Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.12785/sjm/020202

Approximate Analytical Solution of ZK-BBM Equation

Orkun Tasbozan and Ali Kurt*

Department of Mathematics, Faculty of Science and Art, Mustafa Kemal University, Hatay, 31000, Turkey.

Received: 27 Oct. 2014, Revised: 25 Jan. 2015, Accepted: 28 Jan. 2015
Published online: 1 May 2015

Abstract: In this paper,we handle ZK-BBM equation and to obtain its approximate analytical solutions, we use the homotopy analysis method (HAM), whose solution includes an auxiliary parameter $\hbar$. This parameter provides a suitable way for setting and controlling the convergence region of solution series. So we investigate a suitable choice of the auxiliary parameter in the model problem.

Keywords: Homotopy analysis method; Approximate analytical solution; ZK-BBM equation

1 Introduction

Mathematical modeling of many physical events in various fields of physics and engineering generally lead to nonlinear ordinary or partial differential equations. As it is well known that searching and finding exact and numerical solutions of these equations are very important in applied mathematics. The HAM which is a powerful tool for investigating the approximate solutions of nonlinear evolution equations (NLEEs) was first proposed by Liao [1,2] . Differently from perturbation techniques, the HAM is not restricted to any small physical parameters in the considered equation. For this reason, the HAM can overcome the preceding restrictions and limitations of perturbation techniques so that it provides us a powerful tool to analyze nonlinear problems[3]. Many authors have been applying this method successfully to solve several nonlinear problems arising in science and engineering[1-18] and the references therein. In this paper, we apply the HAM to the ZK-BBM equation.

2 Fundamentals of the HAM

In the manuscript, HAM has been applied to the treated problem.Let us consider the following differential equation to give fundamentals of the method,

$$\mathscr{N}[u(x,t)] = 0$$

where $\mathscr{N}$ is a nonlinear operator, x and t denote independent variables, $u(x,t)$ is an unknown function. Using generalization of the HAM, Liao [1,2] has formulated zero-order deformation equation

$$(1-p)\mathscr{L}[\phi(x,t;p) - u_0(x,t)] = p\hbar\mathscr{N}[\phi(x,t;p)] \quad (1)$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $\mathscr{L}$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$, $\phi(x,t;p)$ is an unknown function, respectively. Obviously, we have great freedom to choose auxiliary things in HAM. If we choose $p = 0$ and $p = 1$ then we obtain

$$\phi(x,t;0) = u_0(x,t), \phi(x,t;1) = u(x,t)$$

respectively. So, as the embedding parameter p increases from 0 to 1, the solutions $\phi(x,t;p)$ differ from the initial value $u_0(x,t)$ to the solution $u(x,t)$. If $\phi(x,t;p)$ is expanded in Taylor series with respect to the embedding parameter p, we get

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)p^m$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \quad (2)$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are chosen in a suitable way, the series which are denoted above, converges at $p = 1$, and we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)$$

* Corresponding author e-mail: orkun.tasbozan@gmail.com

which must be one of the solutions of the original nonlinear equation, as proved by Liao [2,5]. According to (2), the governing equation can be reduced from the zero-order deformation equation (1). Define the vector

$$\mathbf{u}_n = \{u_0(x,t), u_1(x,t), ..., u_n(x,t)\}.$$

If we differentiate Eq. (1) m times with respect to the embedding parameter p and then get $p = 0$ and divide by $m!$, we obtain the mth-order deformation equation

$$\mathscr{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\mathbf{u}_{m-1}) \tag{3}$$

where

$$R_m(\mathbf{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_{p=0}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Finally, we emphasize that $u_m(x,t)$ for $m \ge 1$ is governed by the Eq. (3) with the boundary condition which comes from problem. It can be easily solved using symbolic computation software such as Mathematica.

3 Application of the HAM

We first consider the ZK-BBM equation [19] in the following form

$$u_t + u_x + a(u^2)_x + b(u_{xt} + u_{yy})_x = 0 \tag{4}$$

with initial condition

$$u(x,y,0) = \frac{1-c}{2a}\left(1 - 3\tanh^2\left(\frac{1}{\sqrt{2b}}(x+y)\right)\right). \tag{5}$$

In this paper, a, b and c are going to be taken as 1, 1/2 and -1, respectively for all calculations. To search the series solution of Eq. (4) with initial condition (5), the linear operator is chosen

$$\mathscr{L}[\phi(x,y,t;p)] = \frac{\partial\phi(x,y,t;p)}{\partial t}$$

with the property

$$\mathscr{L}[c] = 0$$

where c is constant. From Eq. (4), we define a nonlinear operator as

$$\begin{aligned}\mathscr{N}[\phi(x,y,t;p)] = {} & \frac{\partial\phi(x,y,t;p)}{\partial t} + \frac{\partial\phi(x,y,t;p)}{\partial x} \\ & + 2\phi(x,y,t;p)\frac{\partial\phi(x,y,t;p)}{\partial x} \\ & + \frac{1}{2}\left(\frac{\partial^3\phi(x,y,t;p)}{\partial x^2\partial t} + \frac{\partial^3\phi(x,y,t;p)}{\partial y^2\partial x}\right).\end{aligned}$$

So, the zero-order deformation equation is constructed as

$$(1-p)\mathscr{L}[\phi(x,y,t;p) - u_0(x,y,t)] = p\hbar\mathscr{N}[\phi(x,y,t;p)]. \tag{6}$$

Obviously, if we choose $p = 0$ and $p = 1$ then we get

$$\phi(x,y,t;0) = u_0(x,y,t) = u(x,y,0), \phi(x,y,t;1) = u(x,y,t)$$

respectively. Thus, as the embedding parameter p increases from 0 to 1, the solutions $\phi(x,y,t;p)$ vary from the initial value $u_0(x,y,t)$ to the solution $u(x,y,t)$. If we expand $\phi(x,t;p)$ in Taylor series with respect to the embedding parameter p, we get

$$\phi(x,y,t;p) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t)p^m \tag{7}$$

where

$$u_m(x,y,t) = \frac{1}{m!}\frac{\partial^m\phi(x,y,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{8}$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter $\hbar$ are chosen in a suitable way, the series which are denoted above, converges at $p = 1$, and one has

$$u(x,y,t) = u_0(x,y,t) + \sum_{m=1}^{\infty} u_m(x,y,t)$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [2,5]. The mth-order deformation equation is obtained by differentiating Eq. (6) m times with respect to the embedding parameter p,

$$\mathscr{L}[u_m(x,y,t) - \chi_m u_{m-1}(x,y,t)] = \hbar R_m(\mathbf{u}_{m-1}) \tag{9}$$

where

$$\begin{aligned}R_m(\mathbf{u}_{m-1}) = {} & \frac{\partial u_{m-1}}{\partial t} + \frac{\partial u_{m-1}}{\partial x} + 2\sum_{n=0}^{m-1} u_n \frac{\partial u_{m-1-n}}{\partial x} \\ & + \frac{1}{2}\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n}\frac{\partial u_k}{\partial x}\frac{\partial u_{n-k}}{\partial x}\right)\frac{\partial u_{m-1-n}}{\partial t} \\ & + \frac{1}{2}\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n}\frac{\partial u_k}{\partial y}\frac{\partial u_{n-k}}{\partial y}\right)\frac{\partial u_{m-1-n}}{\partial x}.\end{aligned}$$

The solution of the mth-order deformation Eq. (9) for $m \ge 1$ leads to

$$u_m(x,y,t) = \chi_m u_{m-1}(x,y,t) + \hbar\mathscr{L}^{-1}[R_m(\mathbf{u}_{m-1})]. \tag{10}$$

By using Eq.(10) with initial condition given by (5) we successively obtain

$$
\begin{aligned}
&u_0(x,y,t) = 1 - 3\tanh(x+y)^2,\\
&u_1(x,y,t) = 6\hbar t sech(x+y)^2 \tanh(x+y),\\
&u_2(x,y,t) = \frac{3}{2}\hbar t sech(x+y)^5\\
&\times(-3\hbar t\cosh(x+y) + \hbar t\cosh(3(x+y))\\
&+(1-21\hbar)\sinh(x+y) + (1+3\hbar)\sinh(3(x+y))),\\
&\vdots
\end{aligned}
$$

Therefore, we can write the series solutions expressed by HAM in the following form

$$u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + \dots . \quad (11)$$

To show the efficiency of the method, the HAM solutions of ZK-BBM equation given by Eq. (4) are compared with the exact solutions [19]

$$u(x,y,t) = \frac{1-c}{2a}\left(1 - 3\tanh^2\left(\frac{1}{\sqrt{2b}}(x+y-ct)\right)\right). \quad (12)$$

Note that auxiliary parameter $\hbar$ which our HAM solution series contains, provides us with a simply way to arrange and control the convergence of the solution series. To obtain an suitable range for $\hbar$, we consider the so-called $\hbar$-curve to choose a proper value of $\hbar$ which provides that the solution series is convergent, as pointed by Liao [2], by discovering the valid region of $\hbar$ which corresponds to the line segments nearly parallel to the horizontal axis.

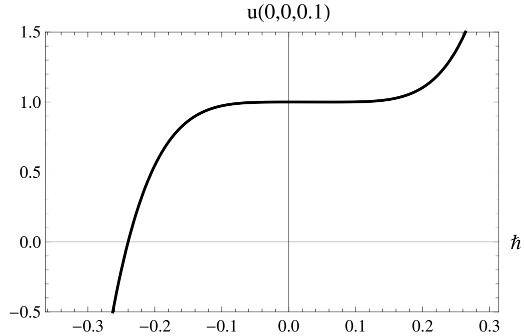

Fig. 1: The $\hbar$-curve of 5th-order approximate solution obtained by the HAM.

In Fig.1, we present the $\hbar$-curve of $u(0,0,0.1)$ given by 5th-order HAM solution (11). From the figure we can see that the valid range of $\hbar$ is approximately $-0.2 \le \hbar \le 0.2$.

Fig. 2 shows the numerical solutions of $u(x,y,t)$ at $x = 0.5$ and $y = 1$ during $0 \le t \le 1$ for $\hbar = -0.1, -0.5, -0.7$ and -0.9 obtained by 5th-order HAM and analytical solutions, respectively. Between $t = 0$ and $t = 1$, it can be seen obviously from this figure that the choice of $\hbar = -0.1$ is a suitable value.

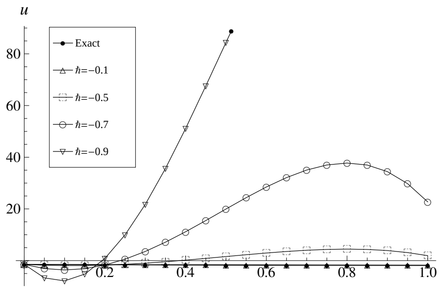

Fig. 2: The results obtained by the HAM for various $\hbar$ by 5th-order approximate solution, in comparison with the exact solution at $x = 0.5$ and $y = 1$.

4 Conclusion

In this paper, we apply the HAM successfully to obtain approximate analytical solution of ZK-BBM equation. We also see that the HAM solution of the problem converges very rapidly to the exact one by choosing a suitable auxiliary parameter $\hbar$. Consequently, as it is seen that the HAM is a powerful and efficient technique for finding the approximate analytical solution of ZK-BBM equation and also many other nonlinear evolution equations which arise in science and engineering.

References

[1] S.J. Liao, The proposed homotopy analysis tecnique for the solution of nonlinear problems, Ph.D thesis, Shanghai Jiao Tong University, 1992.
[2] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC Press, Boca Raton, 2003.
[3] S.J. Liao, Homotopy analysis method: A new analytical technique for nonlinear problems, Commun. Nonlinear Sci. Numer. Simulat. 2(2) (1997) 95-100.
[4] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput., 147, (2004) 499-513.
[5] S.J. Liao, Notes on the homotopy analysis method: Some definitions and theorems, Commun. Nonlinear Sci. Numer. Simulat., 14 (2009) 983-997.
[6] S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota-Satsuma coupled KdV equation, Phys. Lett. A, 361, (2007), 478-483.
[7] E. Babolian, J. Saeidian, Analytic approximate solutions to Burgers, Fisher, Huxley equations and two combined

forms of these eqautions, Commun. Nonlinear. Sci. Numer. Simulat., 14 (2009) 1984-1992.
[8] A. Fakhari, G. Domairry, Ebrahimpour, Approximate explicit solutions of nonlinear BBMB equations by homotopy analysis method and comparison with the exact solution, Phys. Lett. A 368 (2007) 64-68.
[9] M.M. Rashidi, G. Domairry, A. DoostHosseini, S. Dinarvand, Explicit Approximate Solution of the Coupled KdV Equations by using the Homotopy Analysis Method, Int. Journal of Math. Analysis, 2 (12) (2008) 581-589.
[10] M. Inc, On numerical solution of Burgers equation by homotopy analysis method, Phys Lett A, 372 (2008) 356-360.
[11] A.S. Bataineh, M.S.M. Noorani, I. Hashim, Approximate analytical solutions of systems of PDEs by homotopy analysis method, Comp. Math. Appl., 55 (2008) 2913-2923.
[12] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A, 360 (2006) 109-113.
[13] T. Hayat, M. Sajid, On analytic solution for thin film flow of a forth grade fluid down a vertical cylinder, Phys. Lett. A, 361 (2007) 316-322.
[14] S.J. Liao, A.Y. Tan, A general approach to obtain series solutions of nonlinear differential equations, Commun. Nonlinear. Sci. Numer. Simulat., 14 (2009) 983-997.
[15] A. Esen, O.Tasbozan and N.M. Yagmurlu, Approximate Analytical Solutions of the Fractional Sharmo-Tasso-Olver Equation Using Homotopy Analysis Method and a Comparison with Other Methods, Çankaya University Journal of Science and Engineering, 9 (2) (2012) 139-147.
[16] A. Esen, N.M. Yagmurlu and O. Tasbozan, Approximate Analytical Solution to Time-Fractional Damped Burger and Cahn-Allen Equations, Applied Mathematics Information Sciences, 7 (5) (2013) 1951-1956.
[17] O. Tasbozan, A. Esen and N.M. Yagmurlu, Approximate analytical solutions of fractional coupled mKdV equation by homotopy analysis method, Open Journal of Applied Science, 2 (3) (2012) 193-197.
[18] M. Kurulay, A. Secer and M. A. Akinlar, A New Approximate Analytical Solution of Kuramoto Sivashinsky Equation Using Homotopy Analysis Method, Applied Mathematics and Information Sciences, 7, No. 1, 267-271 (2013).
[19] Abdul-Majid Wazwaz, Compact and noncompact physical structures for the ZK-BBM equation, Applied Mathematics and Computation 169 (2005) 713-725.

Orkun Tasbozan graduated from the department of Mathematics of the Afyon Kocatepe University. He has completed his M.Sc. degree in applied mathematics. He is currently studying his Pd.D. thesis. His main interest areas include analytical solutions, fractional differential equations, HAM, FEM etc.

Ali Kurt graduated from the department of Mathematics of the Pamukkale University. He has completed his M.Sc. degree in applied mathematics. He is currently studying his Pd.D. thesis. His main interest areas include analytical solutions for fractional differential equations, HAM, etc.