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Analysis and Control of a Three-Dimensional Autonomous Chaotic System

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Abstract: In this paper, a novel three-dimensional (3D) autonomous chaotic system is investigated, which displays complicated dynamical behaviors. Basic dynamical properties are analyzed by means of phase portraits and equilibria. Also, an optimal control law is designed for the novel chaotic system, based on the Pontryagin minimum principle (PMP). Furthermore, an adaptive and feedback control law is introduced to stabilize the new chaotic system with unknown parameters. The adaptive control results are established using the Lyapunov stability theory. Numerical simulations are included to demonstrate the efficiency and high accuracy of the proposed method.

Keywords: Autonomous chaotic system, Optimal control, Pontryagin's minimum principle (PMP), Spectral methods, Adaptive and feedback control, Lyaponuv stability theory.

1 Introduction

Chaos is an interesting phenomenon in nonlinear system dynamics, which has theoretical and practical applications in many disciplines of laser [1], nonlinear circuit [2], power systems [3], etc. The chaotic systems are dynamic systems described by nonlinear differential equations, which are strongly sensitive to the initial conditions [4]. This means that even if the system mathematical description is deterministic, its behavior is still unpredictable. In 1963, Lorenz [5] discovered a simple three-dimensional (3D) smooth autonomous chaotic system as the first chaotic model. The dynamic properties of Lorenz system are well investigated in many papers and monographs. Later, many Lorenz-like chaotic systems were reported and analyzed, such as Rössler systems [6], Chen system [7], Lü system [8], Liu systems [9], and so on. Notice that, the family of Lorenz systems has two cross-product terms on the right-hand side of governing equations. More recently, Li et al. [10] introduced a new 3D smooth autonomous chaotic system with three cross-product terms. Also, they have analyzed the different dynamic behaviors of the proposed chaotic system, especially when changing each system parameter.

In recent years, the control of chaotic systems has been received more attention due to its potential applications in physics, chemical reactor, biological networks, artificial neural networks, telecommunications, etc [11]. In the case of chaos control, some useful methods have been developed. These include optimal control [12,14], synchronization [13], adaptive control [15], state-feedback control [16], sliding mode control [17], time-delayed feedback control [11], etc. Sarkar and Banerjee [18] used a stochastic approach for chaos and optimal control of cancer self-remission and tumor unstable equilibrium states. El-Gohary and Al-Ruzaiza [19] discussed the chaos and adaptive control of three species continuous time prey-predator model. El-Gohary [20] used a feedback control approach for chaos and optimal control of cancer self-remission and tumor unstable equilibrium states. El-Gohary and Alwasel [12] studied the chaos and optimal control of cancer model with complete unknown parameters. They have also discussed the stability analysis of the biologically feasible steady-states of this model. More recently, Sundarapandian and Pehlivan [21] designed an adaptive control law to stabilize a novel 3D chaotic system with unknown parameters.

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Most of chaotic dynamical systems do not have exact analytic solutions, so approximation and numerical techniques must be used. Numerical methods can be used for finding explicit expressions for the orbits, simulating dynamical systems and computing their Lyapunov Characteristic Exponents (LCE). The class of solution methods based on orthogonal polynomials have become known as spectral methods. Spectral methods are one of the principal methods of discretization for the numerical solution of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns. The three most widely used spectral versions are the Galerkin, collocation, and tau methods [22,23]. Collocation methods [22,23,24] have become increasingly popular for solving differential equations, also we can apply the method in the search of limit cycles and isolated cycles emerging from a Hopf bifurcation.

In this paper, some basic dynamical characters of the present chaotic system are investigated by means of phase portraits and equilibria. Also, an optimal control law is designed for the novel chaotic system, based on the Pontryagin's minimum principle (PMP) [25]. Furthermore, an adaptive and feedback control law is introduced to stabilize the new chaotic system with unknown parameters. The adaptive control results derived in this paper are established using the Lyapunov stability theory [26].

The rest of this paper is as follows. Section 2 introduces and analyzes the novel 3D chaotic system. Section 3 discusses the problem of optimal control for the novel chaotic system. In Section 4, an adaptive control law is designed to stabilize the new chaotic system with unknown parameters. Finally, conclusions are given in the last section.

2 Analysis of the novel 3D chaotic system

The novel chaotic system is described by the following autonomous nonlinear system of ordinary differential equations (ODEs) [10]:

$$\begin{cases} \dot{x}_1 = -ax_1 + fx_2x_3, \\ \dot{x}_2 = cx_2 - dx_1x_3, \\ \dot{x}_3 = -bx_3 + ex_2^2, \end{cases} \tag{1}$$

where x_1, x_2, and x_3 are the state variables, and $a, b, c, d, e,$ and f are positive constant parameters. Note that, the new system (1) consists of two quadratic cross-product terms and a square term.

2.1 Chaotic phase portraits and time responses

When $a = 16$, $b = 5$, $c = 10$, $d = 6$, $e = 18$, and $f = 0.5$, the new system (1) is chaotic with the Lyapunov exponents $L_1 = 1.86852 > 0$, $L_2 = 0$, and $L_3 = -17.73664 < 0$. The corresponding time responses and phase portraits are depicted in Figs. 1-2, respectively. It appears from Figs. 1-2 that the novel attractor displays abundantly complicated behaviors of chaotic dynamics. What is more, the attractor resembles the butterfly shape, which is different from that of the Lorenz-like systems or any existing chaotic systems.

2.2 Dissipation and attractor existence

The vector field on the right-hand side of Eq. (1) is defined by:

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{bmatrix} = \begin{bmatrix} -ax_1 + fx_2x_3 \\ cx_2 - dx_1x_3 \\ -bx_3 + ex_2^2 \end{bmatrix}. \tag{2}$$

The divergence of the vector field F is easily calculated as:

$$\nabla.F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} = -a + c - b. \tag{3}$$

A necessary and sufficient condition for system (1) to be dissipative is that the divergence of the vector field F is negative. In view of Eq. (3), it is immediate that system (1) is dissipative if and only if $-a + c - b < 0$. Under this condition, the new system (1) converges exponentially; that is:

$$\frac{dF}{dt} = (-a + c - b)F \Rightarrow F = F_0 e^{(-a+c-b)t}. \tag{4}$$

Thus, a volume element F_0 in the dynamical system (1) is apparently contracted by the flow into a volume element $F_0 e^{(-a+c-b)t}$ in time t. This means that each volume containing the system trajectory shrinks to zero as $t \to \infty$ at an exponential rate, $-a + c - b$. Therefore, all the orbits of dynamical system (1) are ultimately confined to a specific subset of zero volume, and the asymptotic motion of the system (1) settles onto an attractor.

Here, with $a = 16$, $b = 5$, $c = 10$, $d = 6$, $e = 18$ and $f = 0.5$, the exponential contraction rate of the forced dissipative system is calculated as:

$$\frac{dF}{dt} = (-16 + 10 - 5)F = -11F \Rightarrow F = F_0 e^{-11t}. \tag{5}$$

2.3 Equilibria and stability analysis

The new system (1) has three equilibrium points, given by:

$$\begin{cases} E_1(0,0,0), \\ E_2(\frac{ef}{ab}x_{2*}^3, x_{2*}, \frac{e}{b}x_{2*}^2), \\ E_3(-\frac{ef}{ab}x_{2*}^3, -x_{2*}, \frac{e}{b}x_{2*}^2), \end{cases} \tag{6}$$

where

$$x_{2*} = \left(\frac{ab^2c}{de^2f}\right)^{\frac{1}{4}}. \tag{7}$$

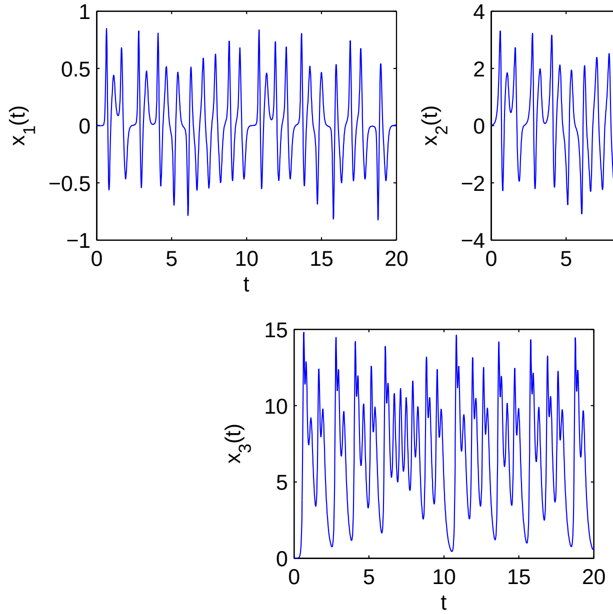

Fig. 1: Time response of the system states with $a = 16$, $b = 5$, $c = 10$, $d = 6$, $e = 18$, and $f = 0.5$.

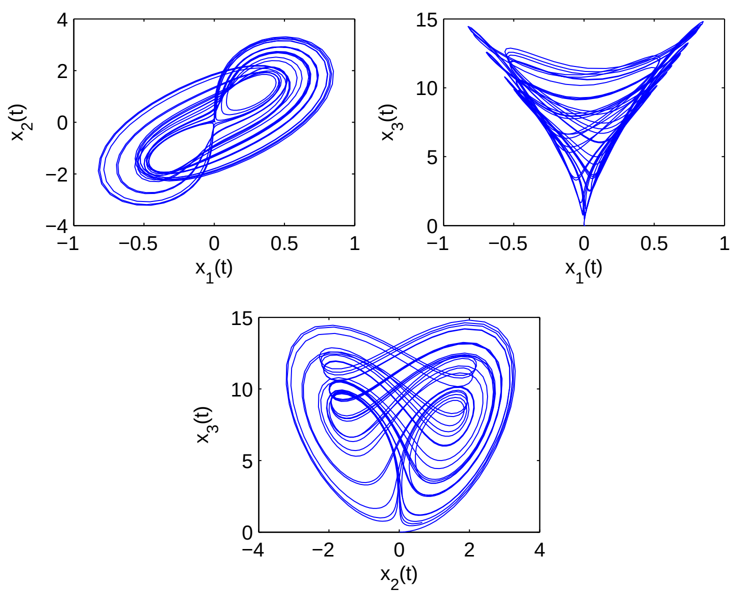

Fig. 2: Phase portraits of the system with $a = 16$, $b = 5$, $c = 10$, $d = 6$, $e = 18$, and $f = 0.5$.

Clearly, E_1 is an equilibrium of the system (1) for all values of the parameters a, b, c, d, e, and f.

Proposition 2.1. The equilibrium points E_1, E_2, and E_3 of system (1) are unstable when $a=16$, $b=5$, $c=10$, $d=6$, $e=18$, and $f=0.5$.

Proof. The Jacobian matrix of system (1) is given by:

$$J=\begin{bmatrix} -a & fx_3 & fx_2 \\ -dx_3 & c & -dx_1 \\ 0 & 2ex_2 & -b \end{bmatrix}. \tag{8}$$

When $a=16, b=5, c=10, d=6, e=18$, and $f=0.5$, system (1) has three equilibrium points, given by:

$$\begin{cases} E_1(0,0,0), \\ E_2(0.325, 1.4243, 7.303), \\ E_3(-0.325, -1.4243, 7.303). \end{cases} \tag{9}$$

The Jacobian matrix for system (1) at equilibrium $E_1(0,0,0)$ is easily obtained as:

$$J(E_1)=\begin{bmatrix} -16 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \tag{10}$$

which has the eigenvalues:

$$\lambda_1^{(1)}=-16,\ \lambda_2^{(1)}=10,\ \lambda_3^{(1)}=-5. \tag{11}$$

The Jacobian matrix for system (1) at equilibrium $E_2(0.325, 1.4243, 7.303)$ is easily obtained as:

$$J(E_2)=\begin{bmatrix} -16 & 3.6515 & 0.7121 \\ -43.8178 & 10 & -1.9503 \\ 0 & 51.2744 & -5 \end{bmatrix}, \tag{12}$$

which has the eigenvalues:

$$\lambda_1^{(2)}=-15.7009,\ \lambda_2^{(2)}=2.3505+14.0814i,$$
$$\lambda_3^{(2)}=2.3505-14.0814i. \tag{13}$$

The Jacobian matrix for system (1) at equilibrium $E_3(-0.325, -1.4243, 7.303)$ is easily obtained as:

$$J(E_3)=\begin{bmatrix} -16 & 3.6515 & -0.7121 \\ -43.8178 & 10 & 1.9503 \\ 0 & -51.2744 & -5 \end{bmatrix}, \tag{14}$$

which has the eigenvalues:

$$\lambda_1^{(3)}=-15.7009,\ \lambda_2^{(3)}=2.3505+14.0814i,$$
$$\lambda_3^{(3)}=2.3505-14.0814i. \tag{15}$$

Since the Jacobian matrices $J(E_1)$, $J(E_2)$, and $J(E_3)$ have eigenvalues with positive real parts, it follows from Lyapunov stability theory [26] that the equilibrium points E_1, E_2, and E_3 are unstable. Thus, the trajectories of system (1) diverge from the three equilibrium points, and the proof is complete. □

3 Optimal control of the novel 3D chaotic system

In this section, we study the optimal control problem of the novel 3D autonomous chaotic system (1). For the purpose of optimal control, we will apply the PMP [25].

3.1 Theoretical results

Let us consider the novel 3D chaotic system (1) to have the form:

$$\begin{cases} \dot{x}_1=-ax_1+fx_2x_3+u_1, \\ \dot{x}_2=cx_2-dx_1x_3+u_2, \\ \dot{x}_3=-bx_3+ex_2^2+u_3, \end{cases} \tag{16}$$

where u_1, u_2, and u_3 are the control inputs, which will be satisfied the optimality conditions, obtained via the PMP. The proposed control strategy is to design the optimal control inputs u_1, u_2, and u_3 such that the state trajectories tend to an unstable equilibrium point in a given finite time interval $[0,t_f]$. Hence, the boundary conditions are considered as:

$$\begin{cases} x_1(0)=x_{0,1},\ x_1(t_f)=\bar{x}_{i,1}, \\ x_2(0)=x_{0,2},\ x_2(t_f)=\bar{x}_{i,2}, \\ x_3(0)=x_{0,3},\ x_3(t_f)=\bar{x}_{i,3}, \end{cases} \tag{17}$$

where $\bar{x}_{i,j}$ $(j=1,2,3)$ denotes the coordinates of equilibrium point E_i $(i=1,2,3)$. In addition, we define the following cost functional, which also penalizes the use of control with large magnitude:

$$J_i=\frac{1}{2}\int_0^{t_f}\sum_{j=1}^{3}(\alpha_j(x_j-\bar{x}_{i,j})^2+\beta_j {u_j}^2)dt, \tag{18}$$

where α_j and β_j $(j=1,2,3)$ are positive constants.

The optimal control problem is to find the control inputs u_1, u_2, and u_3, and the corresponding state trajectories x_1, x_2, and x_3, which minimize the cost functional (18), and satisfy the dynamical system (16) and boundary conditions (17). To solve this problem, we will derive the optimality conditions as a nonlinear two-point boundary value problem (TPBVP) via the PMP. In the following, we shall find it convenient to use the function $\mathscr{H}$, called the Hamiltonian, defined as:

$$\begin{aligned} \mathscr{H} \triangleq & -\frac{1}{2}\sum_{j=1}^{3}(\alpha_j(x_j-\bar{x}_{i,j})^2+\beta_j {u_j}^2) \\ & +\lambda_1[-ax_1+fx_2x_3+u_1] \\ & +\lambda_2[cx_2-dx_1x_3+u_2] \\ & +\lambda_3[-bx_3+ex_2^2+u_3], \end{aligned} \tag{19}$$

where λ_1, λ_2, and λ_3 are the co-state variables. Using this notation, the optimality conditions can be written as follows:

$$\begin{cases} \dot{x}_1=\frac{\partial\mathscr{H}}{\partial\lambda_1}, \\ \dot{x}_2=\frac{\partial\mathscr{H}}{\partial\lambda_2}, \\ \dot{x}_3=\frac{\partial\mathscr{H}}{\partial\lambda_3}, \end{cases} \tag{20}$$

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial \mathscr{H}}{\partial x_1}, \\ \dot{\lambda}_2 = -\frac{\partial \mathscr{H}}{\partial x_2}, \\ \dot{\lambda}_3 = -\frac{\partial \mathscr{H}}{\partial x_3}, \end{cases} \tag{21}$$

$$\begin{cases} \frac{\partial \mathscr{H}}{\partial u_1} = 0, \\ \frac{\partial \mathscr{H}}{\partial u_2} = 0, \\ \frac{\partial \mathscr{H}}{\partial u_3} = 0. \end{cases} \tag{22}$$

Substituting the Hamiltonian function $\mathscr{H}$ from (19) into (20)-(22), the optimality conditions are derived in the form:

$$\begin{cases} \dot{x}_1 = -ax_1 + fx_2x_3 + u_1, \\ \dot{x}_2 = cx_2 - dx_1x_3 + u_2, \\ \dot{x}_3 = -bx_3 + ex_2^2 + u_3, \end{cases} \tag{23}$$

$$\begin{cases} \dot{\lambda}_1 = \alpha_1(x_1 - \overline{x}_{i,1}) + a\lambda_1 + d\lambda_2x_3, \\ \dot{\lambda}_2 = \alpha_2(x_2 - \overline{x}_{i,2}) - f\lambda_1x_3 - c\lambda_2 - 2e\lambda_3x_2, \\ \dot{\lambda}_3 = \alpha_3(x_3 - \overline{x}_{i,3}) - f\lambda_1x_2 + d\lambda_2x_1 + b\lambda_3, \end{cases} \tag{24}$$

$$\begin{cases} \beta_1u_1 + \lambda_1 = 0, \\ \beta_2u_2 + \lambda_2 = 0, \\ \beta_3u_3 + \lambda_3 = 0. \end{cases} \tag{25}$$

Let us solve Eq. (25) to obtain the expressions for ${u_1}^*(t)$, ${u_2}^*(t)$, and ${u_3}^*(t)$; that is:

$$\begin{cases} {u_1}^* = -\frac{\lambda_1}{\beta_1}, \\ {u_2}^* = -\frac{\lambda_2}{\beta_2}, \\ {u_3}^* = -\frac{\lambda_3}{\beta_3}. \end{cases} \tag{26}$$

If these expressions are substituted into Eqs. (23) and (24), we have a set of first order nonlinear ODEs as:

$$\begin{cases} \dot{x}_1 = -ax_1 + fx_2x_3 - \frac{\lambda_1}{\beta_1}, \\ \dot{x}_2 = cx_2 - dx_1x_3 - \frac{\lambda_2}{\beta_2}, \\ \dot{x}_3 = -bx_3 + ex_2^2 - \frac{\lambda_3}{\beta_3}, \\ \dot{\lambda}_1 = \alpha_1(x_1 - \overline{x}_{i,1}) + a\lambda_1 + d\lambda_2x_3, \\ \dot{\lambda}_2 = \alpha_2(x_2 - \overline{x}_{i,2}) - f\lambda_1x_3 - c\lambda_2 - 2e\lambda_3x_2, \\ \dot{\lambda}_3 = \alpha_3(x_3 - \overline{x}_{i,3}) - f\lambda_1x_2 + d\lambda_2x_1 + b\lambda_3. \end{cases} \tag{27}$$

The boundary conditions for these equations are given by Eq. (17). Notice that, as expected, we are confronted by a nonlinear TPBVP. Solving this problem, we can obtain the optimal control law and the optimal state trajectories. In the next section, we will discuss the numerical solution of the above-mentioned nonlinear TPBVP using the MATLAB in-built solver `bvp4c`, which is a finite difference code to solve TPBVPs.

3.2 Numerical results

This section presents the numerical solution of the nonlinear TPBVP (27) with the boundary conditions (17). The results show how the optimal solution is possible for the controlled nonlinear chaotic system (16). In the following numerical simulations, the MATLAB in-built solver `bvp4c` is used, which is a finite difference code to solve TPBVPs. The system parameters are chosen as $a = 16$, $b = 5$, $c = 10$, $d = 6$, $e = 18$, and $f = 0.5$. By these values, the new autonomous chaotic system exhibits a chaotic behavior if no control is applied. Also, the positive constants in the cost functional J_i are chosen as $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\alpha_3 = 0.001$, $\beta_1 = 5$, $\beta_2 = 8$, $\beta_3 = 10$. Figures 3-5 show the optimal control and state trajectories of the new chaotic system for initial states $x_1(0) = 0.05$, $x_2(0) = 0.05$, $x_3(0) = 0.0001$, and $t_f = 0.2$.

4 Adaptive control of the novel 3D autonomous chaotic system

In this section, we obtain the new results for adaptive and feedback control of the novel 3D autonomous chaotic system (1), which is based on the Lyapunov stability theory [26].

4.1 Theoretical results

Let us describe the controlled novel chaotic system by:

$$\begin{cases} \dot{x}_1 = -ax_1 + fx_2x_3 + u_1, \\ \dot{x}_2 = cx_2 - dx_1x_3 + u_2, \\ \dot{x}_3 = -bx_3 + ex_2^2 + u_3, \end{cases} \tag{28}$$

where a, b, c, d, e, and f are now considered to be unknown parameters, and u_1, u_2, and u_3 are the adaptive controllers to be designed.

Theorem 4.1. The novel chaotic system (28) with unknown system parameters is globally and exponentially stabilized for all initial states $(x_1(0), x_2(0), x_3(0)) \in \mathbb{R}^3$ by the adaptive control law:

$$\begin{cases} u_1 = \hat{a}x_1 - \hat{f}x_2x_3 - k_1(x_1 - \overline{x}_{i,1}), \\ u_2 = -\hat{c}x_2 + \hat{d}x_1x_3 - k_2(x_2 - \overline{x}_{i,2}), \\ u_3 = \hat{b}x_3 - \hat{e}x_2^2 - k_3(x_3 - \overline{x}_{i,3}), \end{cases} \tag{29}$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{e}$, and $\hat{f}$ are the estimate values of unknown parameters a, b, c, d, e, and f, respectively, and k_y $(y = 1, 2, 3)$ are positive constants. Moreover, the update law for the estimates of system parameters is given by:

$$\begin{cases} \dot{\hat{a}} = -(x_1 - \overline{x}_{i,1})x_1 + k_4(a - \hat{a}), \\ \dot{\hat{b}} = -(x_3 - \overline{x}_{i,3})x_3 + k_5(b - \hat{b}), \\ \dot{\hat{c}} = (x_2 - \overline{x}_{i,2})x_2 + k_6(c - \hat{c}), \\ \dot{\hat{d}} = -(x_2 - \overline{x}_{i,2})x_1x_3 + k_7(d - \hat{d}), \\ \dot{\hat{e}} = (x_3 - \overline{x}_{i,3})x_2^2 + k_8(e - \hat{e}), \\ \dot{\hat{f}} = (x_1 - \overline{x}_{i,1})x_2x_3 + k_9(f - \hat{f}), \end{cases} \tag{30}$$

where k_y $(y = 4, \ldots, 9)$ are positive constants.

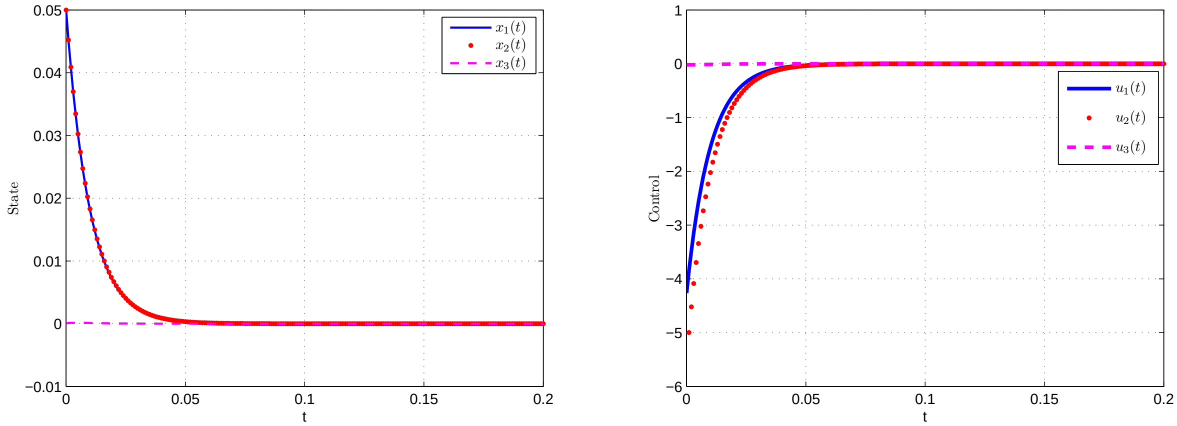

Fig. 3: The behavior of state and control functions for equilibrium point E_1.

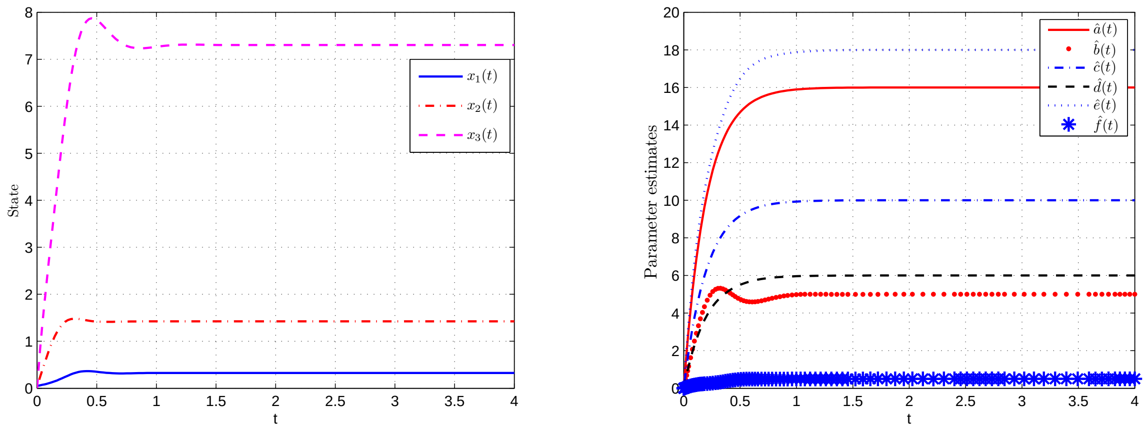

Fig. 4: Time history of the state trajectories and parameter estimates for equilibrium point E_2.

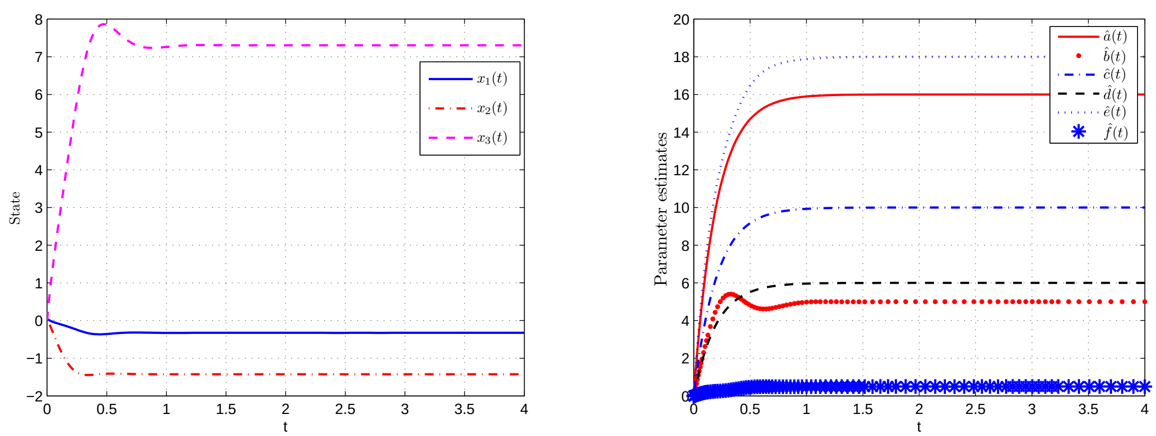

Fig. 5: Time history of the state trajectories and parameter estimates for equilibrium point E_3.

Proof. Substituting (29) into (28), we get the closed-loop system as:

$$\begin{cases} \dot{x}_1 = -(a-\hat{a})x_1 + (f-\hat{f})x_2x_3 - k_1(x_1-\overline{x}_{i,1}), \\ \dot{x}_2 = (c-\hat{c})x_2 - (d-\hat{d})x_1x_3 - k_2(x_2-\overline{x}_{i,2}), \\ \dot{x}_3 = -(b-\hat{b})x_3 + (e-\hat{e})x_2^2 - k_3(x_3-\overline{x}_{i,3}). \end{cases} \tag{31}$$

Let us define the parameter estimation error as:

$$\begin{cases} e_a = a-\hat{a}, \ \ e_b = b-\hat{b}, \ \ e_c = c-\hat{c}, \\ e_d = d-\hat{d}, \ \ e_e = e-\hat{e}, \ \ e_f = f-\hat{f}. \end{cases} \tag{32}$$

Using (32), the closed-loop dynamics (31) can be simplified as:

$$\begin{cases} \dot{x}_1 = -e_a x_1 + e_f x_2x_3 - k_1(x_1-\overline{x}_{i,1}), \\ \dot{x}_2 = e_c x_2 - e_d x_1x_3 - k_2(x_2-\overline{x}_{i,2}), \\ \dot{x}_3 = -e_b x_3 + e_e x_2^2 - k_3(x_3-\overline{x}_{i,3}). \end{cases} \tag{33}$$

For the derivation of the update law for adjusting the parameter estimates, the Lyapunov approach is used.
We consider the quadratic Lyapunov function:

$$V(x_1,x_2,x_3,e_a,e_b,e_c,e_d,e_e,e_f) = \frac{1}{2}((x_1-\overline{x}_{i,1})^2 + (x_2-\overline{x}_{i,2})^2 + (x_3-\overline{x}_{i,3})^2 + {e_a}^2 + {e_b}^2 + {e_c}^2 + {e_d}^2 + {e_e}^2 + {e_f}^2). \tag{34}$$

Note that:

$$\begin{cases} \dot{e_a} = -\dot{\hat{a}}, \ \ \dot{e_b} = -\dot{\hat{b}}, \ \ \dot{e_c} = -\dot{\hat{c}}, \\ \dot{e_d} = -\dot{\hat{d}}, \ \ \dot{e_e} = -\dot{\hat{e}}, \ \ \dot{e_f} = -\dot{\hat{f}}. \end{cases} \tag{35}$$

Differentiating V along the trajectories of (33), and using (35), we obtain:

$$\begin{aligned} \dot{V} &= -k_1(x_1-\overline{x}_{i,1})^2 - k_2(x_2-\overline{x}_{i,2})^2 - k_3(x_3-\overline{x}_{i,3})^2 \\ &\quad + e_a(-(x_1-\overline{x}_{i,1}) - \dot{\hat{a}}) + e_b(-x_3(x_3-\overline{x}_{i,3}) - \dot{\hat{b}}) \\ &\quad + e_c(x_2(x_2-\overline{x}_{i,2}) - \dot{\hat{c}}) + e_d(-x_1x_3(x_2-\overline{x}_{i,2}) - \dot{\hat{d}}) \\ &\quad + e_e({x_2}^2(x_3-\overline{x}_{i,3}) - \dot{\hat{e}}) + e_f(x_2x_3(x_1-\overline{x}_{i,1}) - \dot{\hat{f}}). \end{aligned} \tag{36}$$

Substituting (30) into (36), the time derivative of the Lyapunov function becomes:

$$\begin{aligned} \dot{V} &= -k_1(x_1-\overline{x}_{i,1})^2 - k_2(x_2-\overline{x}_{i,2})^2 - k_3(x_3-\overline{x}_{i,3})^2 \\ &\quad -k_4e_a^2 - k_5e_b^2 - k_6e_c^2 - k_7e_d^2 - k_8e_e^2 - k_9e_f^2. \end{aligned} \tag{37}$$

Since the Lyapunov function V in (34) is a positive definite function on $\mathbb{R}^9$ and $\dot{V}$ in (37) is a negative definite function on $\mathbb{R}^9$, according to the Lyapunov stability theory [26], it follows that:

$$\begin{cases} x_1(t) \to \overline{x}_{i,1}, \ \ x_2(t) \to \overline{x}_{i,2}, \ \ x_3(t) \to \overline{x}_{i,3}, \\ e_a \to 0, \ \ e_b \to 0, \ \ e_c \to 0, \ \ e_d \to 0, \ \ e_e \to 0, \ \ e_f \to 0, \end{cases} \tag{38}$$

exponentially as $t \to \infty$. This completes the proof. □

4.2 Numerical results

In this section, we consider the controlled novel chaotic system (28) with the adaptive control law (29) and the parameter update law (30). In the following numerical simulations, the MATLAB in-built solver `ode45` is used to solve the present initial value problem. The initial states and initial values of the parameter estimates are selected as $x_1(0) = 0.05$, $x_2(0) = 0.05$, $x_3(0) = 0.0001$, $\hat{a}(0) = 0$, $\hat{b}(0) = 0$, $\hat{c}(0) = 0$, $\hat{d}(0) = 0$, $\hat{e}(0) = 0$, $\hat{f}(0) = 0$. For the adaptive and update laws, we take $k_y = 5$ for $y = 1,\ldots,9$. Simulation results are depicted in Figs. 6-8. These figures show that the state trajectories of the controlled chaotic system (28) converges to E_i ($i = 1,2,3$) exponentially with time. Also, these figures demonstrate that the parameter estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t), \hat{e}(t)$, and $\hat{f}(t)$ converge to the system parameter values $a = 16$, $b = 5$, $c = 10$, $d = 6$, $e = 18$, and $f = 0.5$ exponentially with time.

5 Conclusion

This paper analyzed the basic properties of a novel 3D autonomous chaotic system by means of phase portraits and equilibria. Also, an optimal control law was designed for the novel chaotic system, based on the PMP. Furthermore, an adaptive and feedback control law was introduced to stabilize the new chaotic system with unknown parameters. Simulation results not only demonstrate the efficiency and high accuracy of the suggested approach, but also indicate its effectiveness in practical use.

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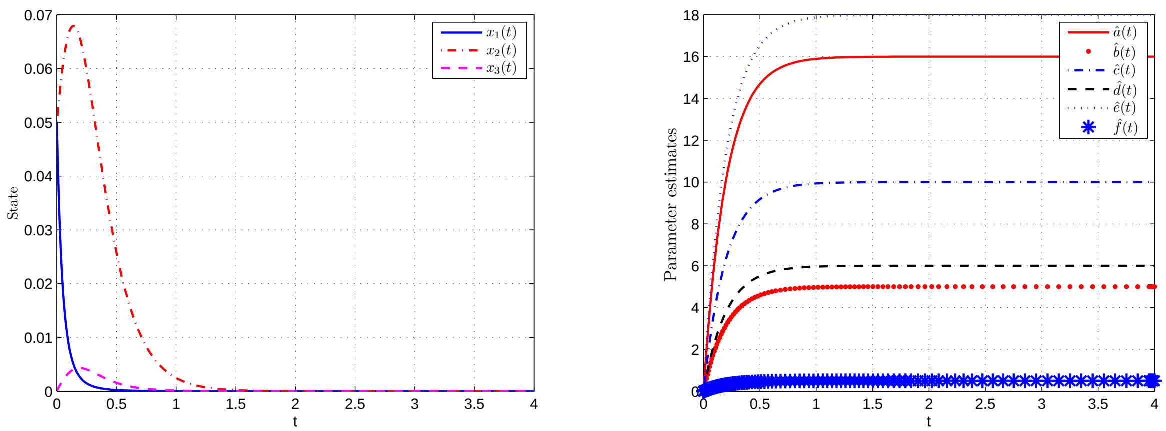

Fig. 6: Time history of the state trajectories and parameter estimates for equilibrium point E_1.

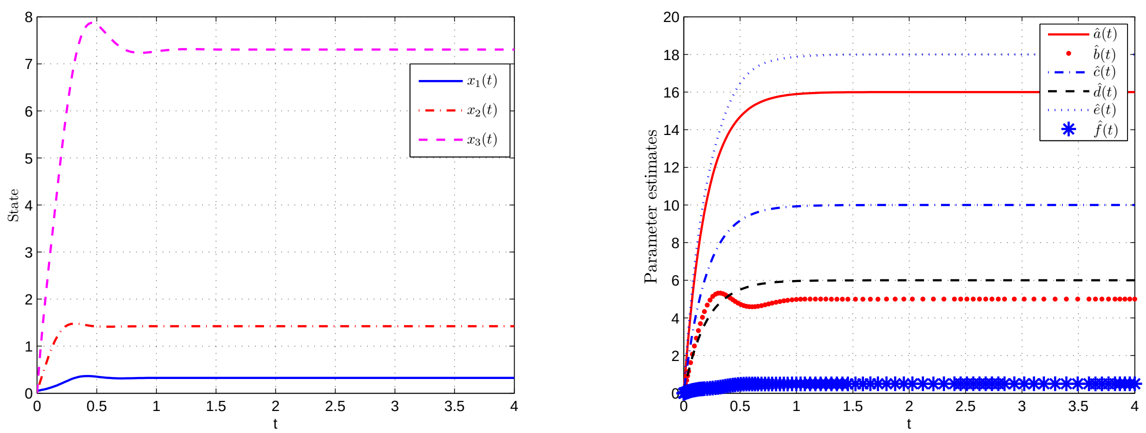

Fig. 7: Time history of the state trajectories and parameter estimates for equilibrium point E_2.

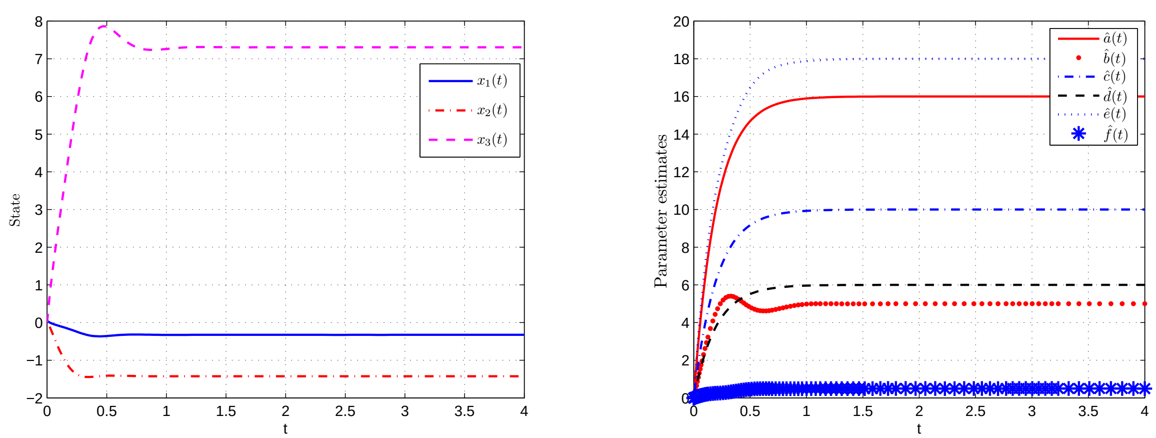

Fig. 8: Time history of the state trajectories and parameter estimates for equilibrium point E_3.

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