 \leq q < 1$, which is a contradiction, as $0 \leq a < 1$ and $0 < c < \frac{1}{5}$ which in turn implies that $ft = t = gt$.

In particular, we can say that t is common fixed point of f and g. Uniqueness follows easily.

Example 2.2. Let $X = [0,1]$ with the G-metric defined as follows:

$$G(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\}, \text{ for all } x,y,z \in X.$$

Define: $f(x) = \frac{x}{7}$ and $g(x) = \frac{x}{2}$ for all $x \in X$.

Then (X,G) is G-metric space and $f(X) \subseteq g(X)$. Moreover, f and g has 0 as unique common fixed point and satisfy all the conditions of the Theorem 2.1, for $a = \frac{3}{4}$, $b = \frac{1}{4}$ and $c = \frac{1}{8}$.

Corollary 2.3. *Let f and g be weakly compatible self maps of a G-metric space (X,G) satisfying (2.1) and (2.2) and the following condition:*

$$\begin{aligned}
&G(f^m(x), f^m(y), f^m(y))\\
&\leq \max\{(aG(g^m(x),g^m(y),g^m(y)),\\
&\quad b[G(g^m(x), f^m(x), f^m(x))+2G(g^m(y), f^m(y), f^m(y))],\\
&\quad c[G(g^m(x), f^m(y), f^m(y))+\\
&\quad G(g^m(y), f^m(y), f^m(y))+G(g^m(y), f^m(x), f^m(x))])\},
\end{aligned}$$

for all $x,y \in X$ and $0 \leq a < 1$, $0 \leq b < \frac{1}{3}$ and $0 \leq c < \frac{1}{5}$.

Then f and g have a unique common fixed point in X.

Proof. From Theorem 2.1, we have f^m and g^m have unique fixed point (say t), that is $f^m(t) = t = g^m(t)$. But $f(t) = f(f^m(t)) = f^{(m+1)}(t) = f^m(f(t))$, so $f(t)$ is another fixed point of f^m and by uniqueness $f(t) = t$ and similarly $g(t) = t$. Hence the result follows.

Theorem 2.4. *Let f and g be weakly compatible self maps of a G-metric space (X,G) satisfying (2.1) and (2.2) and* *the following condition:*

$$
\begin{aligned}
&G(fx,fy,fz)\\
&\le \max\{aG(gx,gy,gz), b[G(gx,fx,fx)\\
&\quad +G(gy,fy,fy)+G(gz,fz,fz)],\\
&\quad c[G(gx,fy,fy)+G(gy,fz,fz)+G(gz,fx,fx)]\},
\end{aligned}
$$

for all $x,y\in X$ *and* $0\le a<1$, $0\le b<\frac{1}{3}$ *and* $0\le c<\frac{1}{5}$.

Then f *and* g *have a unique common fixed point in* X.

*Proof.*Taking $z=y$ in condition (2.3) and result follows from Theorem 2.1.

Example 2.5. Let $X=[0,1]$ with the G-metric defined as follows:

$$G(x,y,z)=|x-y|+|y-z|+|x-z|,\ \text{for all } x,y,z\in X.$$

Define: $f(x)=\frac{x}{10}$ and $g(x)=\frac{x}{2}$ for all $x\in X$.

Then (X,G) is G-metric space and $f(X)\subseteq g(X)$. Moreover, f and g has 0 as unique common fixed point and satisfy all the conditions of the Theorem 2.4, for $a=\frac{4}{5}$, $b=\frac{1}{5}$ and $c=\frac{1}{5}$.

Theorem 2.6. *Let* f *and* g *be weakly compatible self maps of a G-metric space* (X,G) *satisfying (2.1) and (2.2) and the following condition:*

$$
\begin{aligned}
&G(fx,fy,fy)\\
&\le k\max\{([G(gx,fx,fx)+2G(gy,fy,fy)],\\
&\quad [G(gx,fy,fy)+G(gy,fy,fy)+G(gy,fx,fx)],\\
&\quad [G(gy,gy,fx)+G(gy,gy,fy)+G(gx,gx,fy)])\}
\end{aligned} \tag{2.5}
$$

for all $x,y\in X$ *and* $0\le k<\frac{1}{9}$.

Then f *and* g *have a unique common fixed point in* X.

*Proof.*Let x_0 be an arbitrary point in X. By (2.1), one can choose a point x_1 in X such that $fx_0=gx_1$. In general one can choose x_{n+1} such that $y_n=fx_n=gx_{n+1}$, $n=0,1,2,\ldots$.

From (2.5), we have

$$
\begin{aligned}
&G(fx_n,fx_{n+1},fx_{n+1})\\
&\le k\max\{([G(gx_n,fx_n,fx_n)+2G(gx_{n+1},fx_{n+1},fx_{n+1})],\\
&\quad [G(gx_n,fx_{n+1},fx_{n+1})+G(gx_{n+1},fx_{n+1},fx_{n+1})\\
&\quad +G(gx_{n+1},fx_n,fx_n)],[G(gx_{n+1},gx_{n+1},fx_n)\\
&\quad +G(gx_{n+1},gx_{n+1},fx_{n+1})+G(gx_n,gx_n,fx_{n+1})])\}
\end{aligned}
$$

or

$$
\begin{aligned}
&G(y_n,y_{n+1},y_{n+1})\\
&\le k\max\{([G(y_{n-1},y_n,y_n)+2G(y_n,y_{n+1},y_{n+1})],\\
&\quad [G(y_{n-1},y_{n+1},y_{n+1})+G(y_n,y_{n+1},y_{n+1})\\
&\quad +G(y_n,y_n,y_n)],[G(y_n,y_n,y_n)+G(y_n,y_n,y_{n+1})\\
&\quad +G(y_{n-1},y_{n-1},y_{n+1})])\}
\end{aligned}
$$

By using Proposition 1.6, we have

$$
\begin{aligned}
&G(y_n,y_{n+1},y_{n+1})\\
&\le k\max\{([G(y_{n-1},y_{n+1},y_{n+1})+G(y_{n+1},y_n,y_n)\\
&\quad +2G(y_n,y_{n+1},y_{n+1})],[G(y_{n-1},y_{n+1},y_{n+1})\\
&\quad +G(y_n,y_{n+1},y_{n+1})],2[G(y_n,y_{n+1},y_{n+1})\\
&\quad +G(y_{n-1},y_{n+1},y_{n+1})]])\}\\
&\le k\max\{([G(y_{n-1},y_{n+1},y_{n+1})+4G(y_n,y_{n+1},y_{n+1})],\\
&\quad 2[G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_{n+1},y_{n+1})]])\}
\end{aligned}
$$

Case 1. If

$$
\begin{aligned}
&\max\{([G(y_{n-1},y_{n+1},y_{n+1})+4G(y_n,y_{n+1},y_{n+1})],\\
&2[G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_{n+1},y_{n+1})]])\}\\
&=[G(y_{n-1},y_{n+1},y_{n+1})+4G(y_n,y_{n+1},y_{n+1})].
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
&[1-4k]G(y_n,y_{n+1},y_{n+1})\\
&\quad \le kG(y_{n-1},y_{n+1},y_{n+1})\\
&\quad \le k[G(y_{n-1},y_n,y_n)+G(y_n,y_{n+1},y_{n+1})].
\end{aligned}
$$

or

$$G(y_n,y_{n+1},y_{n+1})\le k/(1-5k)G(y_{n-1},y_n,y_n),$$

i.e.,

$$G(y_n,y_{n+1},y_{n+1})\le qG(y_{n-1},y_n,y_n),$$

where $q=\frac{k}{1-5k}$, $q<1$ as $0\le k<\frac{1}{9}$.

Now using (2.4) we conclude that $\{y_n\}$ is a G-Cauchy sequence in X.

Case 2. If

$$
\begin{aligned}
&\max\{([G(y_{n-1},y_{n+1},y_{n+1})+4G(y_n,y_{n+1},y_{n+1})],\\
&\quad 2[G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_{n+1},y_{n+1})]])\}\\
&=2[G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_{n+1},y_{n+1})].
\end{aligned}
$$

Then by using Proposition 1.6, we have

$$
\begin{aligned}
&[1-2k]G(y_n,y_{n+1},y_{n+1})\\
&\quad \le 2kG(y_{n-1},y_{n+1},y_{n+1})\\
&\quad \le 2k[(y_{n-1},y_n,y_n)+G(y_n,y_{n+1},y_{n+1})],
\end{aligned}
$$

yielding that

$$G(y_n,y_{n+1},y_{n+1})\le 2k/(1-4k)G(y_{n-1},y_n,y_n),$$

i.e.,

$$G(y_n,y_{n+1},y_{n+1})\le qG(y_{n-1},y_n,y_n),$$

where $q=\frac{2k}{1-4k}$, $q<1$ as $0\le k<\frac{1}{9}$.

In view of (2.4), $\{y_n\}$ is a G-Cauchy sequence in X.

Hence in all cases the sequence $\{y_n\}$ is a G-Cauchy sequence in X.

Since either $f(X)$ or $g(X)$ is complete, for definiteness assume that $g(X)$ is complete subspace of X then the subsequence of $\{y_n\}$ must get a limit in $g(X)$. Call it be t. Let $u \in g^{-1}t$. Then $gu = t$, as $\{y_n\}$ is a G-Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also convergent implying thereby the convergence of subsequence of the convergent sequence. Now we show that $fu = t$.

On setting $x = u$, $y = x_n$ in (2.5), we have

$$\begin{aligned}
&G(fu, fx_n, fx_n)\\
&\le k\max\{([G(gu, fu, fu) + 2G(gx_n, fx_n, fx_n)],\\
&\quad [G(gu, fx_n, fx_n) + G(gx_n, fx_n, fx_n) + G(gx_n, fu, fu)],\\
&\quad [G(gx_n, gx_n, fu) + G(gx_n, gx_n, fx_n) + G(gu, gu, fx_n)])\}.
\end{aligned}$$

Proceeding limit as $n \to \infty$ and in view of Proposition 1.6, we have

$$G(fu, gu, gu) \le 2kG(fu, gu, gu),$$

which is a contradiction, as $k < \frac{1}{9}$.

Hence $fu = gu = t$. Thus u is a coincident point of f and g. Since f and g are weakly compatible, it follows that $fgu = gfu$, i.e., $ft = gt$.

We now show that $ft = t$. Suppose that $ft \neq t$, therefore $G(ft, t, t) > 0$.

From (2.5), on setting $x = t$, $y = u$, we have

$$\begin{aligned}
&G(ft, fu, fu)\\
&\quad \le k\max\{([G(gt, ft, ft) + 2G(gu, fu, fu)],\\
&\qquad [G(gt, fu, fu) + G(gu, fu, fu) + G(gu, ft, ft)],\\
&\qquad [G(gu, gu, ft) + G(gu, gu, fu) + G(gt, gt, fu)])\}\\
&\quad = k\max\{[G(ft, fu, fu) + G(fu, ft, ft)],\\
&\qquad [G(fu, fu, ft) + G(ft, ft, fu)]\}.
\end{aligned}$$

Then by using Proposition 1.6, we have

$$G(ft, t, t) \le kG(ft, t, t) + 2G(ft, t, t) = 3kG(ft, t, t),$$

which is a contradiction, as $k < \frac{1}{9}$, yielding that $ft = t = gt$.

i.e., t is common fixed point of f and g. Uniqueness follows easily.

Corollary 2.7. *Let f and g be weakly compatible self maps of a G-metric space (X, G) satisfying (2.1) and (2.2) and the following condition:*

$$\begin{aligned}
&G(f^m(x), f^m(y), f^m(y))\\
&\le k\max\{([G(g^m(x), f^m(x), f^m(x))\\
&\quad + 2G(g^m(y), f^m(y), f^m(y))],\\
&\quad [G(g^m(x), f^m(y), f^m(y)) + G(g^m(y), f^m(y), f^m(y))\\
&\quad + G(g^m(y), f^m(x), f^m(x))], [G(g^m(y), g^m(y), f^m(x))\\
&\quad + G(g^m(y), g^m(y), f^m(y)) + G(g^m(x), g^m(x), f^m(y))])\}
\end{aligned}$$

for all $x, y \in X$ and $0 \le k < \frac{1}{9}$.

Then f and g have a unique common fixed point in X.

*Proof.*The proof follows from Theorem 2.6 and the result follows similarly as in Corollary 2.3.

Theorem 2.8. *Let f and g be weakly compatible self maps of a G-metric space (X, G) satisfying (2.1) and (2.2) and the following condition:*

$$\begin{aligned}
&G(fx, fy, fz)\\
&\le k\max\{[G(gx, fx, fx) + G(gy, fy, fy) + G(gz, fz, fz)],\\
&\quad [G(gx, fy, fy) + G(gy, fz, fz) + G(gy, fx, fx)],\\
&\quad [G(gy, gy, fx) + G(gz, gz, fy) + G(gx, gx, fz)]\}
\end{aligned}$$

for all $x, y \in X$ and $0 \le k < \frac{1}{9}$.

Then f and g have a unique common fixed point in X.

*Proof.*Taking $z = y$ in condition (2.5) and result follows from Theorem 2.6.

3 (CLRg) property in G-metric spaces

Recently, Sintunavarat and Kumam [12] introduced a new property which is so called "Common Limit in the Range of g property" (i.e., (CLRg) property). (CLRg) property relaxes the condition of closeness of range of mappings.

Definition 3.1. ([12]) Suppose that (X, d) is a metric space and $f, g : X \to X$. Two mappings f and g are said to satisfy the common limit in the range of g property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \text{ for some } x \in X.$$

The common limit in the range of g property will be denoted by the (CLRg) property.

Example 3.1. ([12]) Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \to X$ by $fx = x + 1$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{1 + \frac{1}{n}\}$. Since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = g_1 = 2,$$

therefore f and g satisfy the (CLRg) property.

Example 3.2. ([12])Let $X = [0,\infty)$ be the usual metric space. Define $f,g : X \to X$ by $fx = \frac{x}{4}$ and $gx = \frac{3x}{4}$ for all $x \in X$. Consider the sequence $\{x_n\} = \{\frac{1}{n}\}$. Since

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = g_0 = 0,$$

therefore f and g satisfy the (CLRg) property.

In similar mode, we use (CLRg) property in G-metric spaces.

Theorem 3.1. *Let f and g be weakly compatible self maps of a G-metric space (X,G) satisfying condition (2.3). Then f and g have a unique common fixed point in X provided f and g satisfy the (CLRg) property.*

*Proof.*Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu \text{ for some } u \in X.$$

First we show that $gu = fu$.

Now on setting $x = u$, $y = x_n$ in (2.3), we have

$$\begin{aligned}
&G(fu, fx_n, fx_n)\\
&\leq \max\{aG(gu,gx_n,gx_n), b[G(gu, fu, fu)\\
&\quad + 2G(gx_n, fx_n, fx_n)], c[G(gu, fx_n, fx_n)\\
&\quad + G(gx_n, fx_n, fx_n) + G(gx_n, fu, fu)]\}.
\end{aligned}$$

Proceeding limit as $n \to \infty$ and in view of Proposition 1.6, we have

$$\begin{aligned}
&G(fu,gu,gu)\\
&\leq \max\{aG(gu,gu,gu), b[G(gu, fu, fu) + 2G(gu,gu,gu)],\\
&c[G(gu,gu,gu) + G(gu,gu,gu) + G(gu, fu, fu)]\}\\
&= 2(b+c)G(fu,gu,gu),
\end{aligned}$$

which is a contradiction, as $2(b+c) < 1$.

Hence $fu = gu = t$ (say). Thus u is a coincident point of f and g and the pair (f,g) is weakly compatible, it follows that $fgu = gfu$, i.e., $ft = gt$.

We now show that $ft = t$. Suppose that $ft \neq t$, therefore $G(ft,t,t) > 0$.

From (2.3), on setting $x = t$, $y = u$, we have,

$$\begin{aligned}
&G(ft, fu, fu)\\
&\leq \max\{aG(gt,gu,gu), b[G(gt, ft, ft) + 2G(gu, fu, fu)],\\
&c[G(gt, fu, fu) + G(gu, fu, fu) + G(gu, ft, ft)]\}\\
&= \max\{aG(ft, fu, fu), c[G(ft, fu, fu) + G(fu, ft, ft)]\}.
\end{aligned}$$

In view of Proposition 1.6, we have

$$G(ft,t,t) \leq \max\{aG(ft,t,t), 2cG(ft,t,t)\}.$$

In both cases $G(ft,t,t) \leq qG(ft,t,t)$, where $0 \leq q < 1$ as $0 \leq a < 1$, and $0 \leq c < \frac{1}{5}$, a contradiction, yielding that $ft = t = gt$.

i.e., t is common fixed point of f and g. Uniqueness follows easily.

Theorem 3.2. *Let f and g be weakly compatible self maps of a G-metric space (X,G) satisfying condition (2.5). Then f and g have a unique common fixed point in X provided f and g satisfy the (CLRg) property.*

*Proof.*Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu \text{ for some } u \in X.$$

First we show that $gu = fu$.

Now on setting $x = u$, $y = x_n$ in (2.5), we have

$$\begin{aligned}
&G(fu, fx_n, fx_n)\\
&\leq k\max\{[G(gu, fu, fu) + 2G(gx_n, fx_n, fx_n)],\\
&\quad [G(gu, fx_n, fx_n) + G(gx_n, fx_n, fx_n)\\
&\qquad + G(gx_n, fu, fu)], [G(gx_n,gx_n, fu)\\
&\qquad + G(gx_n,gx_n, fx_n) + G(gu,gu, fx_n)]\}.
\end{aligned}$$

Proceeding limit as $n \to \infty$ and in view of Proposition 1.6, we have

$$\begin{aligned}
&G(fu,gu,gu)\\
&\leq k\max\{G(gu, fu, fu), G(gu,gu, fu)\}\\
&= k\max\{2G(fu,gu,gu), G(fu,gu,gu)\}\\
&= 2kG(fu,gu,gu),
\end{aligned}$$

which is a contradiction, as $k < \frac{1}{9}$, yielding that $fu = gu = t$ (say).

Thus u is a coincident point of f and g. Since f and g are weakly compatible, it follows that $fgu = gfu$, i.e., $ft = gt$.

We now show that $ft = t$. Suppose that $ft \neq t$, therefore $G(ft,t,t) > 0$.

From (2.5), on setting $x = t$, $y = u$, we have

$$\begin{aligned}
&G(ft, fu, fu)\\
&\leq k\max\{[G(gt, ft, ft) + 2G(gu, fu, fu)],\\
&\quad [G(gt, fu, fu) + G(gu, fu, fu) + G(gu, ft, ft)],\\
&\quad [G(gu,gu, ft) + G(gu,gu, fu) + G(gt,gt, fu)]\}.
\end{aligned}$$

In view of Proposition 1.6, we have

$$G(ft,t,t) \leq k\{G(ft,t,t) + 2G(ft,t,t)\} = 3kG(ft,t,t).$$

Which is a contradiction, as $k < \frac{1}{9}$, yielding that $ft = t = gt$.

i.e., t is common fixed point of f and g. Uniqueness follows easily.

Remark 3.1. "Common limit in the range" property (i.e. (CLRg) property) does not require condition of closeness of range and so our theorems generalize, unify and extend many results in literature.

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